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$\psi^*\alpha$-Closed Sets in Bitopological Spaces

Research Article

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Abstract: In this paper we introduce $\psi^*\alpha$-closed sets in bitopological spaces and obtain the relationship between the other existing closed sets. Also we study the notion of (i,j)-$\psi^*\alpha$-closure operator and some of its properties. As applications we introduce (i,j)-${}_{\psi^*\alpha}T_c$-space, (i,j)-${}_{\psi^*\alpha}T_\alpha$-space and study some of their properties.

Keywords: $\psi^*\alpha$-closed set, $\psi^*\alpha$-open set, ψg-open set, τ_i-open set.

1. Introduction

Levine [10] introduced the concepts of generalized closed sets in topological spaces and studied their basic properties. Several authors have introduced and investigated various types generalized closed sets in topological spaces. Only a few class of generalized closed sets form a topology. The class of $\psi^*\alpha$-closed sets in topological spaces is one among them and it was introduced by Balamani and Parvathi [1]. The study of bitopological spaces was initiated by Kelly [7] and thereafter topological concepts have been generalized to bitopological setting. Fukutake [5] introduced g-closed sets in bitopological spaces. In this paper we introduce a new class of sets in bitopological spaces called (i,j)-$\psi^*\alpha$-closed sets and study their basic properties. Also we define (i,j)-$\psi^*\alpha$-closure of a set and prove that the closure operator (i,j)-$\psi^*\alpha$-closure is the Kuratowski closure operator on (X,τ_1,τ_2).

2. Preliminaries

The interior, closure and complement of a subset A of a space (X,τ) are denoted by $int(A)cl(A)$ and A^c respectively. Throughout this paper (X,τ_1,τ_2) represents bitopological space on which no separation axioms are assumed, unless otherwise mentioned.

Definition 2.1. *A subset A of a topological space (X,τ) is called*

(1). g-closed set [10] if $cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in (X,τ).

(2). sg-closed [3] if $scl(A)\subseteq U$ whenever $A\subseteq U$ and U is semi-open in (X,τ).

(3). ψ-closed set [13] if $scl(A)\subseteq U$ whenever $A\subseteq U$ and U is sg-open in (X,τ).

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(4). ψg-closed set [11] if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).

(5). $\psi^*\alpha$-closed set [1] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ψg-open in (X, τ).

Definition 2.2. *For $i, j = 1, 2$ and $i \neq j$, a subset A of a bitopological space (X, τ_1, τ_2) is called*

(1). (i, j)-g-closed [5] if τ_j-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_i-open in X.

(2). (i, j)-gp-closed [4] if τ_j-$pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_i-open in X.

(3). (i, j)-gpr-closed [6] if τ_j-$pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_i-regular open in X.

(4). (i, j)-ω-closed [6] if τ_j-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_i-semi open in X.

(5). (i, j)-g^*-closed [12] if τ_j-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_i-g-open in X.

(6). (i, j)-$g\alpha$-closed [8] if τ_j-$\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_i-α-open in X.

(7). (i, j)-αg-closed [4] if τ_j-$\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_i-open in X.

(8). (i, j)-$\widetilde{g}_a$-closed [9] if τ_j-$\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_i-$^{\#}gs$-open in X.

(9). (i, j)-ψg-closed [11] if τ_j-$\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_i-open in X.

Definition 2.3. *A topological space (X, τ) is said to be a*

(1). ${}_{\psi^*\alpha}T_c$-space if every $\psi^*\alpha$-closed subset of (X, τ) is closed in (X, τ) [2].

(2). ${}_{\psi^*\alpha}T_\alpha$-space if every $\psi^*\alpha$-closed subset of (X, τ) is α-closed in (X, τ) [2].

3. (i, j)-$\psi^*\alpha$-Closed Sets

Definition 3.1. *A subset A of a bitopological space (X, τ_i, τ_j) is called (i, j)-$\psi^*\alpha$-closed if τ_j-$\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_i-ψg-open in (X, τ_i, τ_j), where $i, j = 1, 2$ and $i \neq j$. The family of all (i, j)-$\psi^*\alpha$-closed sets in (X, τ_i, τ_j) is denoted by $\psi^*\alpha C(i, j)$.*

Remark 3.2. *By setting $\tau_i = \tau_j$ in Definition 3.1, an (i, j)-$\psi^*\alpha$-closed set reduces to a $\psi^*\alpha$-closed set.*

Example 3.3. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then ϕ, $\{c\}$, $\{a, c\}$, $\{b, c\}$, X are $(1, 2)$-$\psi^*\alpha$-closed.*

Proposition 3.4. *Every τ_j-closed (resp. τ_j-α- closed) set in $(X, \tau_1, \tau_2)$is(i, j)-$\psi^*\alpha$-closed but not conversely.*

Proof. Let A be τ_j-closed (resp. τ_j-α-closed) in (X, τ_1, τ_2) such that $A \subseteq U$, where U is τ_i-ψg-open. Since A is τ_j-closed (resp. τ_j-α-closed) τ_j-$cl(A)$ (resp. τ_j-$\alpha cl(A)$) $= A \subseteq U$. But τ_j-$\alpha cl(A) \subseteq \tau_j - cl(A)$. Therefore τ_j-$\alpha cl(A) \subseteq U$. Hence A is an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2). □

Example 3.5. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{a, b\}, X\}$. The subset $\{b\}$ is $(1, 2)$-$\psi^*\alpha$-closed but not τ_2-closed.*

Example 3.6. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$ and $t_2 = \{\phi, \{a, b\}, X\}$. The subset $\{b, c\}$ is $(1, 2)$-$\psi^*\alpha$-closed but not τ_2-α-closed.*

Proposition 3.7. *Every (i,j)-$\psi^*\alpha$-closed set in (X,τ_1,τ_2) is (i,j)-gp-closed but not conversely.*

Proof. Let $A \subseteq U$ and U be τ_i-open in (X,τ_1,τ_2). Since every τ_i-open set is τ_i-ψg-open and A is (i,j)-$\psi^*\alpha$-closed in $(X,\tau_1,\tau_2), \tau_j$-$\alpha cl(A) \subseteq U$. We know that τ_j-$pcl(A) \subseteq \tau_j$-$\alpha cl(A) \subseteq U$. Therefore A is (i,j)-gp -closed. □

Example 3.8. *Let $X = \{a,b,c,d\}$, $\tau_1 = \{\phi,\{a\},X\}$ and $\tau_2 = \{\phi,\{c\},\{a,b\},\{a,b,c\},X\}$. The subset $\{a,c,d\}$ is $(1,2)$-gp-closed but not $(1,2)$-$\psi^*\alpha$-closed.*

Proposition 3.9. *Every (i,j)-$\psi^*\alpha$-closed set in (X,τ_1,τ_2) is (i,j)-gpr -closed but not conversely.*

Proof. Let $A \subseteq U$ and U be τ_i-regular open in (X,τ_1,τ_2). Since every τ_i-regular open set is τ_i-ψg-open and A is (i,j)-$\psi^*\alpha$-closed in $(X,\tau_1,\tau_2), \tau_j$-$\alpha cl(A) \subseteq U$. We know that τ_j-$pcl(A) \subseteq \tau_j$-$\alpha cl(A) \subseteq U$. Therefore A is (i,j)-gpr-closed. □

Example 3.10. *Let $X = \{a,b,c,d\}$, $\tau_1 = \{\phi,\{a,b\},X\}$ and $\tau_2 = \{\phi,\{a\},\{b\},\{a,b\},\{a,b,c\},\{a,b,d\},X\}$. The subset $\{a,d\}$ is $(1,2)$-gpr-closed but not $(1,2)$-$\psi^*\alpha$-closed.*

Proposition 3.11. *Every (i,j)-$\psi^*\alpha$-closed set in (X,τ_1,τ_2) is (i,j)-$\widetilde{g}_a$-closed but not conversely.*

Proof. Let $A \subseteq U$ and U be τ_i-$^{\#}gs$-open in (X,τ_1,τ_2). Since every τ_i-$^{\#}gs$-open set is τ_i-ψg-open and A is (i,j)-$\psi^*\alpha$-closed in $(X,\tau_1,\tau_2), \tau_j$-$\alpha cl(A) \subseteq U$. Therefore A is (i,j)-$\widetilde{g}_a$- closed. □

Example 3.12. *Let $X = \{a,b,c,d\}$, $\tau_1 = \{\phi,\{a\},\{a,b\},X\}$ and $\tau_2 = \{\phi,\{d\},\{a,b\},\{a,b,d\},X\}$. The subset $\{b,c\}$ is $(1,2)$-$\widetilde{g}_a$-closed but not $(1,2)$-$\psi^*\alpha$-closed.*

Proposition 3.13. *Every (i,j)-$\psi^*\alpha$-closed set in (X,τ_1,τ_2) is (i,j)-$g\alpha$-closed but not conversely.*

Proof. Let $A \subseteq U$ and U be τ_i-α- open in (X,τ_1,τ_2). Since every τ_i-α-open set is τ_i-ψg-open and A is (i,j)-$\psi^*\alpha$-closed in $(X,\tau_1,\tau_2), \tau_j$-$\alpha cl(A) \subseteq U$. Therefore A is (i,j)-$g\alpha$-closed. □

Example 3.14. *Let $X = \{a,b,c\}$, $\tau_1 = \{\phi,\{a\},\{b,c\},X\}$ and $\tau_2 = \{\phi,\{a\},\{a,b\},\{a,c\},X\}$. The subsets $\{a,b\}$ and $\{a,c\}$ are $(1,2)$-$g\alpha$-closed but not $(1,2)$-$\psi^*\alpha$-closed.*

Proposition 3.15. *Every (i,j)-$\psi^*\alpha$-closed set in (X,τ_1,τ_2) is (i,j)-αg-closed but not conversely.*

Proof. Let $A \subseteq U$ and U be τ_i-open in (X,τ_1,τ_2). Since every τ_i-open set is τ_i-ψg-open and A is (i,j)-$\psi^*\alpha$-closed in $(X,\tau_1,\tau_2), \tau_j$-$\alpha cl(A) \subseteq U$. Therefore A is (i,j)-αg-closed. □

Example 3.16. *Let $X = \{a,b,c\}$, $\tau_1 = \{\phi,\{a\},\{b,c\},X\}$ and $\tau_2 = \{\phi,\{a\},X\}$. The subsets $\{a,b\}$ and $\{a,c\}$ are $(1,2)$-αg-closed but not $(1,2)$-$\psi^*\alpha$-closed.*

Proposition 3.17. *Every (i,j)-$\psi^*\alpha$-closed set in (X,τ_1,τ_2) is (i,j)-ψg-closed but not conversely.*

Proof. Let $A \subseteq U$ and U be τ_i-open in (X,τ_1,τ_2). Since every τ_i-open set is τ_i-ψg-open and A is (i,j)-$\psi^*\alpha$-closed in (X,τ_1,τ_2), τ_j-$\alpha cl(A) \subseteq U$. We know that τ_j-$\psi cl(A) \subseteq \tau_j$-$\alpha cl(A) \subseteq U$ and so τ_j-$\psi cl(A) \subseteq U$. Therefore A is (i,j)-ψg-closed. □

Example 3.18. *Let $X = \{a,b,c\}$, $\tau_1 = \{\phi,\{a\},X\}$ and $\tau_2 = \{\phi,\{a,b\},X\}$. The subsets $\{b\},\{c\},\{a,b\}$ and $\{a,c\}$ are $(1,2)$-ψg-closed but not $(1,2)$-$\psi^*\alpha$-closed.*

Remark 3.19. *The following example show that (i,j)-$\psi^*\alpha$-closed set is independent of (i,j)-g-closed set, (i,j)-g^*-closed set and (i,j)-ω-closed set.*

Example 3.20. *Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, \{c\}, \{a, b\}, \{a, b, c\}, X\}$ and $\tau_2 = \{\phi, \{a\}, X\}$. The subset $\{a, c, d\}$ is $(1, 2)$-g-closed, $(1, 2)$-g^*-closed and $(1, 2)$-ω-closed but not $(1, 2)$-$\psi^*\alpha$-closed. The subset $\{b\}$ is $(1, 2)$-$\psi^*\alpha$-closed but not $(1, 2)$-g-closed, not $(1, 2)$-g^*-closed and not $(1, 2)$-ω-closed.*

Theorem 3.21. *If A is τ_i-ψg-open and (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2) then A is τ_j-α-closed.*

Proof. Let A be τ_i-ψg-open and (i, j)-$\psi^*\alpha$-closed. Since $A \subseteq A$, then τ_j-$\alpha cl(A) \subseteq A$. Therefore τ_j-$\alpha cl(A) = A$. Consequently A is τ_j-α-closed. □

Theorem 3.22. *If A is (i, j)-$\psi^*\alpha$-closed and τ_i-ψg-open and F is τ_j-α-closed in (X, τ_1, τ_2) then $A \cap F$ is τ_j-α-closed.*

Proof. Since A is (i, j)-$\psi^*\alpha$-closed and τ_i-ψg-open in (X, τ_1, τ_2), A is τ_j-α-closed (by Theorem 3.21). Since F is τ_j-α-closed, $A \cap F$ is τ_j-α-closed in (X, τ_1, τ_2). □

Theorem 3.23. *Union of two (i, j)-$\psi^*\alpha$-closed sets is (i, j)-$\psi^*\alpha$-closed.*

Proof. Let A and B are (i, j)-$\psi^*\alpha$-closed sets and U be any ψg-open set in (X, τ_i) containing A and B. Then $\tau_j - \alpha cl(A) \subseteq U$, $\tau_j - \alpha cl(B) \subseteq U$, $\tau_j - \alpha cl(A \cup B) = \tau_j - \alpha cl(A) \cup \tau_j - \alpha cl(B) \subseteq U$. Hence $A \cup B$ is (i, j)-$\psi^*\alpha$-closed. □

Remark 3.24. *The intersection of two (i, j)-$\psi^*\alpha$-closed sets need not be (i, j)-$\psi^*\alpha$-closed set as seen from the following example.*

Example 3.25. *Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{c\}, \{a, b\}, \{a, b, c\}, X\}$. The subsets $A = \{a, b, d\}$ and $B = \{b, c, d\}$ are $(1, 2)$-$\psi^*\alpha$-closed but their intersection $A \cap B = \{b, d\}$ is not $(1, 2)\psi^*\alpha$-closed.*

Theorem 3.26. *If a subset A of a bitopological space (X, τ_1, τ_2) is (i, j)-$\psi^*\alpha$-closed then τ_j-$\alpha cl(A)$-A contains no nonempty τ_j-ψg-closed set.*

Proof. Let A be an (i, j)-$\psi^*\alpha$-closed set and F be a τ_j-ψg-closed set such that $F \subseteq \tau_j - \alpha cl(A) - A$. Therefore $A \subseteq F^c$ and $F \subseteq \tau_j - \alpha cl(A)$. Since F^c is τ_i-ψg-open and A is (i, j)-$\psi^*\alpha$-closed, $\tau_j - \alpha cl(A) \subseteq F^c$. Thus $F \subseteq [\tau_j - \alpha cl(A)]^c$. Hence $F \subseteq [\tau_j - \alpha cl(A)] \cap [\tau_j - \alpha cl(A)]^c = \phi$. Therefore $F = \phi$. Hence τ_j-$\alpha cl(A)$-A contains no nonempty τ_i-ψg-closed set. □

Remark 3.27. *The converse of the above theorem is not true as seen from the following example.*

Example 3.28. *Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, \{a, b, c\}, X\}$ and $\tau_2 = \{\phi, \{c\}, \{a, b\}, \{a, b, c\}, X\}$. $\psi gO(X, \tau_1) = \{\{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X\}$, $\psi^*\alpha(1, 2) = \{\phi, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, c, d\}, X\}$. If $A = \{a, b\}$, $\tau_j - \alpha cl(A) - A = \{a, b, d\} - \{a, b\} = \{d\}$. But $\{a, b\}$ is not $(1, 2)$-$\psi^*\alpha$-closed.*

Theorem 3.29. *Let A be an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2). Then A is τ_j-α-closed if and only if τ_j-$\alpha cl(A)$-A is τ_i-ψg-closed in (X, τ_1, τ_2).*

Proof. Suppose that A is (i, j)-$\psi^*\alpha$-closed. Let A be τ_j-α-closed. Then $\tau_j - \alpha cl(A) = A$. Therefore τ_j-$\alpha cl(A) - A = \phi$ is τ_i-ψg-closed in (X, τ_1, τ_2).

Conversely, suppose that A is (i, j)-$\psi^*\alpha$-closed and τ_j-$\alpha cl(A)$-A is τ_i-ψg-closed. Since A is (i, j)-$\psi^*\alpha$-closed, τ_j-$\alpha cl(A)$-A contains no nonempty τ_i-ψg-closed set (by Theorem 3.26). Since τ_j-$\alpha cl(A)$-A is τ_i-ψg-closed, $\tau_j - \alpha cl(A) - A = \phi$. Then $\tau_j - \alpha cl(A) = A$. Hence A is τ_j-α-closed. □

Theorem 3.30. *Let A and B be subsets of (X, τ_1, τ_2) such that $A \subseteq B \subseteq \tau_j - \alpha cl(A)$. If A is (i, j)-$\psi^*\alpha$-closed then B is (i, j)-$\psi^*\alpha$-closed.*

Proof. Let A and B be subsets such that $A \subseteq B \subseteq \tau_j - \alpha cl(A)$. Suppose that A is (i,j)-$\psi^*\alpha$-closed. Let $B \subseteq U$ and U be τ_i-ψg-open in (X, τ_1, τ_2). Then $A \subseteq U$. Since A is (i,j)-$\psi^*\alpha$-closed, $\tau_j - \alpha cl(A) \subseteq U$. Since $B \subseteq \tau_j - \alpha cl(A)$, $\tau_j - \alpha cl(B) \subseteq \tau_j - \alpha cl[\tau_j - \alpha cl(A)] = \tau_j - \alpha cl(A) \subseteq U$. Therefore B is (i,j)-$\psi^*\alpha$-closed. □

Theorem 3.31. *Let $B \subseteq A \subseteq X$ and suppose that B is (i,j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2), then B is (i,j)-$\psi^*\alpha$-closed relative to A. The converse is true if A is τ_i-open and (i,j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2).*

Proof. Let B be (i,j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Let $B \subseteq U$ and U be τ_i-ψg-open in A. Since U is τ_i-ψg-open in A, $U = V \cap A$, where V is τ_i-ψg-open in (X, τ_1, τ_2). Hence $B \subseteq U \subseteq V$. Since B is (i,j)-$\psi^*\alpha$-closed in $(X, \tau_1, \tau_2), \tau_j - \alpha cl(B) \subseteq V$. Hence $\tau_j - \alpha cl(B) \cap A \subseteq V \cap A$, which in turn implies that $\tau_j - \alpha cl_A(B) \subseteq V \cap A = U$. Therefore B is (i,j)-$\psi^*\alpha$-closed relative to A.

Now to prove the converse, assume the given condition. Let $B \subseteq U$ and U be τ_i-ψg-open in (X, τ_1, τ_2). Then $A \cap U$ is τ_i-ψg-open in A. Since $B \subseteq A$ and $B \subseteq U$, $B \subseteq A \cap U$. Since B is (i,j)-$\psi^*\alpha$-closed relative to A, $\tau_j - \alpha cl_A(B) \subseteq A \cap U$. Since A is τ_i-open, it is τ_i-ψg-open in (X, τ_1, τ_2). Since $A \subseteq A$ and A is (i,j)-$\psi^*\alpha$-closed in $(X, \tau_1, \tau_2), \tau_j - \alpha cl(A) \subseteq A$. Since $B \subseteq A$, $\tau_j - \alpha cl(B) \subseteq \tau_j - \alpha cl(A)$. Hence $\tau_j - \alpha cl(B) \subseteq A$. Therefore, $\tau_j - \alpha cl(B) \cap A = \tau_j - \alpha cl(B) \Rightarrow \tau_j - \alpha cl_A(B) = \tau_j - \alpha cl(B)$. Hence $\tau_j - \alpha cl(B) \subseteq A \cap U \subseteq U$. Thus B is (i,j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). □

Remark 3.32. *In general $\psi^*\alpha C(\tau_i, \tau_j) \neq \psi^*\alpha C(\tau_j, \tau_i)$ which can be seen from the following example.*

Example 3.33. *Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then $\psi^*\alpha C(\tau_i, \tau_j) = \{\phi, \{c\}, \{b, c\}, \{a, c\}, X\}$ and $\psi^*\alpha C(\tau_j, \tau_i) = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$. This shows that $\psi^*\alpha C(\tau_i, \tau_j) \neq \psi^*\alpha C(\tau_j, \tau_i)$.*

Theorem 3.34. *If $\tau_1 \subseteq \tau_2$ in (X, τ_1, τ_2) then $\psi^*\alpha C(2, 1) \subseteq \psi^*\alpha C(1, 2)$.*

Proof. Let $A \in \psi^*\alpha C(2, 1)$. Let $U \in \psi gO(X, \tau_1)$ such that $A \subseteq U$. Since $\psi gO(X, \tau_1) \subseteq \psi gO(X, \tau_2)$, $U \in \psi gO(X, \tau_2)$. Since A is $(2,1)$-$\psi^*\alpha$-closed, $\tau_1 - \alpha cl(A) \subseteq U$. Since $\tau_1 \subseteq \tau_2$, $\tau_2 - \alpha cl(A) \subseteq \tau_1 - \alpha cl(A)$. Thus $\tau_2 - \alpha cl(A) \subseteq U$. Hence A is $(1,2)$-$\psi^*\alpha$-closed. That is, $A \in \psi^*\alpha C(1, 2)$. □

The converse of the above theorem need not be true as seen from the following example:

Example 3.35. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{a, b\}, \{a, c\}\}$ and $\tau_2 = \{\phi, \{a\}, \{b, c\}, X\}$. Then $\psi^*\alpha C(2, 1) \subseteq \psi^*\alpha C(1, 2)$ but $\tau_1 \nsubseteq t_2$.*

Definition 3.36. *A set A of a bitopological space (X, τ_1, τ_2) is called (i, j)-ψ star alpha open (briefly, (i, j)-$\psi^*\alpha$-open) if its complement is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). The set of all (i, j)-$\psi^*\alpha$-open sets in (X, τ_1, τ_2) is denoted by $\psi^*\alpha O(i, j)$.*

Example 3.37. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then $\phi, \{a\}, \{b\}, \{a, b\}$ are (1, 2)-$\psi^*\alpha$-open.*

Definition 3.38. *An (i, j)-ψ star alpha interior of a subset A (briefly, (i, j)-$\psi^*\alpha$ int(A)) in (X, τ_1, τ_2) is defined as follows.*

$$(i,j) - \psi^*\alpha \ int(A) = \cup\{F \subseteq X : F \subseteq A \ \ and \ \ F \ is \ (i,j)\text{-}\psi^*\alpha\text{-}open \ in \ \ (X, \tau_1, \tau_2)\}.$$

Proposition 3.39.

(1). Every τ_j-open set in (X, τ_1, τ_2) is (i,j)-$\psi^*\alpha$-open.

(2). Every τ_j-α-open set in (X, τ_1, τ_2) is (i,j)-$\psi^*\alpha$-open.

(3). Every (i,j)-$\psi^*\alpha$-open set in (X, τ_1, τ_2) is (i, j)-gp-open.

(4). Every (i, j)-$\psi^*\alpha$-open set in (X, τ_1, τ_2) is (i, j)-gpr-open.

(5). Every (i, j)-$\psi^*\alpha$-open set in (X, τ_1, τ_2) is (i, j)-$\widetilde{g}_\alpha$-open.

(6). Every (i, j)-$\psi^*\alpha$-open set in (X, τ_1, τ_2) is (i, j)-$g\alpha$-open.

(7). Every (i, j)-$\psi^*\alpha$-open set in (X, τ_1, τ_2) is (i, j)-αg-open.

(8). Every (i, j) -$\psi^*\alpha$-open set in (X, τ_1, τ_2) is (i, j)-ψg-open.

The converses of the statements in the above proposition are not true in general as seen from the Examples 3.5, 3.6, 3.8, 3.10, 3.12, 3.14, 3.16 and 3.18.

Theorem 3.40. *A subset A of a bitopological space (X, τ_1, τ_2) is (i, j)-$\psi^*\alpha$-open if and only if $F \subseteq \tau_j - \alpha int(A)$ whenever $F \subseteq A$ and F is τ_i-ψg-closed in (X, τ_1, τ_2).*

Proof. Suppose that A is (i, j)-$\psi^*\alpha$-open. Let $F \subseteq A$ and F be τ_i-ψg-closed. Then $A^c \subseteq F^c$ and F^c is τ_i-ψg-open. Since A^c is (i, j)-$\psi^*\alpha$-closed, $\tau_j - \alpha cl(A^c) \subseteq F^c$. Since $\tau_j - \alpha cl(A^c) = [\tau_j - \alpha int(A)]^c$, $[\tau_j - \alpha int(A)]^c \subseteq F^c$. Hence $F \subseteq \tau_j - \alpha int(A)$. Conversely, suppose that $F \subseteq \tau_j - \alpha int(A)$ whenever $F \subseteq A$ and F is τ_i-ψg-closed in (X, τ_1, τ_2). Let U be τ_i-ψg-open in (X, τ_1, τ_2) and $A^c \subseteq U$. Then U^c is τ_i-ψg-closed and $U^c \subseteq A$. Hence by assumption $U^c \subseteq \tau_j - \alpha int(A)$. Therefore $[\tau_j - \alpha int(A)]^c \subseteq U$. That is $\tau_j - \alpha cl(A^c) \subseteq U$. Therefore A^c is (i, j)-$\psi^*\alpha$-closed. Hence A is (i, j)-$\psi^*\alpha$-open. □

Theorem 3.41. *If a subset A is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2) then τ_j-$\alpha cl(A)$-A is (i, j)-$\psi^*\alpha$-open.*

Proof. Suppose that A is (i, j)-$\psi^*\alpha$-closed in (X, τ_1, τ_2). Let $F \subseteq \tau_j - \alpha cl(A) - A$ and F be τ_i-ψg-closed. Since A is (i, j)-$\psi^*\alpha$-closed, τ_j-$\alpha cl(A)$-A does not contain nonempty τ_i-ψg-closed sets (by Theorem 3.26). Hence $F = \phi$. Thus $F \subseteq \tau_j - \alpha int[\tau_j - \alpha cl(A) - A]$. Hence τ_j-$\alpha cl(A)$-A is (i, j)-$\psi^*\alpha$-open. □

Theorem 3.42. *If a set A is (i, j)-$\psi^*\alpha$-open in (X, τ_1, τ_2) then $G = X$ whenever G is τ_i-ψg-open and $\tau_j - \alpha int(A) \cup A^c \subseteq G$.*

Proof. Suppose that A is (i, j)-$\psi^*\alpha$-open in (X, τ_1, τ_2), G is τ_i-ψg-open and $\tau_j - \alpha int(A) \cup A^c \subseteq G$. Then $G^c \subseteq \{\tau_j - \alpha int(A) \cup A^c\}^c = \tau_j - \alpha cl(A^c) - A^c$. Since A^c is (i, j)-$\psi^*\alpha$-closed, $\tau_j - \alpha cl(A^c) - A^c$ contains no nonempty τ_i-ψg-closed set in (X, τ_1, τ_2) (by Theorem 3.26). Therefore $G^c = \phi$. Hence $G = X$. □

Remark 3.43. *The converse of the above theorem is not true in general as seen from the following example.*

Example 3.44. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{a, b\}, X\}$. Let $A = \{c\}$ and $G = X$. Then G is τ_1-ψg-open, $\tau_2 - \alpha int(A) \cup A^c = \phi \cup \{a, b\} = \{a, b\} \subseteq G$, but $A = \{c\}$ is not $(1, 2)$-$\psi^*\alpha$-open.*

Theorem 3.45. *Let (X, τ_1, τ_2) be a bitopological space. If $x \in X$, then singleton $\{x\}$ is either τ_i-ψg-closed or (i, j)-$\psi^*\alpha$-open.*

Proof. Let $x \in X$ and suppose that $\{x\}$ is not τ_i-ψg-closed. Then $X - \{x\}$ is not τ_i-ψg-open. Consequently, X is the only τ_i-ψg-open set containing the set $X - \{x\}$. Therefore $X - \{x\}$ is (i, j)-$\psi^*\alpha$-closed. Hence $\{x\}$ is (i, j)-$\psi^*\alpha$-open. □

4. (i, j)-$\psi^*\alpha$-closure

Definition 4.1. *An (i, j)-$\psi^*\alpha$-closure of a subset A (briefly, (i, j)-$\psi^*\alpha cl(A)$) of (X, τ_1, τ_2) is defined as $(i, j) - \psi^*\alpha cl(A) = \cap\{F \subseteq X : A \subseteq F$ and F is (i, j)-$\psi^*\alpha$-closed in $(X, \tau_1, \tau_2)\}$.*

Proposition 4.2. *Let E and F be any two subsets of (X, τ_1, τ_2). Then the following results hold.*

(a). (i,j)-$\psi^*\alpha cl(\phi) = \phi$ *and* $(i,j) - \psi^*\alpha cl(X) = X$.

(b). If $E \subseteq F$*, then* $(i,j) - \psi^*\alpha cl(E) \subseteq (i,j) - \psi^*\alpha cl(F)$.

(c). $E \subseteq (i,j) - \psi^*\alpha cl(E) \subseteq \tau_j - cl(E)$.

(d). If A *is* (i,j)-$\psi^*\alpha$*-closed in* (X,τ_1,τ_2) *then* $(i,j) - \psi^*\alpha cl(E) = E$.

(e). $(i,j) - \psi^*\alpha cl(E \cap F) \subseteq (i,j) - \psi^*\alpha cl(E) \cap (i,j) - \psi^*\alpha cl(F)$.

(f). $(i,j) - \psi^*\alpha cl(E \cup F) = (i,j) - \psi^*\alpha cl(E) \cup (i,j) - \psi^*\alpha cl(F)$.

(g). $(i,j) - \psi^*\alpha cl((i,j) - \psi^*\alpha cl(E)) = (i,j) - \psi^*\alpha cl(E)$.

Proof.

(a). Since ϕ and X are (i,j)-$\psi^*\alpha$-closed in (X,τ_1,τ_2), the results follows.

(b). Let $E \subseteq F$. Then by the definition of (i,j)-$\psi^*\alpha$-closure, $(i,j) - \psi^*\alpha cl(E) \subseteq (i,j) - \psi^*\alpha cl(F)$.

(c). From the definition of (i,j)-$\psi^*\alpha$-closure, it follows that $E \subseteq (i,j) - \psi^*\alpha cl(E)$. By Proposition 3.4 every τ_j-closed set is (i,j)-$\psi^*\alpha$-closed. Therefore $E \subseteq (i,j) - \psi^*\alpha cl(E) \subseteq \tau_j - cl(E)$.

(d). Follows from (c) and by the definition of (i,j)-$\psi^*\alpha$-closure.

(e). Since $E \cap F \subseteq E$ and $E \cap F \subseteq F$, by (b) $(i,j) - \psi^*\alpha cl(E \cap F) \subseteq (i,j) - \psi^*\alpha cl(E)$, $(i,j) - \psi^*\alpha cl(E \cap F) \subseteq (i,j) - \psi^*\alpha cl(F)$. Hence $(i,j) - \psi^*\alpha cl(E \cap F) \subseteq (i,j) - \psi^*\alpha cl(E) \cap (i,j) - \psi^*\alpha cl(F)$.

(f). Since $E \subseteq E \cup F$ and $F \subseteq E \cup F$, by (b) $(i,j) - \psi^*\alpha cl(E) \subseteq (i,j) - \psi^*\alpha cl(E \cup F)$ and $(i,j) - \psi^*\alpha cl(F) \subseteq (i,j) - \psi^*\alpha cl(E \cup F)$. To prove the reverse inclusion, let $x \in (i,j) - \psi^*\alpha cl(E \cup F)$ and suppose that $x \notin (i,j) - \psi^*\alpha cl(E) \cup (i,j) - \psi^*\alpha cl(F)$. Then $x \notin (i,j) - \psi^*\alpha cl(E)$ and $x \notin (i,j) - \psi^*\alpha cl(F)$. Therefore there exist (i,j)-$\psi^*\alpha$-closed sets U and V such that $E \subseteq U$, $F \subseteq V$, $x \notin U$ and $x \notin V$. Hence we have $E \cup F \subseteq U \cup V$ and $x \notin U \cup V$. By Theorem 3.23, $U \cup V$ is a (i,j)-$\psi^*\alpha$-closed set and hence $x \notin (i,j) - \psi^*\alpha cl(E \cup F)$, which is a contradiction. Hence $(i,j) - \psi^*\alpha cl(E \cup F) \subseteq (i,j) - \psi^*\alpha cl(E) \cup (i,j) - \psi^*\alpha cl(F)$. Therefore $(i,j) - \psi^*\alpha cl(E \cup F) = (i,j) - \psi^*\alpha cl(E) \cup (i,j) - \psi^*\alpha cl(F)$.

(g). Follows from the definition of (i,j)-$\psi^*\alpha$-closure. □

Theorem 4.3. *The closure operator* (i,j)-$\psi^*\alpha$*-closure is a Kuratowski closure operator on* (X,τ_1,τ_2).

Proof. From $(i,j) - \psi^*\alpha cl(\phi) = \phi$, $A \subseteq (i,j) - \psi^*\alpha cl(A)$, $(i,j) - \psi^*\alpha cl(E \cup F) = (i,j) - \psi^*\alpha cl(E) \cup (i,j) - \psi^*\alpha cl(F)$ and $(i,j) - \psi^*\alpha cl((i,j) - \psi^*\alpha cl(E)) = (i,j) - \psi^*\alpha cl(E)$, we can say that (i,j)-$\psi^*\alpha$-is a Kuratowski closure operator on (X,τ_1,τ_2). □

Definition 4.4. *A bitopological space* (X,τ_1,τ_2) *is called an*

(1) $(i,j) - {}_{\psi^*\alpha}T_c$*-space if every* (i,j)-$\psi^*\alpha$*-closed subset of* (X,τ_1,τ_2) *is* τ_j*-closed in* (X,τ_1,τ_2).

(2) $(i,j) - {}_{\psi^*\alpha}T_\alpha$*-space if every* (i,j)-$\psi^*\alpha$*-closed subset of* (X,τ_1,τ_2) *is* τ_j*-*α*-closed in* (X,τ_1,τ_2).

Proposition 4.5. *Every* $(i,j) - {}_{\psi^*\alpha}T_c$*-space is an* $(i,j) - {}_{\psi^*\alpha}T_\alpha$*-space but not conversely.*

Proof. Assume that (X,τ_1,τ_2) is an $(i,j) - {}_{\psi^*\alpha}T_c$-space. Let A be an (i,j)-$\psi^*\alpha$-closed set in (X,τ_1,τ_2). Then A is τ_j-closed. Since every τ_j-closed set is τ_j-α-closed, A is τ_j-α-closed in (X,τ_1,τ_2). Thus (X,τ_1,τ_2) is an $(i,j) - {}_{\psi^*\alpha}T_\alpha$-space. □

Example 4.6. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{b, c\}, X\}$ and $\tau_2 = \{\phi, \{a\}, X\}$. Then (X, τ_1, τ_2) is an $(i, j) - {}_{\psi^*\alpha}T_\alpha$-space but not an $(i, j) - {}_{\psi^*\alpha}T_c$-space, since the subsets $\{b\}$ and $\{c\}$ are $(1, 2)$-$\psi^*\alpha$-closed but not τ_2-closed in (X, τ_1, τ_2).*

Theorem 4.7. *For a space (X, τ_1, τ_2) the following statements are equivalent.*

(1) (X, τ_1, τ_2) is an $(i, j) - {}_{\psi^*\alpha}T_\alpha$-space.

(2) For each $x \in X$, $\{x\}$ is either τ_i-ψg-closed or τ_j-α-open.

Proof. (1) $\Rightarrow$ (2) Let $x \in X$ and $\{x\}$ be not a τ_i-ψg-closed set in (X, τ_1, τ_2). Then $X - \{x\}$ is not τ_i-ψg-open. Hence X is the only τ_i-ψg-open set containing $X - \{x\}$. This implies that $X - \{x\}$ is an (i, j)-$\psi^*\alpha$-closed set of (X, τ_1, τ_2). Since X is an $(i, j) - {}_{\psi^*\alpha}T_\alpha$-space, $X - \{x\}$ is a τ_j-α-closed set in (X, τ_1, τ_2) or equivalently $\{x\}$ is τ_j-α-open in (X, τ_1, τ_2).
(2) $\Rightarrow$ (1) Let A be an (i, j)-$\psi^*\alpha$-closed set in (X, τ_1, τ_2) and $x \in \tau_j - \alpha cl(A)$. We show that $x \in A$. By (2), $\{x\}$ is either τ_i-ψg-closed or τ_j-α-open.
Case 1: Assume that $\{x\}$ is τ_j-α-open. Then $X - \{x\}$ is τ_j-α-closed. If $x \notin A$, then $A \subseteq X - \{x\}$. Since $x \in \tau_j - \alpha cl(A)$, $x \in [X - \{x\}]$, which is a contradiction. Hence $x \in A$.
Case 2: Assume that $\{x\}$ is τ_i-ψg-closed and $x \notin A$. Then τ_j-$\alpha cl(A)$-A contains a τ_j-ψg-closed set $\{x\}$. This contradicts Theorem 3.26. Therefore $x \in A$. □

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