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The Spectrum of Two New Corona of Graphs and its Applications

Research Article

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Abstract: In this paper we introduced two notions of corona products of graphs such as Duplication vertex corona and Duplication add vertex corona. Here we mainly determine the adjacency, Laplacian and signless Laplacian spectra of the new corona products of two graphs and we prove that the Duplication vertex corona and Duplication add vertex corona are cospectral graphs. In addition to that the Kirchhoff index and number of spanning trees of the new graph corona products were also calculated. Lastly, we focus on the classification of new class of integral graphs.

MSC: 05C50, 05C76.

Keywords: Spectrum, duplication graph, corona of graphs, integral graphs, Kirchhoff index, spanning tree.

1. Introduction

Throughout the paper we consider only simple and undirected graphs. Let G be an arbitrary graph with n vertices and vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. Let $A(G)$ be the adjacency matrix of G. It is an $n \times n$ symmetric matrix, $A(G) = (a_{ij})_{n\times n}$, where $a_{ij} = 1$ if v_i and v_j are connected by an edge in G and 0, elsewhere.

Let the degree of v_i (number of vertices adjacent to v_i) in G be d_i and the diagonal degree matrix of G be $D(G) = diag(d_1, d_2, \dots, d_n)$. *Brouwer* and *Haemers* in [3] defined Laplacian matrix and signless Laplacian matrix as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ respectively. The characteristic polynomial of A or of G is defined as $f_G(A : x) = det(xI_n - A)$, where I_n is the identity matrix of order n. The *eigenvalues* of G are the roots of $f_G(A : x) = 0$. It is denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and usually called adjacency spectrum or $A - spectrum$ of G. Similar manner the eigenvalues of $L(G)$ and $Q(G)$ are denoted by $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ and $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$. They are called the Laplacian and signless Laplacian spectrum (or L - *spectrum* and Q - *spectrum* respectively) of G. The eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are real numbers since the matrices are real and symmetric. The adjacency spectrum of a graph consists of the eigenvalues (together with their multiplicities) and the Laplacian (signless Laplacian) spectrum of G consists of the Laplace (signless Laplace) eigenvalues together with their multiplicities. A - *cospectral* graphs are those graphs with the same A - spectrum. *Frucht* and *Harary* in [5] introduced the concept of *corona* of two graphs and their spectrum by *S. Barik et. al* [2]. In [6] *Gopalapillai* introduced neighborhood corona of graphs and calculated the corresponding spectrum. In [11] *Varghese* and *Susha* defined some new join in duplication graph of an arbitrary graph. Motivated from these, in this paper we define

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two new corona of graphs based on duplication graph of a graph and determined their adjacency, Laplacian and signless Laplacian spectrum.

The organisation of the paper is as follows. In section 2.1 we have some basic results on spectral graph theory which are useful in the succeeding sections. In section 3 we define two new corona product using duplication graph of a graph and find their adjacency, Laplacian and signless Laplacian spectra and we proved that they are cospectral. Then in the last section we discuss some applications such as number of spanning trees and the Kirchhoff index. We also give a brief description on some classification of new class of integral graphs.

2. Preliminaries

Definition 2.1 ([10]). *Suppose G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $U(G) = \{x_1, x_2, \ldots, x_n\}$ be another set corresponding to $V(G)$. Draw x_i adjacent to all the vertices in $N(v_i)$, the neighborhood set of v_i, in G for each i and delete the edges of G only. The graph thus obtained is called the duplication graph of G and we denote it as DG.*

Lemma 2.2 ([4]). *Let $M = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}$ be a block symmetric matrix of order 2×2. Then the eigenvalues of M are those of $M_1 + M_2$ together with $M_1 - M_2$.*

Proposition 2.3 ([4]). *Let $P_1, P_2, P_3,$ and P_4 be matrices of order $n_1 \times n_1, n_1 \times n_2, n_2 \times n_1, n_2 \times n_2$ respectively with P_1 and P_4 are invertible. Then*

$$det \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} = det(P_1)\ det(P_4 - P_3P_1^{-1}P_2)$$
$$= det(P_4)\ det(P_1 - P_2P_4^{-1}P_3).$$

Definition 2.4 ([9]). *Let A be the adjacency matrix of a graph G with n vertices. The determinant $det(xI - A) = f_G(A : x) \neq 0$, is invertible being the characteristic matrix of A. The $A-coronal$, $\chi_A(x)$, of G is defined to be the sum of the entries of the matrix $(xI - A)^{-1}$. We denote this as $\chi_A(x) = \boldsymbol{1}_n^T(xI - A)^{-1}\boldsymbol{1}_n$, where $\boldsymbol{1}_n$ is a $n \times 1$ column vector with all entries equal to 1.*

We use the following results by *McLeman* and *McNicholas* defined in [9].

Let G be an r - regular graph on n vertices. Then

$$\chi_A(x) = \frac{n}{x - r}. \tag{1}$$

Each row sum of the Laplacian matrix $L(G)$ of any graph G with n vertices equal to 0. Then

$$\chi_L(x) = \frac{n}{x}. \tag{2}$$

Let G be the bipartite graph $K_{p,q}$ where $p + q = n$. Then

$$\chi_A(x) = \frac{nx + 2pq}{x^2 - pq}. \tag{3}$$

Let $A = (a_{ij})$ and B be matrices. Then the Kronecker product [4], $A \otimes B$, of A and B is defined as the partition matrix $(a_{ij}B)$. This associative operation has the property that $(A \otimes B)^T = A^T \otimes B^T$, $(A + B) \otimes C = A \otimes C + B \otimes C$ and

$(A\otimes B)(C\otimes D) = AC\otimes BD$ whenever the product AC and BD exist. Also for the non-singular matrix A and $B, (A\otimes B)^{-1}$ $= A^{-1}\otimes B^{-1}$. Moreover if A and B are $n\times n$ and $p\times p$ matrices, then $det\ (A\otimes B) = (det\ A)^p\ (det\ B)^n$. Under these arguments we can substantiate that

$$(\mathbf{1}_n^T\otimes I_n)((xI_n - A)^{-1}\otimes I_n)(\mathbf{1}_n\otimes I_n) = I_n\chi_A(x) \tag{4}$$

$$(\mathbf{1}_n^T\otimes I_n)(((x-1)I_n - A)^{-1}\otimes I_n)(\mathbf{1}_n\otimes I_n) = I_n\chi_A(x-1) \tag{5}$$

3. New Corona Product of Graphs and Their Spectra

The following definitions describes the new graph corona product based on the duplication graph of a graph.

Definition 3.1. *Let G_1 and G_2 be two vertex disjoint graphs with n_1 and n_2 vertices respectively. Let DG_1 be the duplication graph of G_1 with vertex set $V(G_1)\cup U(G_1)$, where $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $U(G_1) = \{x_1, x_2, \ldots, x_{n_1}\}$. Duplication add vertex corona, $G_1\underline{\circledast}G_2$, is the graph obtained from DG_1 and n_1 copies of G_2 by making x_i adjacent to every vertices in the i^{th} copy of G_2 for $i = 1, 2, \ldots, n_1$.*

Definition 3.2. *Let G_1 and G_2 be two vertex disjoint graphs with n_1 and n_2 vertices respectively. Let DG_1 be the duplication graph of G_1 with vertex set $V(G_1)\cup U(G_1)$, where $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $U(G_1) = \{x_1, x_2, \ldots, x_{n_1}\}$. Duplication vertex corona, $G_1\underline{\circledcirc}G_2$, is the graph obtained from DG_1 and n_1 copies of G_2 by making v_i adjacent to every vertices in the i^{th} copy of G_2 for $i = 1, 2, \ldots, n_1$.*

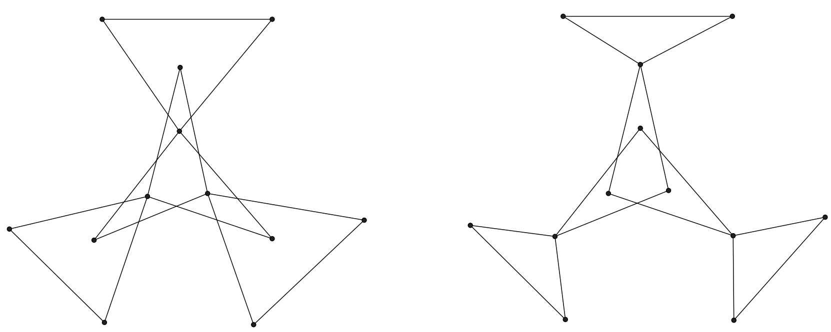

Figure 1. $K_3\underline{\circledcirc}K_2$ **and** $K_3\underline{\circledast}K_2$

If G_1 is a graph with n_1 vertices and m_1 edges and G_2 is a graph with n_2 vertices and m_2 edges, then $G_1\underline{\circledcirc}G_2$ and $G_1\underline{\circledast}G_2$ has $n_1(n_2+2)$ vertices and $2m_1 + n_1(n_2+m_2)$ edges.
Now we find the adjacency, Laplacian and signless Laplacian spectrum of $G_1\underline{\circledast}G_2$.

Theorem 3.3. *Let G_i be two graphs with n_i vertices with spectrum $\lambda_{i1}(G)\geq\lambda_{i2}(G)\geq\cdots\geq\lambda_{in}(G)$, for $i = 1, 2$. Then the characteristic polynomial of duplication add vertex corona, $G_1\underline{\circledast}G_2$, is*

$$f_{G_1\underline{\circledast}G_2}(A:x) = \prod_{j=1}^{n_2}(x-\lambda_{2j})^{n_1}\prod_{i=1}^{n_1}(x^2 - x\chi_{A_2}(x) - \lambda_{1i}^2).$$

Proof. Let G_1 be an r_1 - regular graph on n_1 vertices and m_1 edges. G_2 be an arbitrary graph on n_2 vertices. $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $U(G_1) = \{x_1, x_2, \ldots, x_{n_1}\}$. The vertex in the i^{th} copy of G_2 be $\{u_1^i, u_2^i, \ldots, u_{n_2}^i\}$ and let W_j =

$\{u_j^1, u_j^2, \ldots, u_j^{n_2}\}$ for $j = 1, 2, \ldots, n_2$. Joining x_i to every vertex of the i^{th} copy of G_2. Then $V(G_1) \cup U(G_1) \cup \{W_1 \cup W_2 \cup \ldots \cup W_{n_2}\}$ is a vertex partition of $G_1 \underline{\circledast} G_2$. By these vertex partitioning the adjacency matrix of $G_1 \underline{\circledast} G_2$ is

$$A = \begin{bmatrix} 0 & A_1 & 0_{n_1 \times n_1 n_2} \\ A_1 & 0_{n_1 \times n_1} & \mathbf{1}_{n_2}^T \otimes I_{n_1} \\ 0_{n_1 n_2 \times n_1} & \mathbf{1}_{n_2} \otimes I_{n_1} & A_2 \otimes I_{n_1} \end{bmatrix},$$

where A_1 and A_2 are the adjacency matrix of G_1 and G_2 respectively. $\mathbf{1}_{n_2}$ is a $n_2 \times 1$ column vector with all entries equal to 1. The characteristic polynomial of $G_1 \underline{\circledast} G_2$

$$\begin{aligned} f_{G_1 \underline{\circledast} G_2}(A : x) &= det(xI - A) \\ &= \begin{vmatrix} xI_{n_1} & -A_1 & 0 \\ -A_1 & xI_{n_1} & -\mathbf{1}_{n_2}^T \otimes I_{n_1} \\ 0 & -\mathbf{1}_{n_2} \otimes I_{n_1} & (xI_{n_2} - A_2) \otimes I_{n_1} \end{vmatrix}. \end{aligned}$$

By using Proposition 2.3 we get,

$$f_{G_1 \underline{\circledast} G_2}(A : x) = det\left((xI_{n_2} - A_2) \otimes I_{n_2}\right)\ det\ S,$$

where

$$S = \begin{pmatrix} xI_{n_1} & -A_1 \\ -A_1 & xI_{n_1} \end{pmatrix} - \begin{pmatrix} 0 \\ -\mathbf{1}_{n_2}^T \otimes I_{n_1} \end{pmatrix} \left((xI_{n_2} - A_2) \otimes I_{n_1}\right)^{-1} \begin{pmatrix} 0 & -\mathbf{1}_{n_2} \otimes I_{n_1} \end{pmatrix}.$$

Using the property of Kronecker product and equation (4) we get,

$$\begin{aligned} S &= \begin{pmatrix} xI_{n_1} & -A_1 \\ -A_1 & xI_{n_1} \end{pmatrix} - \begin{pmatrix} 0 \\ -\mathbf{1}_{n_2}^T \otimes I_{n_1} \end{pmatrix} (xI_{n_2} - A_2)^{-1} \otimes I_{n_1} \begin{pmatrix} 0 & -\mathbf{1}_{n_2} \otimes I_{n_1} \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} & -A_1 \\ -A_1 & xI_{n_1} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \chi_{A_2}(x) I_{n_1} \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} & -A_1 \\ -A_1 & xI_{n_1} - \chi_{A_2}(x) I_{n_1} \end{pmatrix}. \end{aligned}$$

Again by Proposition 2.3 we get

$$\begin{aligned} det\ S &= x^{n_1}\ det\left((x - \chi_{A_2}(x)) I_{n_1} - A_1 (xI_{m_1})^{-1} A_1\right) \\ &= x^{n_1}\ det\left((x - \chi_{A_2}(x)) I_{n_1} - \frac{A_1^2}{x}\right). \\ detS &= \prod_{i=1}^{n_1} \left(x^2 - x\chi_{A_2}(x) - \lambda_{1i}^2\right). \end{aligned} \tag{6}$$

Also by the property of Kronecker product,

$$det\ (xI_{n_2} - A_2) \otimes I_{n_1} = (det\ (xI_{n_2} - A_2))^{n_1}\ (det\ (I_{n_1}))^{n_2} \tag{7}$$

$$= \prod_{j=1}^{n_2} (x - \lambda_{2j})^{n_1}. \tag{8}$$

Hence using equations (6) and (7) we arrive that the characteristic equation is,

$$f_{G_1 \underline{\circledast} G_2}(A : x) = \prod_{j=1}^{n_2} (x - \lambda_{2j})^{n_1} \prod_{i=1}^{n_1} (x^2 - x\chi_{A_2}(x) - \lambda_{1i}^2). \tag{9}$$

□

Corollary 3.4. *Let G_1 be a r_1 - regular graph with n_1 vertices and G_2 be a r_2 - regular graph with n_2 vertices. Then the A - spectrum of $G_1 \underline{\circledast} G_2$ consists of,*

(1). λ_{2j}, repeated n_1 times, for $j = 2, 3, \dots, n_2$;

(2). Three roots of the equation, $x^3 - r_2x^2 - (\lambda_{1i}^2 + n_2)x + r_2\lambda_{1i}^2 = 0$ for $i = 1, 2, 3, \dots, n_1$.

Proof. Since G_2 is r_2 - regular, by equation (1),

$$\chi_{A_2}(x) = \frac{n_2}{x - r_2}.$$

From equation (9) the characteristic polynomial,

$$\begin{aligned} det\ (xI - A) &= \prod_{j=1}^{n_2}(x - \lambda_{2j})^{n_1} \prod_{i=1}^{n_1}(x^2 - x\frac{n_2}{x - r_2} - \lambda_{1i}^2) \\ &= \prod_{j=2}^{n_2}(x - \lambda_{2j})^{n_1} \prod_{i=1}^{n_1}(x^3 - r_2x^2 - (\lambda_{1i}^2 + n_2)x + r_2\lambda_{1i}^2). \end{aligned}$$

□

Corollary 3.5. *Let G_1 be a r_1 - regular graph with n_1 vertices and $G_2 = \bar{K_{n_2}}$ (Totally disconnected). Then the A - spectrum of $G_1 \underline{\circledast} G_2$ consists of,*

(1). 0, repeated n_1n_2 times;

(2). $\pm\sqrt{n_2 + \lambda_{1i}^2}$, for $i = 1, 2, \dots, n_1$.

Proof. When $G_2 = \bar{K_{n_2}}$ then, by equation (1),

$$\chi_{A_2}(x) = \frac{n_2}{x}.$$

Also $\lambda_{2i} = 0$ for $i = 1, 2, \dots, n_2$. Hence,

$$f_{G_1 \underline{\circledast} G_2}(A : x) = x^{n_1n_2} \prod_{i=1}^{n_1}(x^2 - n_2 - \lambda_{1i}^2).$$

□

Corollary 3.6. *Let G_1 be a r_1 - regular graph on n_1 vertices and $G_2 = K_{p,q}$, the complete bipartite graph. Then the A - spectrum of $G_1 \underline{\circledast} G_2$ consists of,*

(1). 0, repeated $n_1(p + q - 2)$ times;

(2). Four roots of the equation $x^4 - (p + q + pq + \lambda_{1i}^2)x^2 - 2pq + \lambda_{1i}^2 + pq = 0$, for $i = 1, 2, \dots, n_1$.

Proof. Since $G_2 = K_{p,q}$, by equation (3)

$$\chi_{A_2}(x) = \frac{(p + q)x + 2pq}{x^2 - pq}.$$

The characteristic polynomial can be calculated as,

$$det\ (xI - A) = x^{n_1(n_2-2)} \prod_{i=1}^{n_1}(x^4 - (p + q + pq + \lambda_{1i}^2)x^2 - 2pq + \lambda_{1i}^2 + pq).$$

□

Corollary 3.7.

(1). Let G_1 and G_2 be vertex disjoint regular graph which is cospectral and H is any arbitrary graph, then $G_1\underline{\circledast}H$ and $G_2\underline{\circledast}H$ are A - cospectral.

(2). Let G be a regular graph and H_1 and H_2 be two A - cospectral graphs with $\chi_{A(H_1)}(x) = \chi_{A(H_2)}(x)$ then $G\underline{\circledast}H_1$ and $G\underline{\circledast}H_2$ are A - cospectral.

Theorem 3.8. *Let G_1 be a r_1 - regular graph with n_1 vertices and G_2 be an arbitrary graph on n_2 vertices with Laplacian spectrum $0 = \mu_{j1} \leq \mu_{j2} \leq \cdots \leq \mu_{jn}$, $j = 1, 2$. Then L - spectrum of duplication add vertex corona, $G_1\underline{\circledast}G_2$, consists of*

(1). 0;

(2). $1 + \mu_{2j}$, repeated n_1 times for $j = 2, 3, \ldots, n_2$;

(3). Two roots of the equation $x^2 - (2r_1 + n_2 + 1)x + 2r_1 + n_2r_1 = 0$;

(4). Three roots of the equation $x^3 - (2r_1 + n_2 + 1)x^2 + (n_2r_1 + 2r_1 + 2r_1\mu_{1i} - \mu_{1i}^2)x + (\mu_{1i}^2 - 2r_1\mu_{1i}) = 0$, $i = 2, 3, \ldots, n_1$.

Proof. The degree of the vertices of $G_1\underline{\circledast}G_2$ are $d_{G_1\underline{\circledast}G_2}(v_i) = r_1$, $d_{G_1\underline{\circledast}G_2}(x_i) = n_2+r_1$, $i = 1, 2, \ldots, n_1$ and $d_{G_1\underline{\circledast}G_2}(u_j^i) = d_{G_2}(u_j) + 1$, $j = 1, 2, \ldots, n_2$. The diagonal degree matrix of $G_1\underline{\circledast}G_2$ is,

$$D(G_1\underline{\circledast}G_2) = \begin{bmatrix} r_1I_{n_1} & 0 & 0 \\ 0 & (r_1 + n_2)I_{n_1} & 0 \\ 0 & 0 & (D(G_2) + I_{n_2}) \otimes I_{n_1} \end{bmatrix},$$

where $D(G_2)$ be the diagonal degree matrix of the graph G_2.

$$\begin{aligned} (D(G_2) + I_{n_2}) \otimes I_{n_1} - A_2 \otimes I_{n_1} &= (D(G_2) + I_{n_2} - A_2) \otimes I_{n_1} \\ &= (L_2 + I_{n_2}) \otimes I_{n_1}. \end{aligned}$$

The Laplace matrix of $G_1\underline{\circledast}G_2$ is,

$$\begin{aligned} L &= D - A \\ &= \begin{bmatrix} r_1I_{n_1} & -A_1 & 0 \\ -A_1 & (r_1 + n_2)I_{n_1} & -\mathbf{1}_{n_2}^T \otimes I_{n_1} \\ 0 & -\mathbf{1}_{n_2} \otimes I_{n_1} & (L_2 + I_{n_2}) \otimes I_{n_1} \end{bmatrix}, \end{aligned}$$

where L_2 is the Laplacian matrix of G_2 and $\mathbf{1}_{n_2}$ is a $n_2 \times 1$ column vector with all entries equal to 1. The Laplacian characteristic polynomial of $G_1\underline{\circledast}G_2$,

$$\begin{aligned} f_{G_1\underline{\circledast}G_2}(L : x) &= \begin{vmatrix} (x-r_1)I_{n_1} & A_1 & 0 \\ A_1 & (x-r_1-n_2)I_{n_1} & \mathbf{1}_{n_2}^T \otimes I_{n_1} \\ 0 & \mathbf{1}_{n_2} \otimes I_{n_1} & ((x-1)I_{n_2} - L_2) \otimes I_{n_1} \end{vmatrix} \\ &= det\ (((x-1)I_{n_2} - L_2) \otimes I_{n_1})\ det\ \ S, \end{aligned}$$

$$\text{where, } S = \begin{pmatrix} (x - r_1)I_{n_1} & A_1 \\ A_1 & (x - r_1 - n_2)I_{n_1} \end{pmatrix} - \begin{pmatrix} 0 \\ \mathbf{1}_{n_2}^T \otimes I_{n_1} \end{pmatrix} (((x-1)I_{n_2} - L_2) \otimes I_{n_1})^{-1} \begin{pmatrix} 0 & \mathbf{1}_{n_2} \otimes I_{n_1} \end{pmatrix}.$$

By using the property of Kronecker product and equation (5) we get the following steps.

$$S = \begin{pmatrix} (x-r_1)I_{n_1} & A_1 \\ A_1 & (x-r_1-n_2)I_{n_1} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \chi_{L_2}(x-1)I_{n_1} \end{pmatrix}$$
$$= \begin{pmatrix} (x-r_1)I_n n_1 & A_1 \\ A_1 & (x-r_1-n_2)I_{n_1} - \chi_{L_2}(x-1))I_{n_1} \end{pmatrix}.$$

By applying Proposition 2.3 we get,

$$det \ \ S = (x-r_1)^{n_1} det \left((x - r_1 - n_2 - \chi_{L_2}(x-1))I_{n_1} - \frac{A_1^2}{x-r_1} \right)$$
$$= \prod_{i=1}^{n_1} ((x-r_1-n_2)(x-r_1) - (x-r_1)\chi_{L_2}(x-1) - \lambda_{1i}^2)$$

Since G_2 is r_2 - regular graph on n_2 vertices, using equation (2) we have,

$$\chi_{L_2}(x-1) = \frac{n_2}{x-1}.$$

On substituting these values and simplifying we arrive at the following step.

$$det \ S = \frac{x(x^2-(1+2r_1+n_2)x+(2r_1+n_2r_1))}{(x-1)^{n_1}}$$
$$\prod_{i=2}^{n_1}(x^3-(2r_1+n_2+1)x^2+(r_1^2+2r_1+n_2r_1-\lambda_{1i}^2)x+\lambda_{1i}^2-r_1^2).$$

Since G_1 is r_1 - regular, we use the fact that $\lambda_i = r_1 - \mu_i$ for $i = 2, 3, \ldots, n_1$ and $\mu_1 = 0$. Hence,

$$f_{G_1 \underline{\circledast} G_2}(L:x) = x(x^2-(1+2r_1+n_2)x+(2r_1+n_2r_1)) \prod_{j=2}^{n_2}(x-1-\mu_{2j})^{n_1}$$
$$\prod_{i=2}^{n_1}(x^3-(2r_1+n_2+1)x^2+(n_2r_1+2r_1+2r_1\mu_{1i}-\mu_{1i}^2)x+\mu_{1i}^2-2r_1\mu_{1i}).$$

□

Corollary 3.9.

(1). Let G_1 and G_2 be vertex disjoint regular graph which is Laplacian cospectral and H is any arbitrary graph then $G_1 \underline{\circledast} H$ and $G_2 \underline{\circledast} H$ are Laplacian cospectral.

(2). Let G be a regular graph and H_1 and H_2 be two cospectral graphs then $G \underline{\circledast} H_1$ and $G \underline{\circledast} H_2$ are Laplacian cospectral.

Theorem 3.10. *Let G_1 be a r_1 - regular graph with n_1 vertices and G_2 be an arbitrary graph with n_2 vertices with signless Laplacian spectrum $\nu_{i1} \le \nu_{i2} \le \cdots \le \nu_{in}$ for $i = 1, 2$. Then*

$$f_{G_1 \underline{\circledast} G_2}(Q:x) = \prod_{j=1}^{n_2}(x-1-\nu_{2j})^{n_1}$$
$$\prod_{i=1}^{n_1}(x^2-(2r_1+n_2+\chi_{Q_2}(x-1))x+r_1^2+n_2r_1+r_1\chi_{Q_2}(x-1)-\lambda_{1i}^2).$$

Proof. The signless Laplace adjacency matrix of $G_1 \underline{\circledast} G_2$ is,

$$Q = \begin{bmatrix} r_1 I_{n_1} & A_1 & 0 \\ A_1 & (r_1+n_2)I_{n_1} & \mathbf{1}_{n_2}^T \otimes I_{n_1} \\ 0 & \mathbf{1}_{n_2} \otimes I_{n_1} & (Q_2 + I_{n_2}) \otimes I_{n_1} \end{bmatrix},$$

where Q_2 is the signless Laplacian matrix of G_2. The proof of the theorem is similar to Theorem 3.8. □

Corollary 3.11. *Let G_1 be a r_1 - regular graph with n_1 vertices and G_2 be a r_2 - regular graph with n_2 vertices. Then*

$$f_{G_1 \underline{\circledast} G_2}(Q:x) = \prod_{j=1}^{n_2-1} (x - 1 - \nu_{2j})^{n_1} \prod_{i=1}^{n_1} (x^3 - ax^2 + bx - c),$$

where, $a = 1 + 2r_1 + 2r_2 + n_2, b = 2r_1 + r_1^2 + n_2 r_1 + 2n_2 r_2 + 4r_1 r_2 - \lambda_{1i}^2$ and $c = r_1^2 + 2r_1^2 r_2 + 2n_2 r_1 r_2 - 2r_2 \lambda_{1i}^2 - \lambda_{1i}^2$.

Corollary 3.12.

(1). Let G_1 and G_2 be vertex disjoint regular graph which is cospectral and H is any arbitrary graph then $G_1 \underline{\circledast} H$ and $G_2 \underline{\circledast} H$ are Q - cospectral.

(2). Let G be a regular graph and H_1 and H_2 be two A - cospectral graphs with $\chi_{Q(H_1)}(x) = \chi_{Q(H_2)}(x)$ then $G \underline{\circledast} H_1$ and $G \underline{\circledast} H_2$ are Q - cospectral.

Proposition 3.13. *Let G_1 be a r_1 - regular graph with n_1 vertices and G_2 be an arbitrary graph with n_2 vertices then Duplication vertex corona and Duplication add vertex corona, $G_1 \underline{\circledcirc} G_2$ and $G_1 \underline{\circledast} G_2$, are A - cospectral.*

Proof. Let G_1 be a r_1 - regular graph with n_1 vertices and m_1 edges. G_2 be an arbitrary graph with n_2 vertices. $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $U(G_1) = \{x_1, x_2, \ldots, x_{n_1}\}$. The vertex in the i^{th} copy of G_2 be $\{u_1^i, u_2^i, \ldots, u_{n_2}^i\}$ and let $W_j = \{u_j^1, u_j^2, \ldots, u_j^{n2}\}$ for $j = 1, 2, \ldots, n_2$. Then $V(G_1) \cup U(G_1) \cup \{W_1 \cup W_2 \cup \ldots \cup W_{n_2}\}$ is a vertex partition of $G_1 \underline{\circledcirc} G_2$. By these vertex partitioning the adjacency matrix of Duplication vertex corona, $G_1 \underline{\circledcirc} G_2$, is

$$A = \begin{bmatrix} 0 & A_1 & \mathbf{1}_{n_2}^T \otimes I_{n_1} \\ A_1 & 0_{n_1 \times n_1} & 0_{n_1 \times n_1 n_2} \\ \mathbf{1}_{n_2} \otimes I_{n_1} & 0_{n_1 n_2 \times n_1} & A_2 \otimes I_{n_1} \end{bmatrix},$$

where A_1 and A_2 are the adjacency matrix of G_1 and G_2 respectively. $\mathbf{1}_{n_2}$ is a $n_2 \times 1$ column vector with all entries equal to 1 and I_{n_1} is an identity matrix of order n_1. Interchanging the first and second row and then interchange the first and second column of the above determinant. The characteristic polynomial become

$$\begin{aligned} f_{G_1 \underline{\circledcirc} G_2}(A:x) &= det\,(xI - A) \\ &= \begin{vmatrix} xI_{n_1} & -A_1 & 0 \\ -A_1 & xI_{n_1} & -\mathbf{1}_{n_2}^T \otimes I_{n_1} \\ 0 & -\mathbf{1}_{n_2} \otimes I_{n_1} & (xI_{n_2} - A_2) \otimes I_{n_1} \end{vmatrix} \\ &= f_{G_1 \underline{\circledast} G_2}(A:x). \end{aligned}$$

□

Proposition 3.14. *Let G_1 be a r_1 - regular graph with n_1 vertices and G_2 be an arbitrary graph with n_2 vertices then $G_1 \underline{\circledcirc} G_2$ and $G_1 \underline{\circledast} G_2$ are L - cospectral.*

Proof. The degree of the vertices of $G_1 \underline{\circledcirc} G_2$ are $d_{G_1 \underline{\circledcirc} G_2}(v_i) = n_2 + r_1$, $d_{G_1 \underline{\circledcirc} G_2}(x_i) = r_1$, $i = 1, 2, \ldots, n_1$ and $d_{G_1 \underline{\circledcirc} G_2}(u_j^i) = d_{G_2}(u_j) + 1$, $j = 1, 2, \ldots, n_2$. The diagonal degree matrix of $G_1 \underline{\circledcirc} G_2$ is

$$D(G_1 \underline{\circledcirc} G_2) = \begin{bmatrix} (r_1 + n_2)I_{n_1} & 0 & 0 \\ 0 & r_1 I_{n_1} & 0 \\ 0 & 0 & (D(G_2) + I_{n_2}) \otimes I_{n_1} \end{bmatrix},$$

where $D(G_2)$ be the diagonal degree matrix of the graph G_2.

$$\begin{aligned} (D(G_2) + I_{n_2}) \otimes I_{n_1} - A_2 \otimes I_{n_1} &= (D(G_2) + I_{n_2} - A_2) \otimes I_{n_1} \\ &= (L_2 + I_{n_2}) \otimes I_{n_1}. \end{aligned}$$

The Laplace matrix of $G_1 \underline{\circledcirc} G_2$ is,

$$\begin{aligned} L &= D - A \\ &= \begin{bmatrix} (r_1 + n_2)I_{n_1} & -A_1 & -\mathbf{1}_{n_2}^T \otimes I_{n_1} \\ -A_1 & r_1 I_{n_1} & 0 \\ -\mathbf{1}_{n_2} \otimes I_{n_1} & 0 & (L_2 + I_{n_2}) \otimes I_{n_1} \end{bmatrix}, \end{aligned}$$

where L_1 and L_2 are the Laplacian matrix of G_1 and G_2 respectively. $\mathbf{1}_{n_2}$ is a $n_2 \times 1$ column vector with all entries equal to 1. The Laplacian characteristic polynomial of $G_1 \underline{\circledcirc} G_2$,

$$f_{G_1 \underline{\circledcirc} G_2}(L : x) = \begin{vmatrix} (x - r_1 - n_2)I_{n_1} & A_1 & \mathbf{1}_{n_2}^T \otimes I_{n_1} \\ A_1 & (x - r_1)I_{n_1} & 0 \\ \mathbf{1}_{n_2} \otimes I_{n_1} & 0 & ((x-1)I_{n_2} - L_2) \otimes I_{n_1} \end{vmatrix}$$

Interchanging the first and second row and then interchange the first and second column of the above determinant. The Laplacian charcteristic polynomial become

$$\begin{aligned} f_{G_1 \underline{\circledcirc} G_2}(L : x) &= \begin{bmatrix} (x - r_1)I_{n_1} & -A_1 & 0 \\ -A_1 & (x - r_1 - n_2)I_{n_1} & -\mathbf{1}_{n_2}^T \otimes I_{n_1} \\ 0 & -\mathbf{1}_{n_2} \otimes I_{n_1} & (L_2 + I_{n_2}) \otimes I_{n_1} \end{bmatrix} \\ &= f_{G_1 \underline{\circledast} G_2}(L : x). \end{aligned}$$

□

Proposition 3.15. *Let G_1 be an r_1 - regular graph on n_1 vertices and G_2 be an arbitrary graph on n_2 vertices then $G_1 \underline{\circledcirc} G_2$ and $G_1 \underline{\circledast} G_2$ are Q - cospectral.*

Proof. The proof of the Proposition is exactly same as that of the above Proposition. □

4. Applications

Klein and *Randić* in [8] introduced a new notion named *resistance distance* based on electric resistance in a network corresponding to a graph, in which the resistance distance between any two adjacent vertices is 1 ohm. The sum of the resistance distance between all pairs of the vertices of a graph was conceived as a new graph invariant. The electric resistance is calculated by means of the Kirchhoff laws called *kirchhoff index*. For a graph G with $n(n \geq 2)$ vertices the Kirchhoff index, $Kf(G)$, is defined as

$$Kf(G) = n \sum_{i=2}^{n} \frac{1}{\mu_i}. \tag{10}$$

Theorem 4.1. *Let G_1 be a r_1 - regular graph with n_1 vertices and G_2 be an arbitrary graph with n_2 vertices with Laplacian spectrum $0 = \mu_{j1} \leq \mu_{j2} \leq \cdots \leq \mu_{jn}$, $j = 1, 2$. Then*

$$Kf(G_1 \underline{\circledast} G_2) = n_1(n_2+2)\left[\sum_{i=2}^{n_2} \frac{1}{1+\mu_{2i}} + \sum_{i=2}^{n_1} \frac{n_2r_1 + 2r_1 + 2r_1\mu_{1i} - \mu_{1i}^2}{2r_1\mu_{1i} - \mu_{1i}^2}\right] + \frac{n_1(1+n_2+2r_1)}{r_1}.$$

Proof. Let y_1 and y_2 be the roots of the equation $x^2 - (2r_1 + n_2 + 1)x + 2r_1 + n_2r_1 = 0$,

$$\begin{aligned} \frac{1}{y_1} + \frac{1}{y_2} &= \frac{y_1 + y_2}{y_1y_2} \\ &= \frac{2r_1 + n_2r_1}{r_1(n_2+2)}. \end{aligned}$$

Let y_{i1}, y_{i2} and y_{i3} be the roots of the cubic equation $x^3 - (2r_1 + n_2 + 1)x^2 + (n_2r_1 + 2r_1 + 2r_1\mu_{1i} - \mu_{1i}^2)x + (\mu_{1i}^2 - 2r_1\mu_{1i}) = 0, i = 2, 3, \ldots, n_1$. Then

$$\begin{aligned} \frac{1}{y_{i1}} + \frac{1}{y_{i2}} + \frac{1}{y_{i3}} &= \frac{y_{i2}y_{i3} + y_{i1}y_{i3} + y_{i1}y_{i2}}{y_{i1}y_{i2}y_{i3}} \\ &= \frac{n_2r_1 + 2r_1 + 2r_1\mu_{1i} - \mu_{1i}^2}{2r_1\mu_{1i} - \mu_{1i}^2}. \end{aligned}$$

Substituting these result in the equation (10) we get

$$Kf(G_1 \underline{\circledast} G_2) = n_1(n_2+2)\left[\sum_{j=2}^{n_2} \frac{1}{1+\mu_{2j}} + \sum_{i=2}^{n_1} \frac{n_2r_1 + 2r_1 + 2r_1\mu_{1i} - \mu_{1i}^2}{2r_1\mu_{1i} - \mu_{1i}^2}\right] + \frac{n_1(1+n_2+2r_1)}{r_1}.$$

□

Spanning tree of a graph is a subgraph of it which is also a tree. The number of spanning tree of a graph G is denoted by $t(G)$. If G is a connected graph with n vertices and the Laplacian spectrum $0 = \mu_1(G) \leq \mu_2(G) \cdots \leq \mu_n(G)$ then [4] the number of spanning tree

$$t(G) = \frac{\mu_2(G)\mu_3(G)\cdots\mu_n(G)}{n}. \tag{11}$$

Theorem 4.2. *Let G_1 be a r_1 - regular graph with n_1 vertices and G_2 be an arbitrary graph on n_2 vertices with Laplacian spectrum $0 = \mu_{j1} \leq \mu_{j2} \leq \cdots \leq \mu_{jn}$, $j = 1, 2$. Then*

$$t(G_1 \underline{\circledast} G_2) = \frac{r_1}{n_1} \prod_{i=2}^{n_2} (1+\mu_{2i})^{n_1} \prod_{i=2}^{n_2} (\mu_{1i}^2 - 2r_1\mu_{1i}).$$

Proof. Referring the notations used in Theorem 3.8. Let y_1 and y_2 be the roots of the equation $x^2 - (2r_1 + n_2 + 1)x + 2r_1 + n_2r_1 = 0$. Product of the roots $= y_1y_2 = 2r_1 + n_2r_1$. Let y_{i1}, y_{i2} and y_{i3} be the roots of the cubic equation $x^3 - (2r_1 + n_2 + 1)x^2 + (n_2r_1 + 2r_1 + 2r_1\mu_{1i} - \mu_{1i}^2)x + (\mu_{1i}^2 - 2r_1\mu_{1i}) = 0, i = 2, 3, \ldots, n_1$. Then,

$$\begin{aligned} \text{Product of the roots} &= y_{i1}\ y_{i2}\ y_{i3} \\ &= -(\mu_{1i}^2 - 2r_1\mu_{1i}) \\ &= 2r_1\mu_{1i} - \mu_{1i}^2. \end{aligned}$$

Substituting these result in the equation (11) we get

$$t(G_1 \underline{\circledast} G_2) = \frac{r_1}{n_1} \prod_{i=2}^{n_2} (1+\mu_{2i})^{n_1} \prod_{i=2}^{n_2} (\mu_{1i}^2 - 2r_1\mu_{1i}).$$

□

Corollary 4.3. $t(K_{n_1} \underline{\circledast} K_{n_2}) = (n_1 - 1)n_1^{n_1-2}(n_1+1)^{n_1(n_2-1)}(n_1-2)^{n_1-1}$.

Proof. The notations are same as exactly defined in Theorem 4.2. If $G_1 = K_{n_1}$ and $G_2 = K_{n_2}$, then $r_1 = n_1 - 1$, $\mu_{1i} = n_1$, $i = 2, 3, \ldots, n_1$ and $\mu_{2j} = n_2$, $j = 2, 3, \ldots, n_2$. Proof follows by substituting these values in Theorem 4.2. □

4.1. Infinite Families of Integral Graphs

A graph is said to be an *integral graph* if the spectrum consists only of integers [1, 7]. The following propositions shows the essential conditions for $G_1 \underline{\circledast} G_2$ and $G_1 \underline{\circledcirc} G_2$ to be an integral graph.

Proposition 4.4. *Let G_1 be a r_1 - regular graph with n_1 vertices and G_2 be r_2 - regular graph with n_2 vertices. $G_1 \underline{\circledast} G_2$ (respectively $G_1 \underline{\circledcirc} G_2$) is an integral graph if and only if G_1 and G_2 are integral graphs and the roots of the equation, $x^3 - r_2 x^2 - (\lambda_{1i}^2) + n_2)x + r_2 \lambda_{1i}^2 = 0$ for $i = 2, 3, \ldots, n_1$ are integers.*

In particular if $G_2 = \overline{K_n}$ (totally disconnected) then $G_1 \underline{\circledast} G_2$ (respectively $G_1 \underline{\circledcirc} G_2$) is an integral graph iff G_1 is an integral graph and $n_2 + \lambda_{1i}^2$ for $i = 2, 3, \ldots, n_1$ are perfect squares.

Proposition 4.5. *Let G_i be r_i - regular graph on n_i vertices then, $G_1 \underline{\circledast} K_{p,q}$ (respectively $G_1 \underline{\circledcirc} K_{p,q}$) is an integral graph if and only if $p = q$ and the the roots of the equation $x^4 - (p + q + pq + \lambda_{1i}^2)x^2 - 2pq + \lambda_{1i}^2 + pq = 0$ for $i = 1, 2, \ldots, n_1$, are integers.*

5. Conclusion and Future Research

The concept of corona product of graph has many application in real life. In this paper we introduced two types of corona product of graphs. Also we discussed some applications such as Kirchhoff index and number of spanning trees. We also discuss some infinite family of integral graphs and some class of cospectral graphs. In this paper we are mainly focused on the vertices and define the new corona product. But in future we can define the neighborhood corona and edge corona using the duplication graph and can find the corresponding spectrum.

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