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Contact CR-submanifold with η-parallel F-structure

Research Article

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Abstract: The purpose of this paper is to prove that a contact CR-submanifold of a Sasakian-space-form is a contact CR-product submanifold, by using $\eta-$parallel F-structure. At last, we shall derive cyclic parallelism of contact CR-submanifold and hence obtain some inequalities based on the second fundamental form h.

Keywords: Contact CR-submanifold, Sasakian-space-form, Contact CR-product submanifold mappings.

1. Introduction

Sasakian manifolds with Riemannian metric were introduced in 1960 by the Japanese geometer Shigeo Sasaki [17]. Sasakian geometry is an odd-dimensional counterpart to Kaehler geometry. There was not much activity in this field after the mid-1970s, until the advent of String theory. Since then Sasakian manifolds have gained prominence in physics (including classical mechanics, dynamics, geometric optics and control theory) and algebraic geometry, mostly due to a string of papers by Boyer, Galicki [7] and their co-authors. The study of differential geometry of a contact CR-submanifolds, as a generalization of invariant and anti-invariant submanifolds, of an almost contact metric manifold was initiated by A. Bejancu [3] and was followed by several geometers.

The CR-submanifolds of a Kaehlerian manifold have been defined and studied by A. Bejancu [1, 2] in 1978. K. Yano and M. Kon [9] introduced the notion of a contact CR-submanifolds of a Sasakian manifold which is closely similar to that of a Kaehlerian manifold. And several authors studied contact CR-submanifolds of different classes of almost contact metric manifolds such as Sasakian-space-form [15], Cosymplectic-space-form [6] and Kenmotsu-space-form. Many authors have proved that CR-submanifold of a Kaehlerian manifold, Sasakian manifold is a CR-product, contact CR-product, respectively, with some additional condition. M. Kon [16] proved an inequality

$$\|\nabla A\|^2 \geq \left(\frac{c^2}{8}\right)(\|P\|^2\|t\|^2 + \|FP\|^2)$$

for $n-$dimensional submanifold of complex-space-form. The contact CR-submanifolds are rich and interesting subject. Therefore we continue to work in this branch of submanifolds. So in this present paper we wish to prove these two results in contact CR-submanifold of Sasakian-space-form.

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2. Preliminaries

Let $\overline{\mathcal{M}}$ be a $2n+1$-dimensional Sasakian manifold with structure tensors (ϕ, ξ, η, g). We consider a Riemannian manifold $\mathcal{M}$ isometrically immersed in $\overline{\mathcal{M}}$ with induced metric tensor g. We assume that the submanifold $\mathcal{M}$ is tangent to the structure vector field ξ of $\overline{\mathcal{M}}$. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connection on $\mathcal{M}$ and $\overline{\mathcal{M}}$ respectively. Then the Gauss and Weingarten formulas are given respectively by [11].

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \text{ and } \overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X Y$$

for all X, Y tangent to $\mathcal{M}$ and vector field N normal to $\mathcal{M}$. Here h, $\nabla^{\perp}_X$, A_N denote the second fundamental form, normal connection and the shape operator, respectively. The second fundamental form and the shape operator are related by

$$g(h(X,Y),N) = g(A_N X, Y).$$

Let R and $\overline{R}$ be the curvature tensor of $\mathcal{M}$ and $\overline{\mathcal{M}}$, respectively, then the Gauss equation is given by

$$g(\overline{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z))$$

for any vector fields X, Y, Z, W tangent to $\mathcal{M}$. The mean curvature vector H is defined by

$$H = \frac{1}{n+1}(trace\ h)$$

A submanifold $\mathcal{M}$ is said to be minimal if $H \equiv 0$ and $\mathcal{M}$ is called a totally geodesic submanifold in $\overline{\mathcal{M}}$ if $h = 0$. For any $X \in T_p\mathcal{M}$, we put $\phi X = PX + FX$, where PX and FX are the tangential and normal components of ϕX, respectively. Similarly, for any vector field U normal to $\mathcal{M}$, we put $\phi U = tU + fU$, where tU and fU are the tangential and normal components of ϕU, respectively. The well-known results are as follows:

(A). P and $f-$structures are skew-symmetric,

(B). $g(FX,U) = -g(X,tU)$,

(C). $FP + fF = 0$,

(D). $\phi\xi = 0$ implies $P\xi = 0$, $F\xi = 0$.

Definition 2.1. *A plane section $\prod \in T_p\overline{\mathcal{M}}$ is called a $\phi-$section if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a $\phi-$section is called ϕ -holomorphic sectional curvature.*

Definition 2.2. *A Sasakian manifold with constant $\phi-$holomorphic sectional curvature c is said to be a Sasakian-space-form and it is denoted by $\overline{\mathcal{M}}(c)$.*

Let $\mathcal{M}$ be an $(n+1)-$dimensional submanifold of a Sasakian-space-form $\overline{\mathcal{M}}(c)$. Then we have the following Gauss and Codazzi equations, respectively:

$$\begin{aligned} R(X,Y)Z &= \left(\frac{c+3}{4}\right)[g(Y,Z)X - g(X,Z)Y] + \left(\frac{c-1}{4}\right)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &\quad + g(PY,Z)PX - g(PX,Z)PY + 2g(X,PY)PZ] + A_{h(Y,Z)}X - A_{h(X,Z)}Y \end{aligned} \tag{1}$$

and

$$(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = \left(\frac{c-1}{4}\right)[g(PY,Z)FX - g(PX,Z)FY + 2g(X,PY)FZ] \tag{2}$$

for any vector fields X, Y, Z tangent to $\mathcal{M}$.

Definition 2.3. *Let $\mathcal{M}$ be a submanifold isometrically immersed in a Sasakian manifold $\overline{\mathcal{M}}$ tangent to the structure vector field ξ. Then M is called a CR-submanifold of $\mathcal{M}$ if there exists a differentiable distribution $\mathcal{D} : p \to \mathcal{D}_p \subset T_p\mathcal{M}$ on $\mathcal{M}$ satisfying the following conditions:*

(A). $\mathcal{D}$ is invariant with respect to ϕ, i.e., $\phi\mathcal{D}_p \subset \mathcal{D}_p$ for each $p \in \mathcal{M}$,

(B). the complementary orthogonal distribution $\mathcal{D}^\perp : p \to \mathcal{D}^\perp \subset T_p\mathcal{M}$ is anti-invariant with respect ϕ, i.e., $\phi\mathcal{D}_p^\perp \subset T_p\mathcal{M}^\perp$.

We have the following decomposition of the tangent space $T_p\mathcal{M}$ at each point p of $\mathcal{M}$: $T_p\mathcal{M} = \mathcal{D}_p \oplus \{\xi\} \oplus \mathcal{D}_p^\perp$, where $\phi\mathcal{D}_p \subset \mathcal{D}_p \subset T_p\mathcal{M}$ and $\phi\mathcal{D}_p^\perp \subset F\mathcal{D}_p^\perp \subset T_p\mathcal{M}^\perp$. Similarly, we have $T_p\mathcal{M}^\perp = F\mathcal{D}_p^\perp \oplus \mathcal{N}_p$, where $\mathcal{N}_p$ is the orthogonal complement of $F\mathcal{D}_p^\perp \in T_p\mathcal{M}^\perp$. Then $\phi\mathcal{N}_p = f\mathcal{N}_p = \mathcal{N}_p$.

3. Some Basic Results

This section deals with some basic and very useful lemmas, which are required in proving our main theorems.

Lemma 3.1 ([11]). *In order for a submanifold $\mathcal{M}$, tangent to the structure vector field ξ, of a Sasakian manifold $\overline{\mathcal{M}}$ to be a contact CR-submanifold, it is necessary and sufficient that $FP = 0$.*

Lemma 3.2 ([11]). *Let $\mathcal{M}$ be a contact CR-submanifold of a Sasakian manifold $\overline{\mathcal{M}}$. Then $P^3 + P = 0$ and $f^3 + f = 0$.*

Before going to prove next lemma we would like to give following definition:

Definition 3.3. *A submanifold $\mathcal{M}$ in a Riemannian manifold $\overline{\mathcal{M}}$ is said to be cyclic parallel if its second fundamental form h satisfies*

$$(\nabla_X h)(Y, Z) + (\nabla_Y h)(Z, X) + (\nabla_Z h)(X, Y) = 0 \tag{3}$$

for any vector fields X, Y, Z tangent to $\mathcal{M}$.

Now we can easily prove our following lemma:

Lemma 3.4. *Let $\mathcal{M}$ be a contact CR-submanifold of a Sasakian-space-form $\overline{\mathcal{M}}(c)$. Then cyclic parallelism of $\mathcal{M}$ is equivalent to the following condition:*

$$(\nabla_X h)(Y, Z) = -\left(\frac{c-1}{4}\right)[g(PX, Z)FY + g(PX, Y)FZ] \tag{4}$$

for any vector fields X, Y, Z tangent to $\mathcal{M}$.

Proof. Our assertion (4) can be proved by using equations (3) and (2). □

4. Main Results

In this section, we prove our two main theorems. First we give some basic definitions, which shall be required in our proofs.

Definition 4.1. *The distribution $\mathcal{D}$ is called parallel with respect to the Riemannian connection ∇ on $\mathcal{M}$ if $\nabla_X Y \in \mathcal{D}$ for any vector fields $X, Y \in \mathcal{D}$.*

Definition 4.2. *The $F-$structure in the normal bundle is said to be $\eta-$parallel if it satisfies the following: $g((\nabla_X F)Y, FZ) = 0$ for any vector fields $X, Y, Z \in T_p\mathcal{M}$.*

Definition 4.3. *A contact CR-submanifold $\mathcal{M}$ of a Sasakian manifold is called a contact CR-product submanifold if it is locally a Riemannian product of an invariant submanifold $\mathcal{M}^{\top}$ and an anti-invariant submanifold $\mathcal{M}^{\perp}$.*

Our first main theorem is as follows:

Theorem 4.4. *A contact CR-submanifold $\mathcal{M}$ of a Sasakian space form $\overline{\mathcal{M}}(c)$ is a contact CR-product if $F-$structure in the normal bundle is $\eta-$parallel.*

Proof. Suppose that F is $\eta-$parallel $g((\nabla_X F)Y, FZ) = 0$ for $X, Y, Z \in T_p\mathcal{M}$, then $-g(h(X, PY), FZ) + g(fh(X, Y), FZ) = 0$. This implies that $g(th(X, PY), Z) = 0$ for all vector fields X, Y, Z tangent to $\mathcal{M}$. Now if $X, Y \in \mathcal{D}$ and for any vector field Z tangent to $\mathcal{M}$, we have $g(F\nabla_X Y, FZ) = 0$ which implies that $\nabla_X Y \in \mathcal{D}$ and consequently the distribution $\mathcal{D}$ is parallel. Now if $X, Y \in \mathcal{D}^{\perp}$, and then for any vector field Z tangent to $\mathcal{M}$, we have $g(P\nabla_X Y, PZ) = g(th(X, PZ), Y) = 0$ which implies that $\nabla_X Y \in \mathcal{D}^{\perp}$ and consequently the distribution $\mathcal{D}^{\perp}$ is parallel. Hence $\mathcal{M}$ is a contact CR-product submanifold of $\overline{\mathcal{M}}(c)$. □

Before going to prove our next theorem we would give some notations, which are required in the proof. We put dim $\overline{\mathcal{M}} = 2m + 1$, dim $\mathcal{M} = n + 1$, dim $\mathcal{D} = k$, dim $\mathcal{D}^{\perp} = q$ and codim $\mathcal{M} = 2m - n$.

Theorem 4.5. *Let $\mathcal{M}$ be a contact CR-submanifold of a Sasakian-space-form $\overline{\mathcal{M}}(c)$. Then we have*

$$g(\nabla h, \nabla h) = \|\nabla h\|^2 \geq 2\left(\frac{c-1}{4}\right)^2 kq.$$

The equality holds if and only if (4) holds.

Proof. We define a tensor field T on $\mathcal{M}$ by

$$T(X, Y, Z) = (\nabla_X h)(Y, Z) + \left(\frac{c-1}{4}\right)[g(PX, Z)FY + g(PX, Y)FZ]$$

Then $T = 0$ if and only if (4) holds. Let $\{e_1, \dots, e_{n+1}\}$ be an orthonormal basis of the tangent space $T_p\mathcal{M}$. Therefore, we have

$$\|T\|^2 = \|\nabla h\|^2 + 2\left(\frac{c-1}{4}\right)^2 kq + (c-1)\sum_{i,j=1}^{n+1} g((\nabla_{e_i} h)(e_j, Pe_i), Fe_j). \tag{5}$$

On the other hand, by equation of Codazzi

$$g((\nabla_{e_i} h)(e_j, Pe_i), Fe_j) = -\left(\frac{c-1}{4}\right)kq. \tag{6}$$

Putting (6) into (5), we get

$$\begin{aligned}\|T\|^2 &= \|\nabla h\|^2 + 2\left(\frac{c-1}{4}\right)^2 kq - 4\left(\frac{c-1}{4}\right)^2 kq \\ &= \|\nabla h\|^2 - 2\left(\frac{c-1}{4}\right)^2 kq.\end{aligned}$$

Since $\|\nabla T\|^2 \geq 0$, which implies that

$$\|\nabla h\|^2 \geq 2\left(\frac{c-1}{4}\right)^2 kq$$

This completes the proof of our theorem. □

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