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A Comparison Between Two Approaches to Solve Functional Differential Equations: DTM and DJM

Research Article

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Abstract: Differential Transform Method (DTM) is a Taylor series method for finding Adomian Polynomials along with the method of steps which is used to solve many linear and non-linear differential equations or systems of physical significance. Whereas, the Daftardar-Gejji andJafari Method (DJM) is a method developed by Varsha Daftardar-Gejji and H. Jafari [17] to solve non-linear Integral equations, Fractional differential equations, system of ODEs and so on. Montri Thongmoon and Sasitorn Pusjuso [13] have compared DTM and Laplace transform method to solve system of differential equations. Josef Rebenda et al. [2, 10, 11] have recently shown that DTM can be applied to functional differential equations with delay, whereas, Sachin Bhalekar and Jayvant Patade [16] have recently obtained solutions of system of delay differential equations with proportional delay using DJM in the form of a special function and studied it's properties. We compare these two iterative methods to solve differential equations. We further try to explore the possibility of these two numerical methods on delayed, hybrid dynamical and delayed hybrid models.

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Keywords: Functional Differential Equations, Differential Transform Method (DTM), delayed hybrid models, Daftardar-Gejji and Jafari Method(DJM).

1. Introduction

Hybrid dynamical systems have immense potential as models in many applications in computer science and system control. They have numerous applications in different scientific fields such as computer science and control systems, protocol and in-general software verification, air traffic control, process control, Intelligent Vehicle Highway Systems (IVHS). Functional and fractional differential equations play a central role in understanding these systems. Moreover, delay analysis of these systems is a domain which is still not fully understood. We explore the possibility of solving these systems using Differential Transformation (DTM) [2] and Daftardar-Gejjiand Jafari Method (DJM) [17] which are two widely used iterative techniques to solve different types of differential equations and systems of functional differential equations. In this paper, we review and compare these two techniques and present our understanding.

1.1. Some Preliminaries

The field of Differential equations in general and functional differential equations in particular forms a very rich pool of problems in pure as well as applied mathematics, with applications in ample of domains. The ever increasing demand of closed form and numerically feasible solutions has motivated many researchers to explore various numerical methods and test them for their accuracy and easiness of implementation. Two such recent methods are

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(1). The Differential Transform Method (DTM).

(2). The Daftardar-Gejji and Jafari Method (DJM).

To the best of our knowledge and to our surprise, there is no comparative study of these two of the most well known iterative techniques in literature. This forms the motivation of our study.

2. The Differential Transform Method

Definition 2.1. *We consider a system of p functional differential equations of n-th order with multiple delays* $\alpha_i(t)$, $1 \le i \le r$ *as discussed by Josef Rebenda et al. [2, 10, 11]:*

$$\vec{\mathbf{u}}(t)^n = \overline{f}\left[t,\, \vec{\mathbf{u}}(t),\, \vec{\mathbf{u}}'(t),\, \vec{\mathbf{u}}''(t),\, \vec{\mathbf{u}}'''(t),\, \ldots,\, \vec{\mathbf{u}}^{n-1}(t),\, \vec{\mathbf{u}}_1(\alpha_1(t)),\, \vec{\mathbf{u}}_2(\alpha_2(t)), \ldots\, \vec{\mathbf{u}}_r(\alpha_r(t))\right] \tag{1}$$

where

$$\left.\begin{aligned} &\vec{\mathbf{u}}(t)^n = \left[\vec{\mathbf{u}}_1^n(t),\, \vec{\mathbf{u}}_2^n(t), \ldots\, \vec{\mathbf{u}}_p^n(t)\right]^T, \\ &\vec{\mathbf{u}}(t)^k = \left[\vec{\mathbf{u}}_1^k(t),\, \vec{\mathbf{u}}_2^k(t), \ldots\, \vec{\mathbf{u}}_p^k(t)\right]^T, \\ &k = 0,\, 1,\, 2,\, \ldots,\, n-1 \end{aligned}\right\}$$

$$\vec{\mathbf{f}}(t) = (f_1, f_2, f_3, \ldots.., f_p)^T$$

are p-dimensional vector functions and $u_i(\alpha_i(t)) = \left(\vec{\mathbf{u}}(\alpha_i(t)), \vec{\mathbf{u}}'(\alpha_i(t)), \vec{\mathbf{u}}''(\alpha_i(t)), \ldots, \vec{\mathbf{u}}^m(\alpha_i(t))\right)$ *are* $m_i \cdot p$ *-dimensional vector functions, with* $m_i \le n$, $i = 1, 2, 3, \ldots, r$, $r \in N$ *and* $f_j : [0, \infty) \to \mathbb{R}^{n_p} \times \mathbb{R}^{w_p}$ *are continuous real functions for* $j = 1, 2, 3, \ldots, p$, *where* $w = \sum_{i=0}^{r} m_i$.

The following three types of delays are considered

$$\left.\begin{aligned} \alpha_i &= t - \tau_i, \tau > 0 \\ \alpha_i(t) &= q_i t, q_i \in (0, 1) \\ \alpha_i(t) &= t - \tau_i(t), \tau_i(t) > 0 \end{aligned}\right\} \tag{2}$$

where, τ_i is real constant giving constant delay, $\tau_i(t)$ is a real function giving time dependent or time varying delay and q_i is a proportional delay. The differential transformation of the k-th derivative of function $u(t)$ is given by $U(t) = \frac{1}{k!}\frac{d^k u(t)}{dt^k}$ at $t = 0$, where $u(t)$ gives the original function and $U(k)$ is the transformation function. The inverse differential transformation of $U(k)$ is defined as follows: $u(t) = \sum_{k=0}^{\infty} U(k)(t - t_0)^k$. Hence $u(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!}\frac{d^k u(t)}{dt^k}$ at $t = 0$. Thus, DTM is does not evaluate the symbolic derivatives but relative derivatives are calculated by an iterative process. For more detail on DTM, one may refer the works of Josef Rebenda et al. [2, 10, 11] .

3. The Daftardar-Gejji and Jafari Method

Varsha Daftardar-Gejji and Jafari [17] in their work have proposed an iterative method DJM to solve illustrations on nonlinear functional equations, systems of differential equationsand nonlinear fractional differential equations. They have combined this method with Mathematica and obtained nearly accurate results. A brief outline of DJM is as follows. They have obtained solution of following general functional equation:

$$y = N(y) + f. \tag{3}$$

where, $N : B \to B$ is a nonlinear operator from a Banach Space B and f is a known function. Where as, Sachin Bhalekar et al. [16] have extended their work on functional differential equations to systems with proportional delay by adding an extra linearity operator L.

$$y = L(y) + N(y) + f. \tag{4}$$

The solution of equations (3) and (4) have the form $y = \sum_{i=0}^{\infty} y_i$, where $y_0 = f$, $y_1 = N(y_0)$ along with the following equation (5) holds true:

$$y_{n+1} = N(\sum_{k=0}^{n} y_k) - N(\sum_{k=0}^{n-1} y_k),\ n = 1,\ 2,\ 3,\ \ldots. \tag{5}$$

Both Bhalekar et al. [16] and Daftardar et al. [17] have used the j^{th} term approximate solution for finding exact solution given by

$$y = \sum_{k=0}^{j-1} y_k \tag{6}$$

They have also justified the absolute and uniform convergence of the series solution

$$y = \sum_{k=0}^{\infty} y_k \tag{7}$$

Sachin Bhalekar et al. [16] have applied DJM and obtained series solutions of following linear delay differential equation with proportional delay

$$\left.\begin{aligned} \dot{y}(t) = ay(t) + by(qt), \\ where,\ q \in (0,\ 1),\ y(0) = 1,\ a \in \mathbb{R},\ b \in \mathbb{R}. \end{aligned}\right\} \tag{8}$$

The series solutions thus obtained were shown to be analytical on $\mathbb{R}$. For further reading on DJM one may refer [16] and [17].

4. A Comparison Between DJM and DTM

Josef Rebenda et al. [2, 10, 11] extensively applied the Differential Transform Method (DTM) to different types of delay involving problems in multidimensional partial differential equations and logistic equations. Whereas Xieet. al [14] applied and improved differential transform method to Singular Boundary value problems (SBVPs). These SBVPs, arise frequently in may physical problems related to physics and engineering, such as astrophysics, boundary layer theory, three layer beams, electromagnetic waves or gravity driven flows etc. Daftardar and Bhalekar have applied Daftar Gejji and Jafari Method on Partial differential equations, fractional differential equations, functional differential equations, system of functional differential equations and nonlinear ordinary differential equations [16, 17]. We compare these two methods using some illustrations.

4.1. Illustrations

Some model system of differential equations discussed in [13] are as follows:

Problem 4.1. *Solve for $x(t)$ and $y(t)$*

$$\left.\begin{aligned} &\dot{x} + \dot{y} + x(t) + y(t) = 1, \\ &\dot{y} = 2x(t) + y(t) \\ &with\ initial\ conditions\ x(0) = 0,\ y(0) = 1,\ where\ dotx = \tfrac{dx}{dt}\ and\ \dot{y} = \tfrac{dy}{dt}. \end{aligned}\right\} \tag{9}$$

The exact solutions to these systems using Mathematica, which are same as obtained in [13] using laplace transforms are

$$\left.\begin{aligned} x(t) &= e^{-t} - 1, \\ y(t) &= 2 - e^{-t} \end{aligned}\right\} \tag{10}$$

The solutions of the system (9) using Differential transform method are

$$\left.\begin{aligned} x(t) &= -t + t^2 - \tfrac{5}{6}t^3 + \tfrac{5}{8}t^4 - \tfrac{7}{120}t^5 - \tfrac{19}{144}t^6, \\ y(t) &= 1 + t - \tfrac{1}{2}t^2 + \tfrac{1}{2}t^3 - \tfrac{7}{24}t^4 + \tfrac{23}{120}t^5 + \tfrac{1}{80}t^6 \end{aligned}\right\} \tag{11}$$

The solutions of the system (9) in the form of integral curves obtained using Differential transform method (till some finite terms) and are compared with exact solutions. The comparison of solutions in the form of integral curves is given by Figure 1 below.

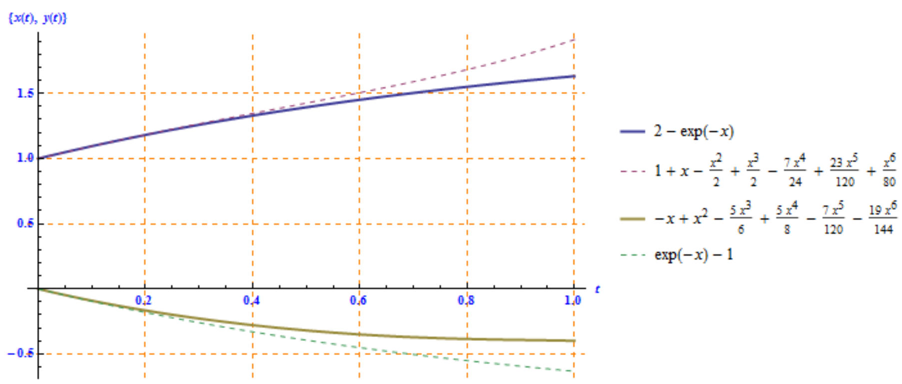

Figure 1. **Integral Curves: Problem 4.1**

Problem 4.2. *Solve for $x(t), y(t)$ and $z(t)$*

$$\left.\begin{aligned} &\dot{x} = z(t) - cos(t), \\ &\dot{y} = z(t) - e^t \\ &\dot{z} = x(t) - y(t) \\ &\textit{with initial conditions } x(0) = 1,\ y(0) = 1,\ z(0) = 2, \textit{ where } \dot{x} = \tfrac{dx}{dt} \textit{ and } \dot{y} = \tfrac{dy}{dt}. \end{aligned}\right\} \tag{12}$$

The exact solutions to these systems using Mathematica, which are same as obtained in [13] using laplace transforms are

$$\left.\begin{aligned} x(t) &= e^t, \\ y(t) &= sin(t), \\ z(t) &= e^t + cos(t) \end{aligned}\right\} \tag{13}$$

The solutions of the system (12) using Differential transform method are

$$\left.\begin{aligned} x(t) &= 1 + t + \tfrac{1}{2}t^2 + \tfrac{1}{6}t^3 + \tfrac{1}{24}t^4 + \tfrac{1}{120}t^5 + \tfrac{1}{720}t^6, \\ y(t) &= t - \tfrac{1}{6}t^3 + \tfrac{1}{120}t^5 \\ z(t) &= 2 + t + \tfrac{1}{6}t^3 + \tfrac{1}{12}t^4 + \tfrac{1}{120}t^5 \end{aligned}\right\} \tag{14}$$

The solutions of the system (12) in the form of integral curves obtained using Differential transform method (till some finite terms) and are compared with exact solutions. The comparison of solutions in the form of integral curves is given by Figure 2 below.

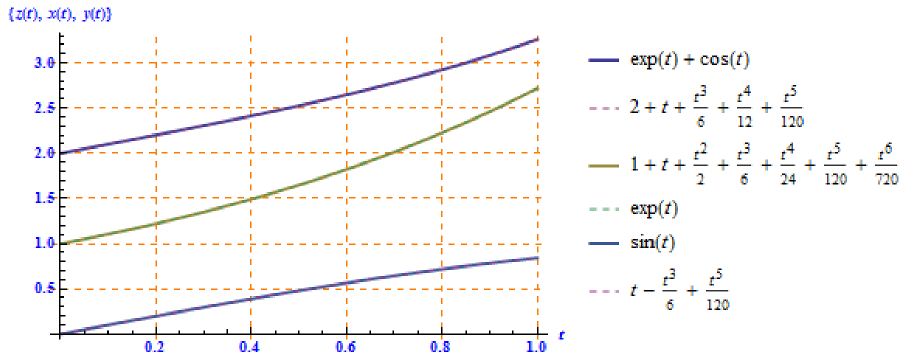

Figure 2. **Integral Curves: Problem 4.2**

Problem 4.3. *Solve for* $x(t)$ *and* $y(t)$

$$\left.\begin{aligned}&\ddot{x} + y(t) = 1,\\&\ddot{y} + x(t) = 0\\&\text{with initial conditions } x(0) = y(0) = \dot{x}(0) = \dot{y}(0) = 0\\&\text{where } \ddot{x} = \tfrac{d^2x}{dt^2} \text{ and } \ddot{y} = \tfrac{d^2y}{dt^2}.\end{aligned}\right\} \tag{15}$$

The exact solutions to these systems using Mathematica, which are same as obtained in [13] using laplace transforms are

$$\left.\begin{aligned}x(t) &= \tfrac{1}{2}t^2 - \tfrac{1}{6}t^3 + \tfrac{1}{12}t^4 + \tfrac{1}{20}t^5 + \tfrac{5}{144}t^6 + \tfrac{121}{5040}t^7,\\ y(t) &= -\tfrac{1}{24}t^4 - \tfrac{1}{120}t^5 + \tfrac{1}{360}t^6 - \tfrac{1}{840}t^7\end{aligned}\right\} \tag{16}$$

The solutions of the system (15) using Differential transform method are

$$\left.\begin{aligned}x(t) &= e^{-t} - 1,\\ y(t) &= 2 - e^{-t}\end{aligned}\right\} \tag{17}$$

The solutions of the system (15) in the form of integral curves obtained using Differential transform method (till some finite terms) and are compared with exact solutions. The comparison of solutions in the form of integral curves is given by Figure 3 below.

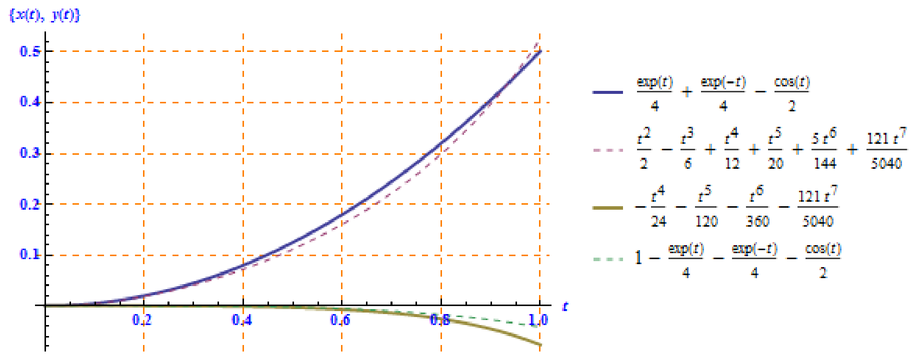

Figure 3. **Integral Curves: Problem 4.3**

5. Solving Delay Differential Equations Using DTM and DJM

We now apply the DJM and DTM to a system of delay differential equation proportional delays. We consider a model problem below:

Problem 5.1. *Solve the following system of differential equations for $y_1(t)$ and $y_2(t)$ using Differential Trasform Method (DTM) and Daftardar-Gejji and Jafari Method (DJM) both:*

$$\left.\begin{array}{l} \dot{y}_1 = y_1(t) + 2y_2(t) + y_1(\frac{t}{2}) + y_2(\frac{t}{2}), \\ \dot{y}_2 = 3y_1(t) + 4y_2(t) + y_2(\frac{t}{2}) \\ \textit{with initial conditions } y_1(0) = 1,\, y_2(0) = 0 \,\textit{where , where } \dot{y_1} = \frac{dy_1}{dt} \textit{ and } \dot{y_2} = \frac{dy_2}{dt}. \end{array}\right\} \qquad (18)$$

The solutions of the system of delayed differential equations (18) obtained by us in the form of integral curves using Differential Transform Method are given by

$$\left.\begin{array}{rcl} y_1(t) & = & 1 + 2t + \frac{21}{4}t^2 + \frac{57}{6}t^3 + \frac{9051}{1536}t^4 + \frac{249423}{7680}t^5, \\ y_2(t) & = & 3t + \frac{39}{4}t^2 + \frac{291}{48}t^3 + \frac{9603}{1536}t^4 + 14.1379t^5 + \end{array}\right\} \qquad (19)$$

The solutions of the system of delayed differential equations (18) obtained by us in the form of integral curves using Daftardar-Gejji and Jafari Method (DJM) are given by

$$\left.\begin{array}{rcl} y_1(t) & = & 1 + 2t + \frac{21}{4}t^2 + \frac{456}{48}t^3 + 5.55125t^4 + 32.1216t^5, \\ y_2(t) & = & 3t + \frac{39}{4}t^2 + \frac{781}{144}t^3 + 12.7144t^4 + 36.52t^5 \end{array}\right\} \qquad (20)$$

We plot these solutions given in (19) and (20) to the system (18) in the form of integral curves using both the methods in the following Figure 4:

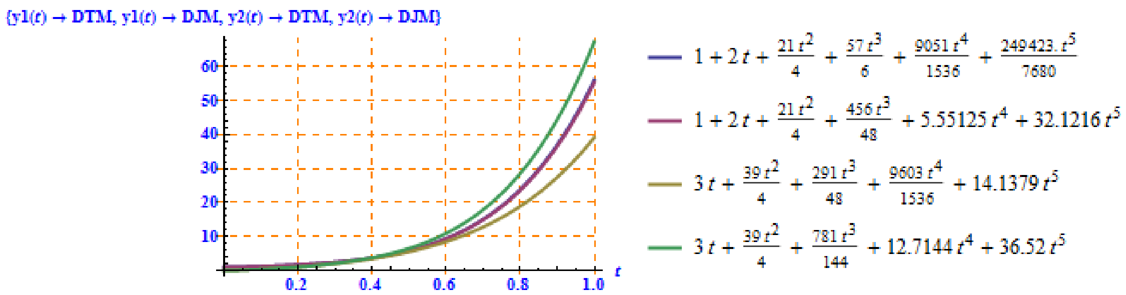

Figure 4. **Integral Curves: Problem 5.1**

6. Conclusion and Future work

We have obtained analytical and numerical solutions for the systems of linear and non-linear differential equations solutions given in [2] and [13] . We have solved the systems in [13] without using delay effects and further try to extend this work with different types of delays as done in [2] using Mathematica. We have obtained the solutions in the form of integral curves. We further have extended this work on delays and obtained some good results. We observe that these two methods are comparable. The figure 5 shows that the solutions $y_1(t)$ of Problem given in 18 exactly coincide where as the solutions $y_2(t)$ obtained by both the methods are comparable. We further intend to explore possible applications of these methods on some other engineering problems with specific interest on hybrid models. This work can also be extended on general Banach algebra of continuous functions on locally compact Abelian groups with Invariant Haar Measure induced and can be applied to integral equations on groups.

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References

[1] V.I.Arnold, *Mathematical Methods of Classical Mechanics*, Springer, (1988).

[2] Josef Rebenda, Zdenek Smarda and Yasir Khan, *A Taylor Method Approach for Solving of Nonlinear Systems of Functional Differential Equations with Delay*, arXiv:1506.05646v1, (2015).

[3] Clement E.Fablo, *Some Elementary Methods for Solving Functional Differential Equations*, Sonoma State University.

[4] Daniel Sepulveda, *Approximation of Linear Differential equations with variable delay on an infinite interval*, arXiv: 607.06960v1, (2016).

[5] Li Zhang, Xinzhi Liu and XueminShen, *Stability of Switched systems with time delay*, Nonlinear Analysis: Hybrid Systems, 1(2007), 44-58.

[6] Michael Stephen Branicky, *Studies in Hybrid System: Modeling, Analysis and Control*, PhD thesis, MIT, (1995).

[7] Daniel Zwillinger, *A Handbook of Differential Equations*, Academic Press, (1989).

[8] Daniel Liberzon, *On quantization and delay effects in nonlinear control systems*, In: Antsaklis P.J., Tabuada P. (eds) Networked Embedded Sensing and Control. Lecture Notes in Control and Information Science, vol 331. Springer, Berlin, Heidelberg.

[9] John Lygeros, *Lecture Notes on Hybrid Systems*, University of Patras Rio, Patras, GR-26500 Greece.

[10] Josef Rebenda and Zdenek Smarda, *An efficient iterative technique for solving initial value problem for partial differential equations*, arXiv. 1414.8756v1, (2014).

[11] Josef Rebenda and Zdenek Smarda, *An efficient numerical approach for delayed logistic models*, arXiv.1610.01528v1, (2016).

[12] Baoquing Liu, Xiaojian Zhou and Qikui Du, *Differential Transform Method for some Delay Differential Equation*, Applied Mathematics, 6(2015), 585-593.

[13] Montri Thongmoon and Sasitron Pusjuso, *Numerical Solutions of Differential Transform Method and Laplace Transform Method for a System of Differential Equations*, Numerical Analysis: Hybrid Systems, Elsevier, 4(2010), 425-431.

[14] Lie-junXie, Cia- lian Zhou and Song Xu *An efficient numerical method to solve a class of nonlinear singular boundary value problems using improved differential transform method*, arXiv: 1601.04922v1, (2016).

[15] Hui Ye and Antony N.Michel, *Stability theory of Hybrid Dynamical Systems*, IEEE transactions on automatic control, 43(4)(1998), 461-474.

[16] Sachin Bhalekar and Jayvant Patade, *Novel Special function obtained from a delay differential equation*, arXiv: 1608.03959v1, (2016).

[17] Varsha Daftardar-Gejji and Hossein Jafari, *An iterative method for solving nonlinear functional equations*, J. Math. Anal.Appl., 316(2)(2006), 753-763.