International Journal of Mathematics And its Applications
Volume 5, Issue 4–D (2017), 499–504.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

A Subclass of Starlike Functions Defined with a Differential Operator

Research Article

Jitendra Awasthi[1*]

1 Department of Mathematics, S.J.N.P.G.College, Lucknow, Uttar Pradesh, India.

Abstract: This paper deals with a new subclass of starlike functions $A^*_{n,\lambda}(p,\mu,\,\beta,\gamma,\delta)$. Coefficients inequality, distortion theorems and closure theorems have been obtained for this class. Further radii of starlikeness and convexity are also obtained for this class.

MSC: 30C45.

Keywords: Analytic functions, univalent, al-oboudi differential operator, extremal functions, Starlikeness, convexity.

1. Introduction

Let A_p denote the class of functions of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} (p \geq 1), \tag{1}$$

which are analytic and univalent in the unit disk $\Delta = \{z : |z| < 1\}$. Al-Oboudi [1] had introduced a differential operator D^n_λ for a function $f(z) \in A_p$ as

$$\begin{aligned} D^0_\lambda f(z) &= f(z), \qquad (2)\\ D^1_\lambda f(z) &= D_\lambda f(z) = (1-\lambda) f(z) + \lambda f'(z),\\ &\ldots\ldots\\ D^n_\lambda f(z) &= D_\lambda (D^{n-1}_\lambda f(z)), \end{aligned}$$

for $n \in N = \{1, 2, 3, ...\}$ and $\lambda \geq 0$. It is easy to see that

$$D^n_\lambda f(z) = z + \sum_{k=1}^{\infty} [1 + (p+k-1)\lambda]^n a_{p+k} z^{p+k},\ (p \geq 1). \tag{3}$$

With the help of differential operator D^n_λ, we define a subclass $A_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$ for $f(z) \in A_p$ such that

$$\left| \frac{\frac{D^{n+1}_\lambda f(z)}{D^n_\lambda f(z)} - \delta}{\mu \frac{D^{n+1}_\lambda f(z)}{D^n_\lambda f(z)} + (1-\gamma)} \right| < \beta,\ (z \in \Delta), \tag{4}$$

* E-mail: drjitendraawasthi@gmail.com

where $0 \leq \mu \leq 1$, $0 < \beta \leq 1$, $0 \leq \gamma \leq 1$, $0 \leq \delta \leq 1$, $\lambda \geq 0$ and $n \geq 0$. In particular, the class $A_{n,1}(1,\mu,\beta,\gamma,1)$, $A_{0,1}(1,\mu,\beta,\gamma,1)$, $A_{0,1}(1,1,\beta,0,1)$ and $A_{0,1}(1,0,1,0,1)$ are studied by Aouf and et. [2] Lee and et. [4] Padmanadban [5] and Singh [6] respectively. Let T_p denote the subclass of A_p whose elements can be expressed in the form

$$f(z) = z - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} (a_{p+k} \geq 0,\ p \geq 1). \tag{5}$$

We denote $A_{n,\lambda}(p,\mu,\beta,\gamma,\delta) \cap T_p = A^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$. Taking $p = 1$, $\delta = 1$, the class $A^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$ reduces to $S^*_{n,\lambda}(\mu,\beta,\gamma)$. Which was defined and studied by Hossen [3]. The object of this paper is to derive several properties of the class $A^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$ such as coefficients inequality, distortion theorem and closure theorems.

2. Coefficient Inequalities

Theorem 2.1. *A function $f(z)$ of the form (5) is in the class $A^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$ if and only if*

$$\sum_{k=1}^{\infty} [\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^n a_{p+k} \leq \{\beta(1-\gamma+\mu)+1-\delta\}. \tag{6}$$

The result is sharp. The extremal function function is given by

$$f(z) = z - \frac{\{\beta(1-\gamma+\mu)+1-\delta\}}{[\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^n} z^{p+k}. \tag{7}$$

Proof. First let $f(z) \in A^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$, then from (4) we have

$$\left| \frac{\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} - \delta}{\mu \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} + (1-\gamma)} \right| = \left| \frac{\{z - \sum_{k=1}^{\infty}[1+(p+k-1)\lambda]^{n+1} a_{p+k} z^{p+k}\} - \delta\{z - \sum_{k=1}^{\infty}[1+(p+k-1)\lambda]^n a_{p+k} z^{p+k}\}}{\mu\{z - \sum_{k=1}^{\infty}[1+(p+k-1)\lambda]^{n+1} a_{p+k} z^{p+k}\} + (1-\gamma)\{z - \sum_{k=1}^{\infty}[1+(p+k-1)\lambda]^n a_{p+k} z^{p+k}\}} \right| < \beta,$$

or

$$\left| \frac{(1-\delta)z - \sum_{k=1}^{\infty}[1+(p+k-1)\lambda-\delta][1+(p+k-1)\lambda]^n a_{p+k} z^{p+k}}{(1-\gamma+\mu)z - \sum_{k=1}^{\infty}[\mu\{1+(p+k-1)\lambda\}+1-\gamma][1+(p+k-1)\lambda]^n a_{p+k} z^{p+k}} \right| < \beta. \tag{8}$$

Since $Re(z) \leq |z|$ for all z, we find from (8) that

$$Re\left\{ \frac{\sum_{k=1}^{\infty}[1+(p+k-1)\lambda-\delta][1+(p+k-1)\lambda]^n a_{p+k} z^{p+k} - (1-\delta)z}{(1-\gamma+\mu)z - \sum_{k=1}^{\infty}[\mu\{1+(p+k-1)\lambda\}+1-\gamma][1+(p+k-1)\lambda]^n a_{p+k} z^{p+k}} \right\} < \beta.$$

Choosing values of z on the real axis so that $\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)}$ is real and letting $z \to 1^-$ through real values, we have

$$\sum_{k=1}^{\infty}[1+(p+k-1)\lambda-\delta][1+(p+k-1)\lambda]^n a_{p+k} - (1-\delta) < \beta(1-\gamma+\mu) - \sum_{k=1}^{\infty}\beta[\mu\{1+(p+k-1)\lambda\}+1-\gamma][1+(p+k-1)\lambda]^n a_{p+k},$$

or

$$\sum_{k=1}^{\infty} [\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^n a_{p+k} \leq \{\beta(1-\gamma+\mu)+1-\delta\}.$$

Conversely let inequality (6) holds. Then

$$\begin{aligned}\left|D_{\lambda}^{n+1}f(z)-\delta D_{\lambda}^{n}f(z)\right|-\beta\left|\mu D_{\lambda}^{n+1}f(z)+(1-\gamma)D_{\lambda}^{n}f(z)\right| &= \left|(1-\delta)z-\sum_{k=1}^{\infty}[1+(p+k-1)\lambda-\delta][1+(p+k-1)\lambda]^{n}a_{p+k}z^{p+k}\right|\\ &-\beta\left|(1-\gamma+\mu)z-\sum_{k=1}^{\infty}[\mu\{1+(p+k-1)\lambda\}+1-\gamma][1+(p+k-1)\lambda]^{n}a_{p+k}z^{p+k}\right|\\ &\leq \sum_{k=1}^{\infty}[1+(p+k-1)\lambda-\delta][1+(p+k-1)\lambda]^{n}a_{p+k}-(1-\delta)\\ &-\beta(1-\gamma+\mu)+\sum_{k=1}^{\infty}\beta[\mu\{1+(p+k-1)\lambda\}+1-\gamma][1+(p+k-1)\lambda]^{n}a_{p+k}\\ &\leq \sum_{k=1}^{\infty}[\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^{n}a_{p+k}-\{\beta(1-\gamma+\mu)+1-\delta\}\\ &\leq 0,\end{aligned}$$

by the hypothesis. Hence by the maximum modulus theorem, we have $f(z)\in A^{*}_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$. □

Corollary 2.2. *Let $f(z)\in T_p$ be in the class $A^{*}_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$. Then*

$$a_{p+k}\leq \frac{\{\beta(1-\gamma+\mu)+1-\delta\}}{[\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^{n}}, \tag{9}$$

for $(k\geq 1, p\geq 1)$. Equality in (9) holds for the function $f(z)$ given by (7).

3. Distortion Theorem

Theorem 3.1. *Let $f(z)\in T_p$ be in the class $A^{*}_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$ with $0\leq\mu\leq 1, 0<\beta\leq 1, 0\leq\gamma\leq 1, 0\leq\delta\leq 1, \lambda\geq 0$ and $n\geq 0, p\geq 1$. Then for $|z|=r<1$,*

$$\begin{aligned} r-\frac{\{\beta(1-\gamma+\mu)+1-\delta\}}{[(1+p\lambda)(1+\beta\mu)+\beta(1-\gamma)-\delta](1+p\lambda)^{n}}r^{p+1} &\leq |f(z)|\\ &\leq r+\frac{\{\beta(1-\gamma+\mu)+1-\delta\}}{[(1+p\lambda)(1+\beta\mu)+\beta(1-\gamma)-\delta](1+p\lambda)^{n}}r^{p+1}.\end{aligned} \tag{10}$$

The result are sharp.

Proof. From inequality (6), it follows that

$$\sum_{k=1}^{\infty}[\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^{n}a_{p+k}\leq\{\beta(1-\gamma+\mu)+1-\delta\}.$$

This implies that

$$\sum_{k=1}^{\infty}a_{p+k}\leq\frac{\{\beta(1-\gamma+\mu)+1-\delta\}}{[(1+p\lambda)(1+\beta\mu)+\beta(1-\gamma)-\delta](1+p\lambda)^{n}}. \tag{11}$$

Consequently, for $|z|=r<1$, we obtain

$$|f(z)|\leq r+r^{p+1}\sum_{k=1}^{\infty}a_{p+k},$$

or

$$|f(z)|\leq r+\frac{\{\beta(1-\gamma+\mu)+1-\delta\}}{[(1+p\lambda)(1+\beta\mu)+\beta(1-\gamma)-\delta](1+p\lambda)^{n}}\ r^{p+1}, \tag{12}$$

and

$$|f(z)|\geq r-r^{p+1}\sum_{k=1}^{\infty}a_{p+k},$$

or

$$|f(z)| \geq r - \frac{\{\beta(1-\gamma+\mu)+1-\delta\}}{[(1+p\lambda)(1+\beta\mu)+\beta(1-\gamma)-\delta](1+p\lambda)^n} r^{p+1}. \tag{13}$$

From (12) and (13) inequality (10) follows. The bounds in (10) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{\{\beta(1-\gamma+\mu)+1-\delta\}}{[(1+p\lambda)(1+\beta\mu)+\beta(1-\gamma)-\delta](1+p\lambda)^n} z^{p+k}. \tag{14}$$

□

4. Closure Theorems

Theorem 4.1. *The class $A^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$ is closed under convex linear combination.*

Proof. Let each of the functions $f_1(z)$ and $f_2(z)$ given by

$$f_j(z) = z - \sum_{k=1}^{\infty} a_{p+k,j} z^{p+k} (a_{p+k,j} \geq 0,\ j = 1, 2.\ p \geq 1), \tag{15}$$

be in the class $A^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$. Then it is sufficient to show that the function $F(z)$ defined by

$$F(z) = tf_1(z) + (1-t)f_2(z)\ (0 \leq t \leq 1), \tag{16}$$

is also in the class $A^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$. Since for $0 \leq t \leq 1$,

$$F(z) = z - \sum_{k=1}^{\infty} [ta_{p+k,1} + (1-t)a_{p+k,2}] z^{p+k}.$$

Then with the aid of Theorem 2.1, we have

$$\sum_{k=1}^{\infty} [\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^n [ta_{p+k,1}+(1-t)a_{p+k,2}] \leq \{\beta(1-\gamma+\mu)+1-\delta\}.$$

Which implies that $F(z) \in A^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$. □

Theorem 4.2. *Let*

$$f_0(z) = z, \quad and$$
$$f_k(z) = z - \frac{\{\beta(1-\gamma+\mu)+1-\delta\}}{[\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^n} z^{p+k}.$$

Then $f(z) \in A^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} t_k f_k(z),\ where \tag{17}$$

$$t_k \geq 0\ (k \geq 1) \quad and \quad \sum_{k=0}^{\infty} t_k = 1. \tag{18}$$

Proof. First let $f(z)$ can be expressed in the form (17). Then

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} t_k f_k(z) \\ &= z - \sum_{k=1}^{\infty} \frac{\{\beta(1-\gamma+\mu)+1-\delta\}}{[\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^n} t_k z^{p+k} \end{aligned}$$

Then, it follows that

$$\sum_{k=1}^{\infty}[\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^{n}$$
$$\times\frac{\{\beta(1-\gamma+\mu)+1-\delta\}}{[\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^{n}}t_k$$
$$=\{\beta(1-\gamma+\mu)+1-\delta\}\sum_{k=1}^{\infty}t_k=\{\beta(1-\gamma+\mu)+1-\delta\}(1-t_0)$$
$$\leq\{\beta(1-\gamma+\mu)+1-\delta\}.$$

Therefore, by Theorem 2.1, $f(z)\in A^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$.

Conversely, let the function $f(z)\in T_p$ is in the class $A^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$, then we have

$$a_{p+k}\leq\frac{\{\beta(1-\gamma+\mu)+1-\delta\}}{[\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^{n}},(k\geq1,p\geq1),$$

Setting

$$t_k=\frac{[\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^{n}}{\{\beta(1-\gamma+\mu)+1-\delta\}}a_{p+k},\quad\text{and}$$
$$t_0=1-\sum_{k=1}^{\infty}t_k.$$

It follows that

$$f(z)=\sum_{k=0}^{\infty}t_kf_k(z)$$

This complete the proof. □

5. Radius of Starlikeness

Theorem 5.1. *If $f(z)\in A^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$ then the function $f(z)$ is starlike in the disk $0<|z|<r=r^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$, where*

$$r=\inf\left[\frac{[\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^{n}}{\{\beta(1-\gamma+\mu)+1-\delta\}(p+k)}\right]^{\frac{1}{p+k-1}},\tag{19}$$

for $k\geq1,\ p\geq1$.

Proof. It is enough to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1\ for\ |z|<1,$$

or

$$\left|\frac{zf'(z)}{f(z)}-1\right|=\left|\frac{-\sum_{k=1}^{\infty}(p+k-1)a_{p+k}z^{p+k}}{z-\sum_{k=1}^{\infty}a_{p+k}z^{p+k}}\right|<1,$$

or

$$\sum_{k=1}^{\infty}(p+k-1)a_{p+k}z^{p+k-1}<1-\sum_{k=1}^{\infty}a_{p+k}z^{p+k-1},$$

or

$$\sum_{k=1}^{\infty}(p+k)a_{p+k}z^{p+k-1}<1.$$

It is easily to see that (19) holds if

$$|z|^{p+k-1}<\left[\frac{[\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^{n}}{\{\beta(1-\gamma+\mu)+1-\delta\}(p+k)}\right].$$

This complete the proof. □

6. Radius of Convexity

Theorem 6.1. *If $f(z) \in A^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$ then the function $f(z)$ is convex in the disk $0 < |z| < r = r^*_{n,\lambda}(p,\mu,\beta,\gamma,\delta)$, where*

$$r = \inf\left[\frac{[\{1+(p+k-1)\lambda\}(1+\beta\mu)+\beta(1-\gamma)-\delta]\{1+(p+k-1)\lambda\}^n}{\{\beta(1-\gamma+\mu)+1-\delta\}(p+k)^2}\right]^{\frac{1}{p+k-1}}, \tag{20}$$

for $k \geq 1$, $p \geq 1$.

Proof. Upon noting the fact that $f(z)$ is convex if and only if $zf\prime(z)$ is starlike, the Theorem 6.1 follows. □

References

[1] F.M.Al-Oboudi, *On univalent functions defined by a generalized Salagean operator*, Int. J. Math. Sci., 27(2004), 1429-1436.

[2] M.K.Aouf, H.M.Hossen and A.Y.Lashin, *Certain class of prestarlike functions defined by Salagean operator*, South. Asian Bull. of Math. Math., 29(2005), 1029-1044.

[3] H.M.Hossen, *Certain class of prestarlike functions defined by Al-Oboudi differential operator*, ARPN Journal of Science and Technology, 5(12)(2015).

[4] S.H.Lee, Y.C.Kim and N.E.Cho, *A subclass of analytic functions with negative coefficients*, Math. Japon., 4(1989), 597-605.

[5] K.S.Padmanabhan, *On certain classes of starlike functions in the unit disc*, J. Indian Math. Soc., 32(1968), 89-103.

[6] R.Singh, *On a class of starlike functions*, J. Indian Math. Soc., 32(1968), 207-213.