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L(2,1)-Labeling for Bloom Graph

Research Article

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Abstract: An $L(2,1)-$labeling of a graph is a function from the vertex set $V(G)$ to the set of all non-negative integers such that $|f(u) - f(v)| \geq 2$ if and are vertices and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$, where $d(u, v)$ denotes the distance between u and v in G. The $L(2,1)$-labeling number of G, denoted by $\lambda(G)$, is the smallest number k such that there is an $L(2,1)$-labeling with maximum label k. In this paper we determine $L(2,1)$-Labeling for Bloom Graph.

Keywords: $L(2,1)$-labeling, $L(2,1)$-numbering, Bloom Graph.

1. Introduction

Vertex coloring is one of the most studied subjects in graph theory. Labeling of Graphs was introduced by Rosa in 1967 [1, 2]. Some more works on labeling is attributed to Graham and Sloane in 1980 [3]. The proper labeling of graph was introduced by Karonski, Luezak and Thomason [4]. The $L(2,1)$-labeling arises from the frequency assigning problems in wireless network services. Here we are given a number of transmitters or stations and we have to give assignment to each frequency so that the transmitters do not interfere with one other. This problem of assigning frequencies to transmitters can be modeled by graphs which is equivalent to assigning an $L(2,1)$-label to each vertex. We try to find the smallest difference between the highest and lowest frequencies assigned to different vertices so that the total bandwidth used could be minimized. Quite many references on the $L(2,1)$-labeling problem are provided by Calamoneri in [5]. A Variation of frequency assignment problem was proposed by Roberts F. S. [6]. Frequency assignment problems was introduced in 1992 by Griggs et al. [7], which was called an $L(2,1)$-labeling problem. To avoid the interference of frequencies the vertices in a graph are labeled in such a way that " close" vertices which are at a distance are given di?erent labels and " very close" vertices which are at a distance are assigned labels that are far. A conjecture by Griggs and Yeh [7] is worth mentioning here which was proposed in 1992, For any graph G with maximum degree $\Delta \geq 2$, $\lambda(G) \leq \Delta^2$. Griggs and Yeh [7] further proved that $\lambda(G) \leq \Delta^2 + 2\Delta$ for general graphs with maximum degree Δ. The bound of an $L(2,1)$-numbering was improved by Chang and Kuo [8] to $\lambda(G) \leq \Delta^2 + \Delta$, which was again reduced by to $\lambda(G) \leq \Delta^2 + \Delta - 1$ by Kr and krekovski [9].

2. Basic Definitions

Definition 2.1 ([7])**.** *An $L(2,1)-$labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all non-negative integers such that $|f(u) - f(v)| \geq 2$ if u and v are vertices and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$, where $d(u, v)$ denotes the*

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distance between u and v in G. The $L(2,1)$-labeling number of G, denoted by $\lambda(G)$, is the smallest number k such that there is an $L(2,1)$-labeling with maximum label k. An $L(2,1)$-labeling having maximum label λ is called optimal.

Definition 2.2 ([10]). *The bloom graph $B_{m,n}$, $m, n > 2$ is defined as follows: $V((B_{m,n}) = \{(x, y) : 0 \leq m-1, 0 \leq y \leq n-1\}$ two distinct vertices (x_1, y_1) and (x_1, y_2) being adjacent if and only if*

(1). $x_2 = x_1 + 1$ and $y_1 = y_2$

(2). $x_1 = x_2 = 0$ and $y_1 \equiv y_1 (mod\ n)$

(3). $x_1 = x_2 = 0$ and $y_1 + 1 \equiv y_2 (mod\ n)$

(4). $x_2 = x_1 + 1$ and $y_1 + 1 \equiv y_2 (mod\ n)$

The first condition describes the vertical edges, the second and third condition describe the horizontal edges in the top most and the lower most rows respectively. Condition four describes the slant edges. Bloom graph has mn vertices and $2mn$ edges. The vertex connectivity and the edge connectivity of bloom graph is 4. Bloom graph is planar, tripartite and 4-regular. (For illustration see Figure 1).

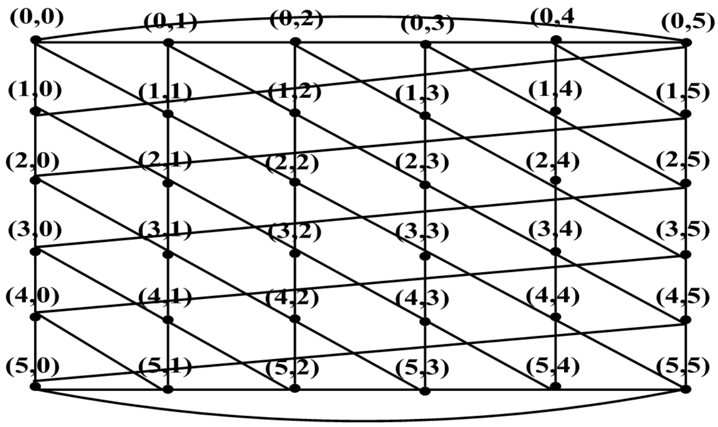

Figure 1. **Bloom Graph** $B_{6,6}$

3. $L(2,1)$-labeling Number of Bloom Graph

Theorem 3.1. *A $m \times n$ Bloom graph admits $L(2,1)$-labeling and $\lambda_{2,1}(B_{m,n}) \leq 13$.*

Proof. Let $d((i,j),(k,l))$ = distance between the vertices (i,j) and (k,l), $i,k = 0,1,\ldots,m-1$, $j,l = 0,1,\ldots,n-1$. And $f(i,j)$ = the label assigned to the vertex (i,j).

Case 1:$n \equiv 0$ mod 3

We partition the vertex set of $B_{m,n}$ as follows:

$$V_1 = (i,j), \quad i = 0,1,2,\ldots,m-1; \quad j = 0,3,6,\ldots,n-3$$

$$V_2 = (i,j), \quad i = 0,1,2,\ldots,m-1; \quad j = 1,4,7,\ldots,n-2$$

$$V_3 = (i,j), \quad i = 0,1,2,\ldots,m-1; \quad j = 2,5,8,\ldots,n-1$$

The vertices of V_1 are labeled as follows:

For $i = 0, 1, 2, \ldots, m-1$; $j = 0, 3, 6, \ldots, n-3$.

$$f(i,j) = \begin{cases} 1, i \equiv 0 \; mod \; 4 \\ 7, i \equiv 1 \; mod \; 4 \\ 2, i \equiv 2 \; mod \; 4 \\ 8, i \equiv 3 \; mod \; 4 \end{cases}$$

The vertices of V_2 are labeled as follows:

For $i = 0, 1, 2, \ldots, m-1$; $j = 1, 4, 7, \ldots, n-2$.

$$f(i,j) = \begin{cases} 3, i \equiv 0 \; mod \; 4 \\ 9, i \equiv 1 \; mod \; 4 \\ 4, i \equiv 2 \; mod \; 4 \\ 10, i \equiv 3 \; mod \; 4 \end{cases}$$

The vertices of V_3 are labeled as follows:

For $i = 0, 1, 2, \ldots, m-1$; $j = 2, 5, 8, \ldots, n-1$.

$$f(i,j) = \begin{cases} 5, i \equiv 0 \; mod \; 4 \\ 11, i \equiv 1 \; mod \; 4 \\ 6, i \equiv 2 \; mod \; 4 \\ 12, i \equiv 3 \; mod \; 4 \end{cases}$$

Subcase (i): When (i,j) and (k,l) are in V_1 we observe that

$$d((i,j),(i,l)) \geq 3, j < l; \; i = 0, 1, 2, \ldots, m-1; \; j, l = 0, 3, 6, \ldots, n-3$$

Then $|f(i,j) - f(i,l)| + d((i,j),(i,l)) \geq 3$. For $i < k$; $i, k = 0, 1, 2, \ldots, m-1$; $j = 0, 3, 6, \ldots, n-1$.

$$d((i,j),(i,l)) = 2$$

then $|f(i,j) - f(k,j)| \geq 1$ and $d((i,j),(k,j)) = 1$ then $|f(i,j) - f(k,j)| \geq 5$. Thus $|f(i,j) - f(k,j)| + d((i,j),(k,j)) \geq 3$.

Subcase (ii): When $(i,j) \in V_1$ and $(k,l) \in V_2$ we observe that

$$d((i,j),(k,l)) \geq 1, i < k, \; i, k = 0, 1, 2, \ldots, m-1; \; j < l, \; j, l = 0, 1, 3, 4, 6, 7, \ldots, n-3, n-2.$$

Then $|f(i,j) - f(k,l)| \geq 2$. Thus $|f(i,j) - f(k,l)| + d((i,j),(k,l)) \geq 3$.

Subcase (iii): When $(i,j) \in V_1$ and $(k,l) \in V_3$ we observe that

$$d((i,j),(k,l)) \geq 1, \; i, k = 0, 1, 2, \ldots, m-1; \; j, l = 0, 2, 3, 5, \ldots, n-3, n-1$$

then $|f(i,j) - f(k,l)| \geq 4$ and when $|f(i,j) - f(k,l)| \geq 1$ then $d((i,j),(k,l)) \geq 3$. Thus $|f(i,j) - f(k,l)| + d((i,j),(k,l)) \geq 3$.

Subcase (iv): When $(i,j) \in V_2$ and $(k,l) \in V_3$ we observe that

$$d((i,j),(k,l)) \geq 1, \; i, k = 0, 1, 2, \ldots, m-1; \; j, l = 1, 2, 4, 5, \ldots, n-3, n-1$$

then $|f(i,j) - f(k,l)| \geq 2$. Thus $|f(i,j) - f(k,l)| + d((i,j),(k,l)) \geq 3$. Hence in all cases this labeling preserves the definition of $L(2,1)$ labeling. (For illustration, see Figure 2).

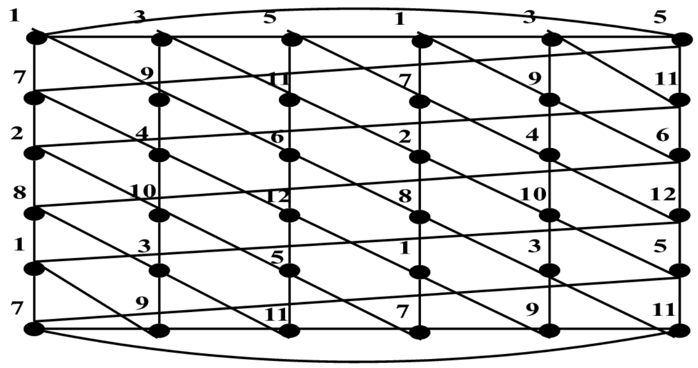

Figure 2. **L(2,1) labeling of** $B_{6,6}$

Case 2: when $n \equiv 1 \ mod \ 3$

We omit the proof since it is similar to Case 1, we only give the vertex partition and the labeling.

Subcase (i): When $m \equiv 2 \ mod \ 4$

We partition the vertex set of $B_{m,n}$ as follows:

$$V_1 = (i,j),\ i = 0,1,2,\ldots,m-1;\ j = 0,3,6,\ldots,n-4$$

$$V_2 = (i,j),\ i = 0,1,2,\ldots,m-1;\ j = 1,4,7,\ldots,n-3$$

$$V_3 = (i,j),\ i = 0,1,2,\ldots,m-1;\ j = 2,5,8,\ldots,n-2$$

$$V_4 = (i,j),\ i = 0,1,2,\ldots,m-1;\ j = n-1$$

The vertices of V_1 are labeled as follows:

For $i = 0,1,2,\ldots,m-1$; $j = 0,3,6,\ldots,n-4$.

$$f(i,j) = \begin{cases} 1, & i \equiv 0 \ mod \ 4 \\ 7, & i \equiv 1 \ mod \ 4 \\ 2, & i \equiv 2 \ mod \ 4 \\ 8, & i \equiv 3 \ mod \ 4 \end{cases}$$

The vertices of V_2 are labeled as follows:

For $i = 0,1,2,\ldots,m-1$; $j = 1,4,7,\ldots,n-3$.

$$f(i,j) = \begin{cases} 3, & i \equiv 0 \ mod \ 4 \\ 9, & i \equiv 1 \ mod \ 4 \\ 4, & i \equiv 2 \ mod \ 4 \\ 10, & i \equiv 3 \ mod \ 4 \end{cases}$$

The vertices of V_3 are labeled as follows:

For $i = 0,1,2,\ldots,m-1$; $j = 2,5,8,\ldots,n-2$.

$$f(i,j) = \begin{cases} 5, & i \equiv 0 \ mod \ 4 \\ 11, & i \equiv 1 \ mod \ 4 \\ 6, & i \equiv 2 \ mod \ 4 \\ 12, & i \equiv 3 \ mod \ 4 \end{cases}$$

The vertices of V_4 are labeled as follows:

For $i = 0, 1, 2, \ldots, m-2$; $j = n-1$.

$$f(i,j) = \begin{cases} 10, & i \equiv 0 \; mod \; 4 \\ 12, & i \equiv 1 \; mod \; 4 \\ 4, & i \equiv 2 \; mod \; 4 \\ 13, & i \equiv 3 \; mod \; 4 \end{cases}$$

And $f(i,j) = 2$, for $i = m-1$, $j = n-1$. (For illustration, see Figure 3).

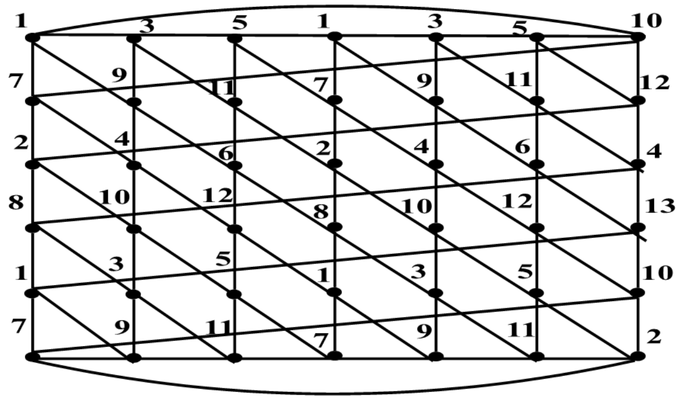

Figure 3. **L(2,1) labeling of** $B_{6,7}$

Subcase (ii): When $m \equiv 3 \; mod \; 4$

Vertex partitioning same as in subcase(i) of Case 2. Vertex labeling is also similar to subcase (i) of Case 2 except for V_4.

The vertices of V_4 are labeled as follows:

For $i = 0, 1, 2, \ldots, m-2$; $j = n-1$.

$$f(i,j) = \begin{cases} 10, & i \equiv 0 \; mod \; 4 \\ 12, & i \equiv 1 \; mod \; 4 \\ 4, & i \equiv 2 \; mod \; 4 \\ 13, & i \equiv 3 \; mod \; 4 \end{cases}$$

And $f(i,j) = 8$, for $i = m-1$, $j = n-1$. (For illustration see Figure 4)

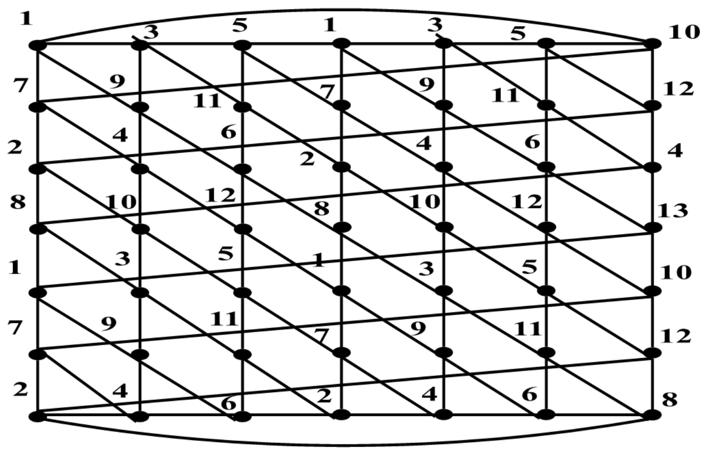

Figure 4. **L(2,1) labeling of** $B_{7,7}$

Subcase (iii): When $m \equiv 0 \; mod \; 4$

Vertex partitioning is same as in subcase (i) of case 2. Vertex labeling is also similar to subcase (i) of Case 2, except for V_4. The vertices of V_4 are labeled as follows:

For $i = 0, 1, 2, \ldots, m-2$; $j = n-1$.

$$f(i,j) = \begin{cases} 10, & i \equiv 0 \; mod \; 4 \\ 12, & i \equiv 1 \; mod \; 4 \\ 4, & i \equiv 2 \; mod \; 4 \\ 13, & i \equiv 3 \; mod \; 4 \end{cases}$$

And $f(i,j) = 1$, for $i = m-1$, $j = n-1$. (For illustration see Figure 5)

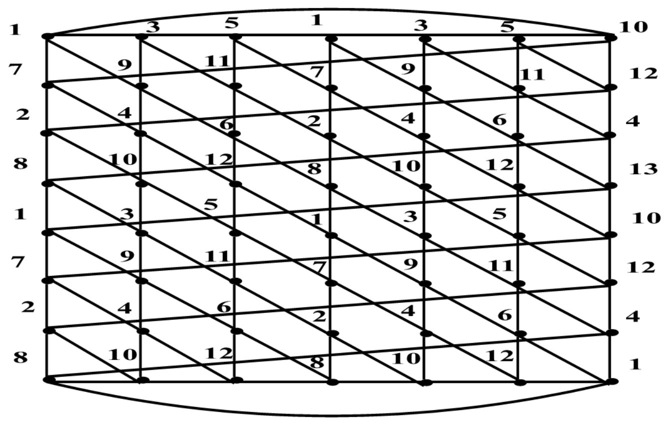

Figure 5. **L(2,1) labeling of $B_{8,7}$**

Subcase (iv): When $m \equiv 1 \; mod \; 4$

Vertex partitioning is same as in subcase (i) of case 2. Vertex labeling is also similar to subcase (i) of Case 2 except for V_4. The vertices of V_4 are labeled as follows:

For $i = 0, 1, 2, \ldots, m-2$; $j = n-1$.

$$f(i,j) = \begin{cases} 10, & i \equiv 0 \; mod \; 4 \\ 12, & i \equiv 1 \; mod \; 4 \\ 4, & i \equiv 2 \; mod \; 4 \\ 13, & i \equiv 3 \; mod \; 4 \end{cases}$$

And $f(i,j) = 7$, for $i = m-1$, $j = n-1$. (For illustration see Figure 6)

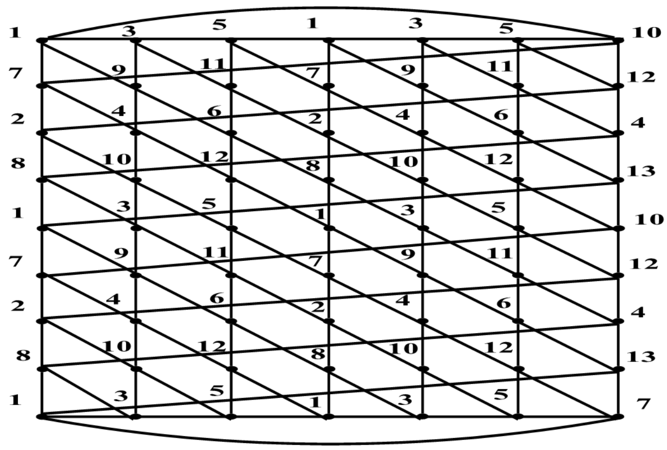

Figure 6. **L(2,1) labeling of $B_{9,7}$**

Case 3 When $n \equiv 2 \; mod \; 3$

Subcase(i): When $m \equiv 2 \; mod \; 4$

Vertex partitioning is same as in subcase (i) of Case 2. Vertex labeling is also similar to subcase (i) of Case 2 except for V_4 and we include V_5. The vertex partition for V_4 is given as below

$$V_4 = (i, j), \; i = 0, 1, 2, \ldots, m-1; \; j = n-2$$

The vertex partition for V_5 is given as

$$V_5 = (i, j), \; i = 0, 1, 2, \ldots, m-1; \; j = n-1$$

The vertices of V_4 are labeled as follows:

For $i = 0, 1, 2, \ldots, m-2$; $j = n-2$.

$$f(i,j) = \begin{cases} 10, & i \equiv 0 \; mod \; 4 \\ 7, & i \equiv 1 \; mod \; 4 \\ 2, & i \equiv 2 \; mod \; 4 \\ 8, & i \equiv 3 \; mod \; 4 \end{cases}$$

And $f(i, j) = 2$, for $i = m-1$, $j = n-2$.

The vertices of V_5 are labeled as follows:

For $i = 0, 1, 2, \ldots, m-2$; $j = n-1$.

$$f(i,j) = \begin{cases} 12, & i \equiv 0 \; mod \; 4 \\ 6, & i \equiv 1 \; mod \; 4 \\ 4, & i \equiv 2 \; mod \; 4 \\ 10, & i \equiv 3 \; mod \; 4 \end{cases}$$

And $f(i, j) = 4$, for $i = m-1$, $j = n-1$. (For illustration see Figure 7)

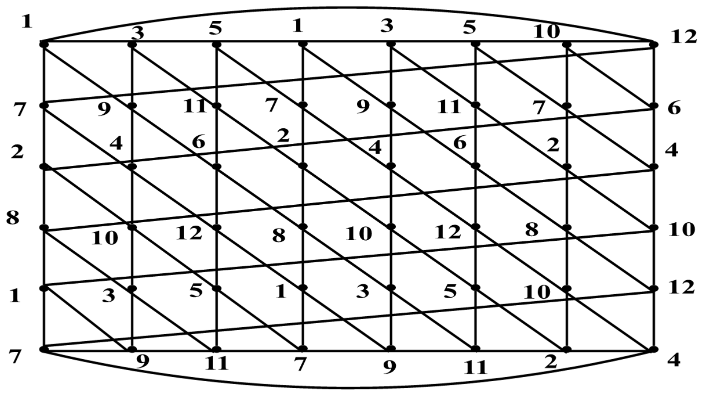

Figure 7. **L(2,1) labeling of** $B_{6,8}$

Subcase (ii): When $m \equiv 3 \; mod \; 4$

Vertex partition in this case is similar to Case (i) except for V_4 and V_5. The vertex partition for V_4 is given as below

$$V_4 = (i, j), \; i = 0, 1, 2, \ldots, m-1; \; j = n-2$$

The vertex partition for V_5 is given as

$$V_5 = (i, j),\ i = 0, 1, 2, \ldots, m-1;\ j = n-1$$

The vertices of V_4 are labeled as follows:

For $i = 0, 1, 2, \ldots, m-2$; $j = n-2$.

$$f(i,j) = \begin{cases} 10, & i \equiv 0 \ mod\ 4 \\ 7, & i \equiv 1 \ mod\ 4 \\ 2, & i \equiv 2 \ mod\ 4 \\ 8, & i \equiv 3 \ mod\ 4 \end{cases}$$

And $f(i,j) = 1$, for $i = m-1$, $j = n-2$.

The vertices of V_5 are labeled as follows:

For $i = 0, 1, 2, \ldots, m-2$; $j = n-1$.

$$f(i,j) = \begin{cases} 12, & i \equiv 0 \ mod\ 4 \\ 6, & i \equiv 1 \ mod\ 4 \\ 4, & i \equiv 2 \ mod\ 4 \\ 10, & i \equiv 3 \ mod\ 4 \end{cases}$$

And $f(i,j) = 13$, for $i = m-1$, $j = n-1$. (For illustration see Figure 8)

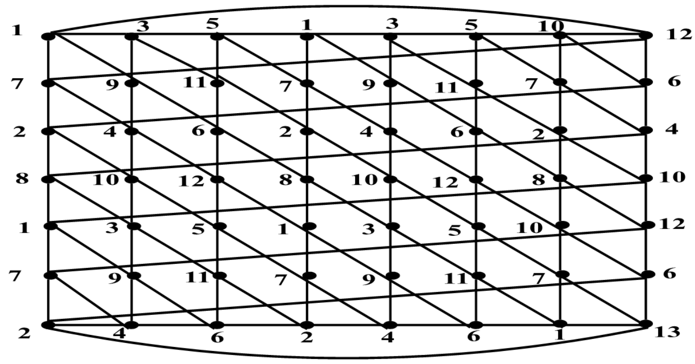

Figure 8. **L(2,1) labeling of** $B_{7,8}$

Subcase (iii): When $m \equiv 0 \ mod\ 4$

Vertex partition is done exactly the same as in subcase (i) except for V_3, V_4 and V_5. The vertex partition for V_3 is given as below

$$V_3 = (i, j),\ i = 0, 1, 2, \ldots, m-1;\ j = 2, 5, 8, \ldots, n-2$$

The vertex partition for V_4 is given as below

$$V_4 = (i, j),\ i = 0, 1, 2, \ldots, m-1;\ j = n-2$$

The vertex partition for V_5 is given as

$$V_5 = (i, j),\ i = 0, 1, 2, \ldots, m-1;\ j = n-1$$

The vertices of V_3 are labeled as follows:

For $i = 0, 1, 2, \ldots, m-2$; $j = 2, 5, 8, \ldots, n-3$.

$$f(i,j) = \begin{cases} 5, & i \equiv 0 \; mod \; 4 \\ 11, & i \equiv 1 \; mod \; 4 \\ 6, & i \equiv 2 \; mod \; 4 \\ 12, & i \equiv 3 \; mod \; 4 \end{cases}$$

And $f(i,j) = 1$, for $i = m-1$, $j = n-3$.

The vertices of V_4 are labeled as follows:

For $i = 0, 1, 2, \ldots, m-2$; $j = n-2$.

$$f(i,j) = \begin{cases} 10, & i \equiv 0 \; mod \; 4 \\ 7, & i \equiv 1 \; mod \; 4 \\ 2, & i \equiv 2 \; mod \; 4 \\ 8, & i \equiv 3 \; mod \; 4 \end{cases}$$

And $f(i,j) = 12$, for $i = m-1$, $j = n-2$.

The vertices of V_5 are labeled as follows:

For $i = 0, 1, 2, \ldots, m-3$; $j = n-1$.

$$f(i,j) = \begin{cases} 12, & i \equiv 0 \; mod \; 4 \\ 6, & i \equiv 1 \; mod \; 4 \\ 4, & i \equiv 2 \; mod \; 4 \\ 10, & i \equiv 3 \; mod \; 4 \end{cases}$$

And $f(i,j) = \begin{cases} 11, & \text{for } i = m-2, \; j = n-1; \\ 4, & \text{for } i = m-1, \; j = n-1. \end{cases}$ (For illustration see Figure 9)

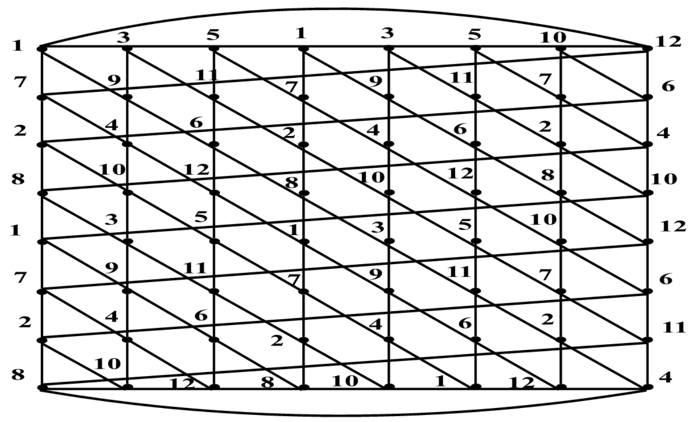

Figure 9. **L(2,1) labeling of $B_{8,8}$**

Subcase (iv): When $m \equiv 1 \; mod \; 4$

Vertex partition is done exactly the same as in subcase (i) except for V_3, V_4 and V_5. The vertex partition for V_3 is given as below

$$V_3 = (i,j), \; i = 0, 1, 2, \ldots, m-1; \; j = 2, 5, 8, \ldots, n-3$$

The vertex partition for V_4 is given as below

$$V_4 = (i,j), \; i = 0, 1, 2, \ldots, m-1; \; j = n-2$$

The vertex partition for V_5 is given as

$$V_5 = (i,j),\ i = 0,1,2,\ldots,m-1;\ j = n-1$$

The vertices of V_3 are labeled as follows:

For $i = 0,1,2,\ldots,m-2$; $j = 2,5,8,\ldots,n-3$.

$$f(i,j) = \begin{cases} 5, & i \equiv 0 \ mod\ 4 \\ 11, & i \equiv 1 \ mod\ 4 \\ 6, & i \equiv 2 \ mod\ 4 \\ 12, & i \equiv 3 \ mod\ 4 \end{cases}$$

And $f(i,j) = 7$, for $i = m-1$, $j = n-3$.

The vertices of V_4 are labeled as follows:

For $i = 0,1,2,\ldots,m-2$; $j = n-2$.

$$f(i,j) = \begin{cases} 10, & i \equiv 0 \ mod\ 4 \\ 7, & i \equiv 1 \ mod\ 4 \\ 2, & i \equiv 2 \ mod\ 4 \\ 8, & i \equiv 3 \ mod\ 4 \end{cases}$$

And $f(i,j) = 5$, for $i = m-1$, $j = n-2$.

The vertices of V_5 are labeled as follows:

For $i = 0,1,2,\ldots,m-2$; $j = n-1$.

$$f(i,j) = \begin{cases} 12, & i \equiv 0 \ mod\ 4 \\ 6, & i \equiv 1 \ mod\ 4 \\ 4, & i \equiv 2 \ mod\ 4 \\ 10, & i \equiv 3 \ mod\ 4 \end{cases}$$

And $f(i,j) = 13$, for $i = m-1$, $j = n-1$. (For illustration see Figure 10)

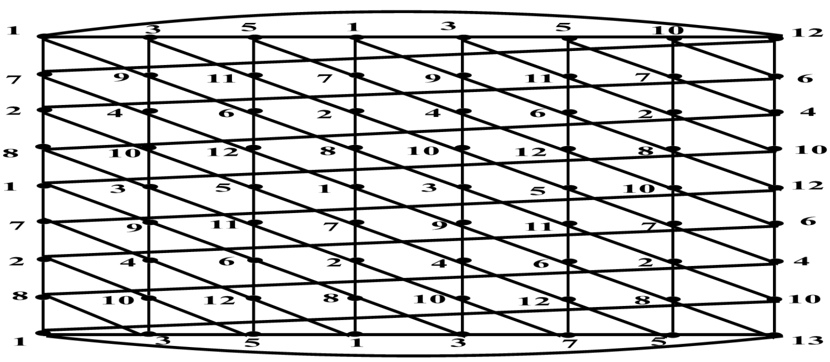

Figure 10. **L(2,1) labeling of** $B_{9,8}$

Thus in all the cases we find that the bloom graph $B_{m,n}$ admits $L(2,1)$-labeling and the upper bound of $L(2,1)$-number for bloom is given by $(B_{m,n}) \leq 13$. □

4. Conclusion

We computed $L(2,1)$-labeling for Bloom graph and found the upper bound of $L(2,1)$-number as 13 for bloom graph, i.e. $\lambda(B_{m,n}) \leq 13$. A lot of work has been done in this area and still work is being carried out for different graphs.

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