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Mathematical Modeling for Poiseuille Flow of Blood Fluid Model with an Axial Velocity and Variation of Apparent Viscosity Slip Along an Artery Wall in Presence of Magnetic Field

Research Article

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Abstract: The effects of non-Newtonian nature of blood and pulsatility on flow through stenosed artery have been investigated. It is of interest to note that the thickness of the viscous flow region with axial distances. Mathematical modeling for poiseuille flow of blood fluid model with an axial velocity slip along an artery wall in presence of magnetic field, is considered. It is observed that when Hartmann number increases the fluid velocity is greatly affected. Many standard result regarding Newtonian fluid flows, uniform and steady flow in an artery can be obtained in the present analysis as the special cases. Applications of this theoretical modelling to cardiovascular diseases and the role of slip in the better functioning of the diseased or occluded arteries are included in brief.

Keywords: Bingham plastic fluid model Reynolds Number, Hartmann number, Velocity profile, Non -Newtonian fluid.

1. Introduction

Cardiovascular system of man and animals is characteristically a branch network of distensible tubes which carry blood from the heart to the periphery and back again [1, 9, 28]. It essential comprises of cardio the heart, blood vessel arteries, veins and capillaries and blood a complex fluid which altogether forms a closed system and regulates body function. The primary function of circulation is to transport nutrients to tissues and to remove metabolic product. Human blood is considered as a suspension of tiny cells or corpuscles that are suspended in a clear fluid plasma [3, 28]. Many researcher [13, 24] have considered blood analyses, presence of different cells in blood has been totally ignored. Further this approach may provide satisfactory results in explaining certain aspects of blood in large arteries, yet to fail to explain the flow behavior of blood in vessels having small diameter [2, 27]. An increase in shear, causes the disaggregation of rouleax and deformation of individual cells with the result that long axes are aligned with direction of flow to minimize the viscous resistance [1–3]. Further blood possesses a finite yield stress and a particular type of Non-Newtonian fluid bearing the property can account for this feature of blood which has been reported by many investigator [1, 3, 7, 11, 22].

Dintenfass [6] and other [12–14, 16] have pointed out that viscosity of significant factor in pathogenesis of ischemia and infraction and may play an important role in hypertension and other cardiovascular diseases. It is therefore necessary to

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measure blood viscosity accurately. In order to measure blood viscosity and flow variables [4, 9, 21, 26, 27] and other [3, 5, 7] have proposed theoretical models on blood flows. In most of the theoretical models on blood flow, usual no-slip condition at vessel wall is considered [3, 16–18, 20] and other [8, 15, 19] have suggested the likely presence of a red cell slip at vessel wall or in its immediate neighbourhood.In view of a possible existences of slip at tube wall [8, 16, 17]and other [10, 13] have considered a velocity slip condition at blood vessel wall or at interface of fluid in their modelling.In view of the importance of slip and its physiological significance for blood flow through an artery, in the present modelling a slip condition for velocity at tube wall of three different location of vs. is employed [25]. Here we consider for one dimensional flow axial velocity $\hat{v} = (0, 0, u_z(r))$ the equation for steady tube flow $0 \leq r \leq R$ of blood in (r, θ, z) co-ordinate system reduce to obtain the following form in presence of transverse magnetic effect.

$$-\frac{\partial \hat{p}}{\partial \hat{r}} = 0 \tag{1}$$

$$-\frac{1}{\hat{r}}\frac{\partial \hat{p}}{\partial \hat{\theta}} = 0 \tag{2}$$

$$-\frac{1}{\rho}\frac{\partial \hat{p}}{\partial \hat{z}} + \frac{\mu}{\hat{r}}\frac{1}{\rho}\frac{\partial}{\partial \hat{r}}\left(\hat{r}\frac{\partial \hat{u}_z}{\partial \hat{r}}\right) + \frac{\sigma}{\rho}\hat{B}^2 u = 0 \tag{3}$$

From which we observed that pressure does not vary in the radial $\hat{r}$ circumferential $\hat{\theta}$ and axial $\hat{z}$ direction and that pressure remain constant across any cross-section of the tube and $\hat{p}$ is a function of only $\hat{z}$ that is $\hat{p} = p(\widehat{z})$ and so pressure gradient term in the last equation above becomes $\frac{d\hat{p}}{d\hat{z}}$. Then (3)

$$-\frac{1}{\rho}\frac{\partial \hat{p}}{\partial \hat{z}} + \frac{\mu}{\hat{r}}\frac{1}{\rho}\frac{\partial}{\partial \hat{r}}\left(\hat{r}\frac{\partial \hat{u}_z}{\partial \hat{r}}\right) + \frac{\sigma}{\rho}\hat{B}^2 u = 0$$

Non-dimensional form

$$r = \frac{\hat{r}}{R_0}, z = \frac{\hat{z}}{R_0}, R = \frac{\hat{R}}{R_0}, P = \frac{\hat{P}}{\rho U_0^2}, U = \frac{\hat{U}}{U_0}, B = \frac{\hat{B}}{U_0}$$

$$-\frac{dp}{dz} + \frac{1}{R_e r}\frac{d(r\tau_{rz})}{dr} + \frac{M^2}{R_e} = 0$$

$$C + \frac{1}{R_e r}\frac{d(r\tau_{rz})}{dr} + \frac{M^2}{R_e} = 0 \tag{4}$$

2. Solving the Equation (4)

$r^2\frac{d^2u_Z}{dr^2} + \frac{du_Z}{dr} = Kr^2$, where $K = -\frac{C}{\mu}R_e r - \frac{M^2 r}{\mu}$

$$u_z = \frac{1}{4}\left[-\frac{C}{\mu}R_e r - \frac{M^2 r}{\mu}\right]r^2 \tag{5}$$

Again, shear stress component at any distance r from the tube axis is given by

$$\tau_{rz} = \mu\frac{du_z}{dr} = \mu e^0 \tag{6}$$

$$\tau_{rz} = \mu\frac{r}{2}\left[-\frac{C}{\mu}R_e - \frac{M^2}{\mu}\right] \tag{7}$$

Express for wall shear stress t_w can be obtained from the formula

$$\tau_w = \tau_{rz}(r = R) \tag{8}$$

$$\tau_w = -\frac{R}{2}\left[CR_e + M^2\right] \tag{9}$$

Using Equation (7) and $\tau_Z(r_0) = \tau_0$ express for τ_0 will lead to the form

$$\tau_0 = -\frac{r_0}{2}\left[CR_e + M^2\right] \tag{10}$$

In between τ_0 and τ_w there may arises two cases wall shear stress is greater and that yield stress. In case $\tau_0\rangle\ \tau_w$ that is if $r_0\rangle R$ then there will occur no flow accordingly velocity function will become

$$u_z = \frac{r^2}{4}\left[-\frac{C}{\mu}R_e - \frac{M^2}{\mu}\right]$$

Also Bingham equation may be described in the following form

$$e^0 = f\left(\tau_{rz}\right) = \frac{1}{\mu}(\tau_{rz} - \tau_0),\ \ \tau_{rz} \geq \tau_0 \tag{11}$$

$$e^0 = \frac{1}{\mu}\left[-\frac{r_0}{2}\left(CR_e + M^2\right) + \frac{r_0}{2}\left(CR_e + M^2\right)\right],\ \ r = r_0$$

$$= 0,\ \ \tau_{rz} \leq \tau_0 \tag{12}$$

In the above, vanishing of strain rate that is e^0

$$u_z = \text{constant} = u_0,\ \text{When}\ \tau = \tau_0 \tag{13}$$

Where u_0 is the core velocity at $r = r_0$ (core radius). As such for blood flow when $r_0 < R$ there arises two region $0 \leq r \leq r_0$ and $r_0 \leq r \leq R$ it is clear for region between 0 and r_0 equation representing the flow is

$$\frac{du_Z}{dr} = 0\ \ 0 \leq r \leq r_0 \tag{14}$$

Which after integration give rise to the form $u_z = u_0$, $0 \leq r \leq r_0$ indicating the velocity profile will become flat in the region and for $r_0 \leq r \leq R$ velocity u_z will show deviation from flat profile and Bingham equation (11) has to be applied for this domain of blood flow the same equation it is easily seen that

$$\frac{du_Z}{dr} = \frac{d}{dr}\left[\frac{r^2}{4\mu}\left(-CR_e - M^2\right)\right]$$

$$\frac{du_Z}{dr} = \left[\frac{r_0 - r}{2\mu}\left(CR_e + M^2\right)\right];\ \ r_0 \leq r \leq R \tag{15}$$

The velocity slip condition at vessels wall is

$$u_Z = u_S r = R \tag{16}$$

Where u_s is the constant slip velocity at tube wall in axial distance. As a result of integration between r and R we have

$$\int_r^R \frac{du_Z}{dr}dr = \int_r^R \frac{r_0 - r}{2\mu}\left(CR_e + M^2\right)dr$$

$$u_z = u_s + \frac{\left(CR_e + M^2\right)(R - r)}{4\mu}\left[(R + r) - 2r_0\right];\ \ r_0 \leq r \leq R \tag{17}$$

At $r_0 = r$ expression for core velocity can be obtained from equation (17)

$$u_z = u_s + \frac{\left(CR_e + M^2\right)(R - r)}{4\mu}\left[(R - r_0)\right] \tag{18}$$

And for all values of r between 0 and r_0 velocity function is

$$u_o = u_s; \quad 0 \leq r \leq r_0 \tag{19}$$

Thus from above expression and consideration velocity distribution u_z can be re-written in the following manners

$$u_z = \begin{cases} u_z(r), \quad r_0 \leq r \leq R \\ u_0(r), 0 \leq r \leq r_0 \\ 0, r\rangle R \end{cases} \tag{20}$$

Where $u_z(r)$ and u_0 are given in equation (17) and (18) respectively. The rate of volume flow can be found from

$$Q = \int_{r=0}^{R} 2\pi r u_z dr$$

By integration after using the equation (16), (17) and (19)

$$Q = 2\pi \int_0^r r u_0 dr + 2\pi \int_{r_0}^{R} r u_z dr$$
$$Q = \pi R^2 u_s + \frac{\pi R^4}{8\mu}\left\{\left(CR_e + M^2\right) - \frac{4}{3}\left(-\frac{2\tau_0}{R}\right) + \frac{1}{3}\left(\frac{-2\tau_0}{R}\right)^4 \left(CR_e + M^2\right)^{-3}\right\} \tag{21}$$

And expression for apparent viscosity μ_a can be found from the formula

$$\mu = \frac{\pi\left(CR_e + M^2\right) R^4}{8\left(\pi R^2 u_s + \frac{\pi R^4}{8\mu}\left\{\left(CR_e + M^2\right) - \frac{4}{3}\left(-\frac{2\tau_0}{R}\right) + \frac{1}{3}\left(\frac{-2\tau_0}{R}\right)^4 \left(CR_e + M^2\right)^{-3}\right\}\right)} \tag{22}$$

And using equation (22), apparent viscosity take the following

$$\mu_a = \left[p(\alpha)\frac{8u_s\mu}{\left(CR_e + M^2\right) R^2}\right]^{-1}, \quad P(a) = \left[1 - \frac{4\left(-\frac{2\tau_0}{R}\right)\left(CR_e + M^2\right)^{-1}}{3} + \frac{1}{3}\left(\left(-\frac{2\tau_0}{R}\right)\left(CR_e + M^2\right)^{-1}\right)^4\right]$$

The parabolic velocity profile for poiseuille flow the takes the form

$$u_z(r) = \frac{\left(CR_e + M^2\right)\left(R^2 - r^2\right)}{4\mu}; \quad 0 \leq r \leq R \tag{23}$$

Employing an axial velocity slip at the tube wall, instead of usual no slip in velocity along the wall the velocity function for poiseuille flow will takes the form

$$u_z(r) = u_s + \frac{\left(CR_e + M^2\right)\left(R^2 - r^2\right)}{4\mu}; \quad 0 \leq r \leq R \tag{24}$$

In the aforesaid cases velocity is maximum at the axis of the tube and expression for maximum velocity obtained from equation (23) and (24) are given by

$$u_{m1} = \frac{\left(CR_e + M^2\right) R^2}{4\mu} \tag{25}$$
$$u_{m1} = u_s + \frac{\left(CR_e + M^2\right) R^2}{4\mu} \tag{26}$$

Expression for rate of volume flow Q can be accordingly obtained for above two cases in the form

$$Q_1 = \frac{\left(CR_e + M^2\right)\pi R^4}{8\mu} \tag{27}$$

$$Q_2 = \pi R^4 u_s + Q_1 \tag{28}$$

Apparent Viscosity: Apparent viscosity μ_a of blood is calculated from equation

$$\mu_a = \left[p(\alpha)\frac{8u_s\mu}{\left(CR_e + M^2\right)R^2}\right]^{-1}, \quad P(a) = \left[1 - \frac{4\left(-\frac{2\tau_0}{R}\right)\left(CR_e + M^2\right)^{-1}}{3} + \frac{1}{3}\left(\left(-\frac{2\tau_0}{R}\right)\left(CR_e + M^2\right)^{-1}\right)^4\right]$$

and variation of μ_a with yields stress for the cases of velocity and no-slip at vessel wall in three different location of CVS is include in the Table 2. The following observation can be seen from the table

(1). As expected, μ_a decreases with an inclusion of velocity slip at tube wall in all three arteries.

(2). t_0 increases, μ_a decreases for both slip and no-slip cases in velocity at an artery wall.

(3). As artery radius from larger artery to smaller one values of, μ_a are found to increases for both slip and non-slip cases. However as tube radius decreases from coronary blood vessel to a smaller tube arteriole, μ_a decreases in the case of slip.

(4). Values of, μ_a are attained the greatest arteriole when $t_0 = 0$ with no-slip and the least also in arteriole when $t_0 = 0$ with no slip.

(5). As μ_a increases as tube radius decreases, therefore apparent viscosity exhibits Inverse Fahraeus-Lindquist effect. Also as tube radius decreases μ_a decreases so μ_a shows Fahraeus-Lindquist. Thus apparent viscosity shows Fahraeus-Lindquist effect and its inversion inverse Fahraeus-Lindquist effect both. Thus Table 1 shows the variation of Apparent Viscosity at three different location.

Carotid		Coronary		Arteriole	
With no slip (u_s=0.0cm/sec)	With slip u_s=0.1cm/sec	With no slip (u_s=0.0cm/sec)	With slip u_s=0.1cm/sec	With no slip (u_s=0.0cm/sec)	With slip u_s=0.1cm/sec
2.0000	1.0000	2.0000	1.3255	2.0143	0.0315066
1.9389	.9870	1.9899	1.3210	1.9447	0.031491
1.9750	.9677	1.9749	1.3144	1.8541	0.031566

Table 1. **Variation of Apparent viscosity μ_a at three diffrerent location of CVS**

S. No.	Name of an artery	Radium $(R^*) \times 10^{-2}$m	Pressure gradient $(C^*) \times 10$kg. m^{-2}.s^{-2}	r^0/R^{**}	
				$\tau_0 = 0.00$	$\tau_0 = 0.10$
01	Carotid	0.40	10.00	0.0000	0.0500
02	Coronary	0.15	139.74	0.0000	0.0095
03	Arteriole	0.008	400.00	0.0000	0.0625

Table 2. **Data for three different locations in Cardiovascular System (CVS)**

3. Conclusion

In this paper we have attempted to study the behavior of poiseuille flow of Bingham plastic fluid model for blood flow with velocity in presence of magnetic effect. A steady one-dimensional flow of blood (-a Bingham fluid) subject to the boundary conditions of velocity slip, suggested in the models of [15, 16] Chaturani and Biswas [4] and, Prahlad and Schultz [21], for three different locations of CVS, in presence of magnetic effect is investigated. Table 2 shows the data for three different location in cvs. Analytic expressions for velocity, flow rate, shear stress at wall, yield stress and apparent viscosity are presented. Axial velocity appears to be a function of pressure gradient C, radial coordinate r, tube semi-diameter R, critical radius r_0 (or yield stress τ_0), Bingham fluid viscosity μ_a and u_s axial velocity slip at the boundary.

Important observations of the present model include the following:

If shear stress τ_{rz} at a distance is not higher than a finite yield stress, blood will not flow. If shear stress is not lower than its yield value, blood flow will be possible. The present model includes Poiseuille flow models with velocity slip and zero ship at vessel wall and steady one-dimensional Bingham plastic fluid model with zero wall slip, as its special cases. Flow variables indicate distinct behavior for vanishing and non-vanishing yield stress.

Flow variables indicate distinct behavior for different Hartmann number at different yield stress. Velocity profiles indicate a parabolic profile in all arteries and for slip and no-slip cases with the usual maximum magnitude at tube axis and a minimum velocity at the boundary in case of vanishing yield stress. These blunted for flat profiles in velocity ($\tau_0 > 0$) clearly exposes the non-Newtonian nature of blood. Assumption that velocity variation in axial direction is negligible as compared to its variation in radial direction, may lead to the implication that the length of the artery is too large as compared to the radius.

Velocity profile increases when Hartmann number M increases in different fluid parameter viz., yield stress τ_0 (≥ 0). The nature of velocity profile is also same in no slip. Velocity profile for a full scale of dimensionless radial co-ordinate $\frac{r}{R}$ from the tube axis to vessel will clearly state that in Figure 1 to Figure 6

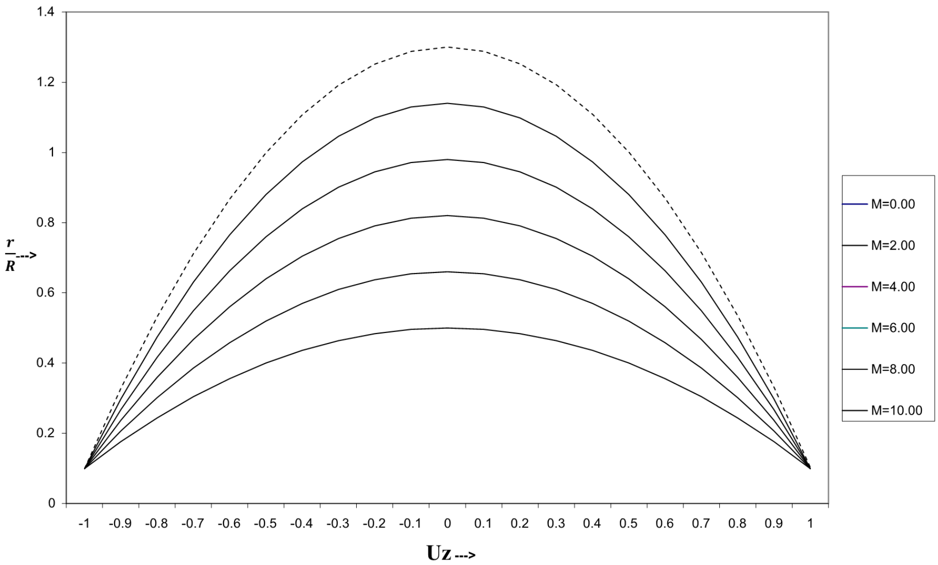

Figure 1. **Variation of velocity profiles U_z with Hartmann number M at Carotid when $\tau_0 = 0$**

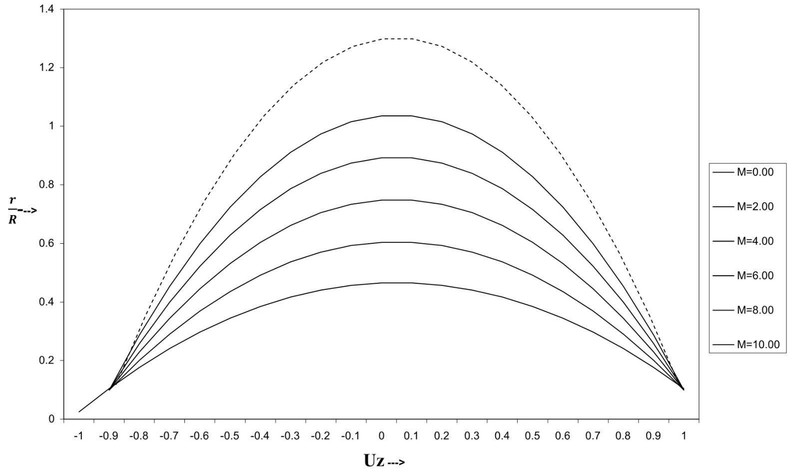

Figure 2. **Variation of velocity profiles U_z with Hartmann number M at Carotid when $\tau_0 = 0.10$**

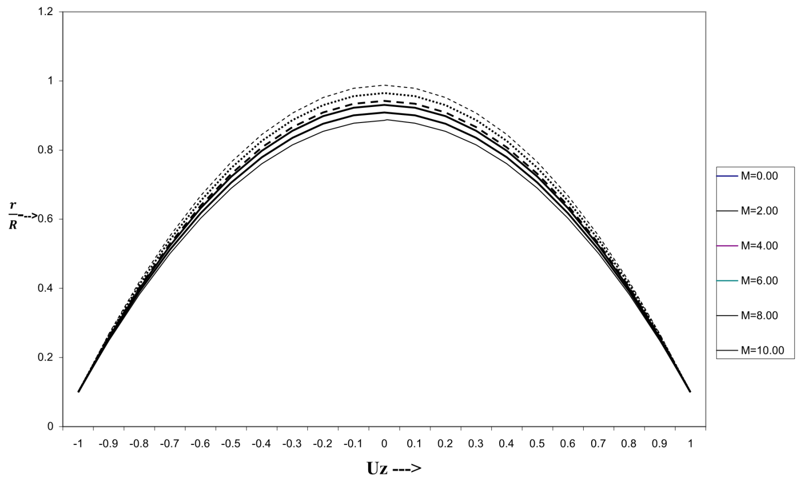

Figure 3. **Variation of velocity profiles U_z at Hartmann number M at Coronary when $\tau_0 = 0$**

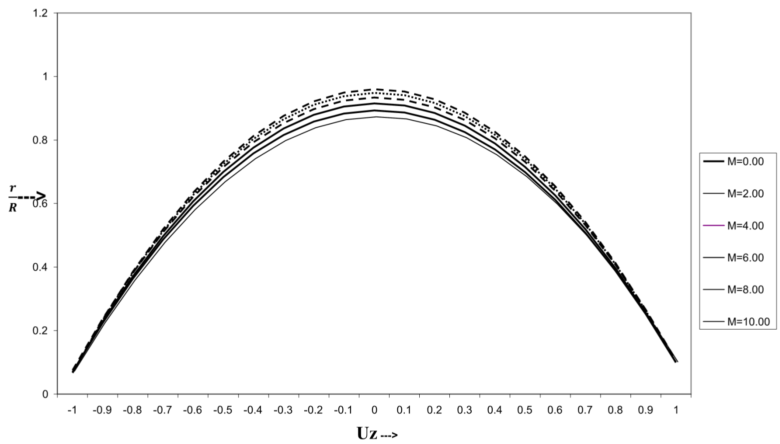

Figure 4. **Variation of velocity profiles U_z with Hartmann number M at coronary when $\tau_0 = 0.10$**

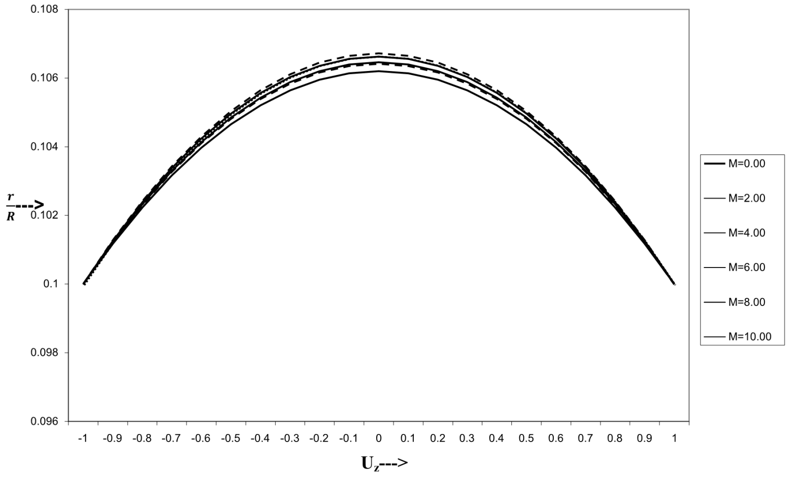

Figure 5. **Variation of velocity profiles U_z with Hartmann number M at Arteriole when $\tau_0 = 0.00$**

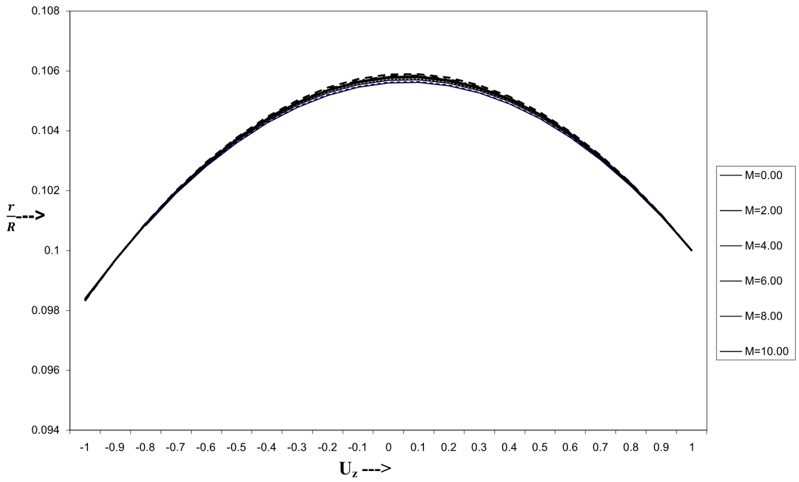

Figure 6. **Variation of velocity profiles U_z with Hartmann number M at Arteriole when $\tau_0 = 0.10$**

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