International Journal of Mathematics And its Applications
Volume 5, Issue 4–B (2017), 259–263.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Affine Semigroups in an Algebra

Research Article

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Abstract: In this paper, we consider certain affine subspaces of an algebra, which are also multiplicative subsemigroups of the algebra. We characterize those lines in an algebra which consists solely of idempotents and which turn out to be subsemigroups of algebra. Also, any such line, if not a semigroup, generates a subsemigroup which is a plane. We also characterize those planes which arise as semigroups generated by such lines and also, those subsemigroups which are generated by such lines.

MSC: 14R10, 20M99.

Keywords: Affine Space, Semigroup, Rectangular Band, Idempotent.

1. Introduction

Throughout this paper, A denotes an associative algebra with unity 1, over the field $\mathbb{K}$, which is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. For a subset X of A, we denote by $\mathsf{A}(X)$, the affine subspace of A generated by X and by $\mathrm{E}(X)$, the set of idempotents in X. By analogy with the geometry of the linear space $\mathbb{R}^2$ over $\mathbb{R}$, we often refer to one-dimensional affine spaces as lines and two-dimensional affine spaces as planes. In particular, $\mathsf{A}(x, y)$ is called the line joining x and y and $\mathsf{A}(x, y, z)$, the plane determined by x, y, z. Also, when using such geometric terminology, elements of A are often referred to as points. An affine subspace of A, which is also a multiplicative subsemigroup of A, will be called an *affine semigroup*

We also use some notations an terminology used in the study of semigroups (See [2, 3] or [4] for details). For elements a and b of a semigroup, we write $a\mathscr{L}b$ iff the principal left ideals generated by a and b are equal and $a\mathscr{R}b$ iff the principal right ideals generated by a and b are equal. It is easily seen that $\mathscr{L}$ and $\mathscr{R}$ are equivalence relations on the semigroup. Also, it can be shown that $\mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$ [2, 3]). We denote this composition by $\mathscr{D}$. For an element a of a semigroup, we denote the $\mathscr{L}$-class containing a by L_a, the $\mathscr{R}$-class containing a by R_a and the $\mathscr{D}$-class containing a by D_a.

2. Characterization of Affine Semigroups

It is easy to see that for idempotents e and f of a semigroup, we have

$$e \,\mathscr{L}\, f \text{ iff } ef = e,\ fe = f \quad \text{and} \quad e \,\mathscr{R}\, f \text{ iff } ef = f,\ fe = e$$

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It follows that any set of idempotents in an $\mathscr{L}$-class or $\mathscr{R}$-class in a semigroup is a subsemigroup. We can show that in the algebra A, the line joining two idempotents in an $\mathscr{L}$-class or $\mathscr{R}$-class contains solely of idempotents in that class. For this we make use of the following results from [1].

Theorem 2.1. *Let e and f be idempotents in the algebra A. Then $\mathsf{A}(e,f) \subseteq \mathrm{E}(A)$ if and only if $(e-f)^2 = 0$.*

From this, we immediately have the following:

Corollary 2.2. *Let e and f be idempotents in A. If $e \mathscr{L} f$, then $\mathsf{A}(e,f) \subseteq \mathrm{E}(L_e)$ and if $e\ \ Rf$, then $\mathsf{A}(e,f) \subseteq \mathrm{E}(R_e)$.*

$\mathscr{L}$-class or a $\mathscr{R}$-class are affine semigroups. We can also prove a sort of converse of this. In the following, any line contained in $\mathrm{E}(A)$ will be called an *idempotent line*.

Theorem 2.3. *An idempotent line in A is an affine semigroup if and only if it is contained in an $\mathscr{L}$-class or $\mathscr{R}$-class in $A(L)$.*

Proof. Let L be an idempotent line which is an affine semigroup and let e and f be points on L. Then $(e-f)^2 = 0$, by Proposition 2.1, so that

$$e + f - ef - fe = 0 \tag{1}$$

Also, since L is a subsemigroup of A, we have $ef \in \mathsf{L} = \mathsf{A}(e,f)$, so that there exists λ in $\mathbb{K}$ such that

$$ef = \lambda e + (1-\lambda)f. \tag{2}$$

Now from the first equation, we have $fef = (e+f-ef)f = f$ and using this in the second equation gives $f = fef = \lambda fe + (1-\lambda)f$. Hence $\lambda(fe - f) = 0$ and so either $\lambda = 0$ or $fe = f$.
Now if $\lambda = 0$, then $ef = f$ from Equation (2) and this gives $fe = e$, by Equation (1). Hence in this case, $e \mathscr{R} f$, so that $\mathsf{L} \subseteq R_e$, by Corollary 2.2. On the other hand if $fe = f$, then $ef = e$ by Equation (1), so that $e \mathscr{L} f$ and hence $\mathsf{L} \subseteq L_e$, again by Corollary 2.2L. ☐

Now an arbitrary idempotent line may not be contained in an $\mathscr{L}$-class or $\mathscr{R}$-class, so that it may not be an affine semigroup, by the above result. For such a line L, we consider the multiplicative subsemigroup of A generated by L. To describe such semigroups, we first prove the following:

Lemma 2.4. *If L is an idempotent line, then $efg = eg$, for all e, f, g in L.*

Proof. Let e, f, g be in L. If $e = f$, then the result is trivially true. Suppose $e \neq f$, so that $\mathsf{L} = \mathsf{A}(e,f)$. Then Equation (1) holds. Also, $g = \lambda e + (1-\lambda)f$ for some λ in $\mathbb{K}$, so that

$$efg = \lambda efe + (1-\lambda)ef = \lambda e + (1-\lambda)ef = e\big(\lambda e + (1-\lambda)f\big) = eg$$

since $efe = e(e+f-ef) = e$, by Equation (1). ☐

It follows that if $e_1, e_2, \ldots, e_n$ is a finite set of points on an idempotent line L, then $e_1 e_2 \cdots e_n = e_1 e_n$. Since for every element e of L, we have $e = ee$ also, the semigroup $\mathsf{S}(\mathsf{L})$ generated by L is given by

$$\mathsf{S}(\mathsf{L}) = \{x \in A \colon x = ef \ \text{ for } \ e, f \in \mathsf{L}\} \tag{3}$$

Again, for e and f in L, Equation (1) holds, so that $efe = e$, as seen earlier, which gives $(ef)^2 = ef$. Hence $\mathrm{S}(\mathsf{L}) \subseteq \mathrm{E}(A)$. Moreover, if e, f, g are in $\mathrm{S}(\mathsf{L})$, then $e = e_1e_2$, $f = f_1f_2$, $g = g_1g_2$ where all factors are in L, so that $efg = e_1g_2 = eg$, by the above lemma.

Now a semigroup in which all elements are idempotent is called a *band* and a band in which $xyz = xz$ for all elements x, y, z is called a *rectangular band*. The name is due to the fact that any such semigroup is isomorphic to a semigroup got by defining on the product $L \times R$ of two non-empty sets, the composition $(l, r)(l', r') = (l, r')$ (see [4], Proposition IV.3.2). Thus $\mathrm{S}(\mathsf{L})$ is a rectangular band. Also, it is contained in a $\mathscr{D}$-class of A, because of the following result proved in [1]:

Theorem 2.5. *If $\mathsf{A}(e, f)$ is an idempotent line in A, then $e \mathrel{\mathscr{R}} ef \mathrel{\mathscr{L}} f$ and $e \mathrel{\mathscr{L}} fe \mathrel{\mathscr{R}} f$. Consequently, e, f, ef and fe are in the same $\mathscr{D}$-class in A.*

Again let e, f be distinct points on an idempotent line L. Then $\mathsf{L} = \mathsf{A}(e, f)$, so that if g and h are in L, then $g = \lambda(e - f) + f$ and $h = \mu(e - f) + f$ for some λ, μ in $\mathbb{K}$. Hence

$$gh = \big(\lambda(e - f) + f\big)\big(\mu(e - f) + f\big) = \lambda(e - f)f + \mu f(e - f) + f = \lambda ef + \mu fe + (1 - \lambda - \mu)f$$

using the fact that $(e - f)^2 = 0$. So, $gh \in \mathsf{A}(ef, fe, f)$. Thus the product of every pair of elements of L is in $\mathsf{A}(ef, fe, f)$. The equation read in reverse also shows that any point in $\mathsf{L}(ef, fe, f)$ is a product of two elements in $\mathsf{A}(e, f) = \mathsf{L}$. Hence $\mathrm{S}(\mathsf{L}) = \mathsf{A}(ef, fe, f)$. Now if L is not contained in an $\mathscr{L}$-class or $\mathscr{R}$-class, then $\mathrm{S}(\mathsf{L})$ is not a line, for if so it would be a line containing L and hence equal to L, which would imply L is contained in an $\mathscr{L}$-class or $\mathscr{R}$-class, by Proposition 2.3. Thus $\mathrm{S}(\mathsf{L}) = \mathsf{A}(ef, fe, f)$ is a plane. Also, $e = ef + fe - f \in \mathsf{A}(ef, fe, f)$, so that $\mathsf{A}(ef, fe, f) = \mathsf{A}(e, f, ef, fe)$. and again, $fe = e + f - fe \in \mathsf{A}(e, f, ef)$, so that $\mathsf{A}(e, f, ef, fe) = \mathsf{A}(e, f, ef)$. Hence $\mathrm{S}(\mathsf{L}) = \mathsf{A}(e, f, ef)$. We summarize this discussion as follows:

Theorem 2.6. *Let L be a line contained in $\mathrm{E}(A)$. Then the subsemigroup $\mathrm{S}(\mathsf{L})$ of A generated by L is an affine rectangular band in A. If L is contained in an $\mathscr{L}$-class or $\mathscr{R}$-class, then $\mathrm{S}(\mathsf{L}) = \mathsf{L}$ and otherwise $\mathrm{S}(\mathsf{L})$ is a plane containing L and contained in the $\mathscr{D}$-class containing L. In either case, $\mathrm{S}(\mathsf{L})$ is the affine subspace of A generated by any two distinct elements of L and their product.*

We can give a purely algebraic formulation of this result. Let e and f be idempotents in A with $(e - f)^2 = 0$. Then $\mathsf{L} = \mathsf{A}(e, f)$ is an idempotent line and we have seen that $\mathrm{S}(\mathsf{L}) = \mathsf{A}(e, f, ef) = \mathsf{A}(e, f, ef, fe)$. Also, ef and fe are idempotents with $efe = e$ and $fef = f$, as seen earlier, so that $\{e, f, ef, fe\}$ is the semigroup generated by e and f, and we denote this as $\mathrm{S}(e, f)$. Thus we can write the above result as follows:

Theorem 2.7. *For $e, f \in \mathrm{E}(A)$, if $(e - f)^2 = 0$ then $\mathrm{S}(\mathsf{A}(e, f))$ is an affine rectangular band in A and $\mathrm{S}(\mathsf{A}(e, f)) = \mathsf{A}(\mathrm{S}(e, f)) = \mathsf{A}(e, f, ef)$.*

In the above discussion, we have seen that every element of $\mathrm{S}(\mathsf{L})$ is a product of a pair of (possibly equal) elements of L, so that we have a map $(e, f) \mapsto ef$ of $\mathsf{L} \times \mathsf{L}$ onto $\mathrm{S}\mathsf{L}$. In the case when L is not contained in an $\mathscr{L}$-class or $\mathscr{R}$-class, it is also one-to-one. To see this, let (e, f) and (g, h) be in $\mathsf{L} \times \mathsf{L}$ with $ef = gh$, so that using Lemma 2.4, $e = efe = ghe = ge$ and $eg = efg = ghg = g^2 = g$ and so $g \mathrel{\mathscr{R}} e$. Hence $g = e$, for if $g \neq e$, then $\mathsf{L} \subseteq R_e$, by Proposition 2.2. Similarly $f = fef = fgh = fh$ and $hf = hef = hgh = h^2 = h$ so that $h \mathrel{\mathscr{L}} f$ and hence $h = f$. Thus $(e, f) \mapsto ef$ is a bijection of $\mathsf{L} \times \mathsf{L}$ onto $\mathrm{S}(\mathsf{L})$. It is in fact an isomorphism, if $\mathsf{L} \times \mathsf{L}$ is equipped with the rectangular band multiplication:

Theorem 2.8. *Let L be an idempotent line, not contained in an $\mathscr{L}$-class or an $\mathscr{R}$-class, then $\mathrm{S}(\mathsf{L})$ is isomorphic to the rectangular band $\mathsf{L} \times \mathsf{L}$ with composition defined by $(e, f)(g, h) = (e, h)$.*

Proof. Let $\phi\colon \mathsf{L}\times\mathsf{L}\to \mathsf{S}(\mathsf{L})$ be defined by $\phi(e,f)=ef$. Then ϕ is a bijection onto $\mathsf{S}(\mathsf{L})$ as seen above. Also, for (e,f), (g,h) in $\mathsf{L}\times\mathsf{L}$, we have $\phi\big((e,f)(g,h)\big)=\phi(e,h)=eh$ and $\phi(e,f)\phi(g,h)=efgh=eh$ so that $\phi\big(e,f)(g,h)\big)=\phi(e,f)\phi(g,h)$. This completes the proof. □

Note that if L is contained in an $\mathscr{L}$-class, then $\mathsf{S}(\mathsf{L})=\mathsf{L}$ is isomorphic to the rectangular band $\mathsf{L}\times\{1\}$ and if L is contained in an $\mathscr{R}$-class, then $\mathsf{S}(\mathsf{L})=\mathsf{L}$ is isomorphic to the rectangular band $\{1\}\times\mathsf{L}$.

From another point of view, Proposition 2.6 describes certain idempotent planes as semigroups generated by idempotent lines. We can characterize such planes as follows:

Theorem 2.9. *Let* P *be a plane contained in* $\mathrm{E}(A)$. *Then* $\mathsf{P}=\mathsf{S}(\mathsf{L})$ *for some line* L *if and only if* P *contains two idempotents, which are neither* $\mathscr{L}$*-related nor* $\mathscr{R}$*-related, and also their product.*

Proof. First suppose that $\mathsf{P}=\mathsf{S}(\mathsf{L})$ for some line L in A and let e, f be elements of L. Then e and f are idempotents, since $\mathsf{L}\subseteq\mathsf{P}\subseteq\mathrm{E}(A)$. Now if e and f are $\mathscr{L}$-related or $\mathscr{R}$-related, then L is contained in an $\mathscr{L}$-class or $\mathscr{R}$-class, by Proposition 2.2 and so $\mathsf{S}(\mathsf{L})=\mathsf{L}$, by Proposition 2.3, contrary to the fact that P is a plane. Thus e and f are not $\mathscr{L}$-related or $\mathscr{R}$-related. Also $ef\in\mathsf{S}(\mathsf{L})=\mathsf{P}$.

Conversely, suppose that P contains e, f and ef with e and f neither $\mathscr{L}$-related nor $\mathscr{R}$-related. Let $\mathsf{L}=\mathsf{A}(e,f)$. Since P is an idempotent affine space, we have $\mathsf{L}=\mathsf{A}(e,f)\subseteq\mathsf{P}\subseteq\mathrm{E}(A)$, and so $e\,\mathscr{R}\,ef$, by Proposition 2.5. Hence if ef is a point on L, then $\mathsf{L}\subseteq R_e$, by Proposition 2.2, and since $f\in\mathsf{L}$, this gives $f\,\mathscr{R}\,e$, contrary to our assumption. Thus $ef\notin\mathsf{L}$ and so e, f, ef are affinely independent points in the plane P. Hence $\mathsf{P}=\mathsf{A}(e,f,ef)$. Again since e, f are distinct elements in L, we have $\mathsf{S}(\mathsf{L})=\mathsf{A}(e,f,ef)$, by Proposition 2.6. So, $\mathsf{P}=\mathsf{S}(\mathsf{L})$. □

Thus among idempotent affine subspaces of A, only certain lines and planes arise as affine semigroups generated by idempotent lines, by Proposition 2.6. Such lines are algebraically characterized in Proposition 2.3 and such planes in Proposition 2.9. A related question is the geometric characterization of those affine semigroups of A which are generated by idempotent lines. This is given by the following result:

Theorem 2.10. *Let* B *be an affine rectangular band in* A. *If* $B=\mathsf{S}(\mathsf{L})$ *for an idempotent line in* A*, then* L *intersects every* $\mathscr{L}$*-class and every* $\mathscr{R}$*-class of* B. *Conversely, if* B *contains an idempotent line* L *which intersects every* $\mathscr{L}$*-class and every* $\mathscr{R}$*-class of* B*, then* $B=\mathsf{S}(\mathsf{L})$.

Proof. First suppose that $B=\mathsf{S}(\mathsf{L})$ for an idempotent line L in A and let $e\in B$. Then $e=fg$ for some f, g in L. Hence $f\,\mathscr{R}\,fg\,\mathscr{L}\,g$ and $f\,\mathscr{L}\,gf\,\mathscr{R}\,g$, by Proposition 2.5. Since $e=fg$, it follows that $f\in\mathsf{L}\bigcap R_e$ and $g\in\mathsf{L}\bigcap L_e$.

Conversely, suppose that B contains an idempotent line L which intersects every $\mathscr{L}$-class and every $\mathscr{R}$-class of B. Since B is a subsemigroup of A containing L and $\mathsf{S}(\mathsf{L})$ is the smallest subsemigroup of A containing L, we have $\mathsf{S}(\mathsf{L})\subseteq B$. To prove the reverse inclusion, let e be an element of B. Then by assumption, there exists f in $\mathsf{L}\bigcap R_e$ and g in $\mathsf{L}\bigcap L_e$. Again by Proposition 2.5, we have $f\,\mathscr{R}\,fg\,\mathscr{L}\,g$ and $f\,\mathscr{L}\,gf\,\mathscr{R}\,g$. Since e and fg are idempotents in $R_f\bigcap L_g$, it follows that $e=fg$ (see [2, 3], Lemma 2.15)). Hence $e\in\mathsf{S}(\mathsf{L})$ and since e is an arbitrary element of B, it follows that $B\subseteq\mathsf{S}(\mathsf{L})$. This proves the result. □

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