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Dhage Iteration Method for Nonlinear First Order Hybrid Functional Differential Equations of Second Type Linear Perturbations

Research Article

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Abstract: In this paper we prove the existence and approximation result for a first order nonlinear initial value problem of hybrid functional differential equations via construction of an algorithm. The main results rely on the Dhage iteration method embodied in a recent hybrid fixed point principle of Dhage (2014). An example is also furnished to illustrate the hypotheses and the abstract result of this paper.

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1. Statement of the Problem

It is well-known that the topic of nonlinear differential and integral equations contributed a lot to the subject of nonlinear analysis and applications. The recent trend ni the theory of nonlinear equations has changed from mere existence theory to other qualitative aspects such as existence and approximate solution via construction of an algorithm for the solutions. In this regard, the Dhage iteration method is a powerful tool for proving above mentioned different aspects of the solutions. This method is successfully applied to variety of nonlinear differential and integral equations in the literature. See Dhage [5–10] and the references therein. Very recently, the method is applied to hybrid first order functional differential equations for approximating the solution via algorithm. In this paper we extend the above method to nonlinear first order hybrid functional differential equations with a linear perturbations of second type. Given the real numbers $r > 0$ and $T > 0$, consider the closed and bounded intervals $I_0 = [-r, 0]$ and $I = [0, T]$ in $\mathbb{R}$ and let $J = [-r, T]$. By $\mathcal{C} = C(I_0, \mathbb{R})$ we denote the space of continuous real-valued functions defined on I_0. We equip the space $\mathcal{C}$ with he norm $\|\cdot\|_{\mathcal{C}}$ defined by

$$\|x\|_{\mathcal{C}} = \sup_{-r\leq\theta\leq 0} |x(\theta)|. \tag{1}$$

Clearly, $\mathcal{C}$ is a Banach space with this supremum norm and it is called the history space of the functional differential equation in question. For any continuous function $x : J \to \mathbb{R}$ and for any $t \in I$, we denote by x_t the element of the space $\mathcal{C}$ defined by

$$x_t(\theta) = x(t + \theta),\ -r \leq \theta \leq 0. \tag{2}$$

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The differential equations involving the history of the dynamic systems are called functional differential equations and it has been recognized long back the importance of such problems in the theory of differential equations. Since then, several classes of nonlinear functional differential equations have been discussed in the literature for different qualitative properties of the solutions. A special class of functional differential equations has been discussed in Bainov and Hristova [1] with usual known method and in Dhage [9] and Dhage and Dhage [13] via a new Dhage iteration method for the existence and approximation of solutions. Therefore, it is desirable to extend this new method to other functional differential equations involving delay in the arguments. The present paper is also an attempt in this direction. In this paper, we consider the following nonlinear first order hybrid functional differential equations (in short HFDE)

$$\left.\begin{aligned} \frac{d}{dt}\big[x(t) - f(t, x(t))\big] = g(t, x_t),\ t \in I, \\ x_0 = \phi, \end{aligned}\right\} \tag{3}$$

where $\phi \in \mathcal{C}$ and $f : I \times \mathbb{R} \to \mathbb{R}$ and $g : I \times \mathcal{C} \to \mathbb{R}$ are continuous functions.

Definition 1.1. *A function $x \in C(J, \mathbb{R})$ is said to be a solution of the HFDE (3) if*

(1). $x_0 \in \mathcal{C}$,

(2). $x_t \in \mathcal{C}$ for each $t \in I$, and

(3). the function $t \mapsto [x(t) - f(t, x(t))]$ is continuously differentiable on I and satisfies the equations in (3),

where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

The HFDE (3) is new and a linear perturbation of second type (see Dhage [3] and the references therein) and can be handled with the hybrid operator theoretic technique involving the sum of two operators in a Banach space. See Dhage [3] and the references therein. The special cases of it are well-known and extensively discussed in the literature for different aspects of the solutions. See Hale [18], Dhage [11], Dhage and Jadhav [16] and the references therein. There is a vast literature on nonlinear functional differential equations for different aspects of the solutions via different approaches and methods. The method of upper and lower solution or monotone method is interesting and well-known, however it requires the existence of both the lower as well as upper solutions as well as certain inequality involving monotonicity of the nonlinearity. In this paper we prove the existence and approximation theorem for the hybrid functional differential equations via a new Dhage iteration method which does not require the existence of both upper and lower solution as well the related monotonic inequality and also obtain the algorithm for the solutions under some natural conditions. The rest of the paper is organized as follows. Section 2 deals with the preliminary definitions and auxiliary results that will be used in subsequent sections of the paper. The main result and an illustrative example is given in Sections 3.

2. Auxiliary Results

Throughout this paper, unless otherwise mentioned, let $(E, \preceq, \|\cdot\|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E and $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Guo and Lakshmikantham [17] and the references therein. We need the following definitions (see Dhage [5, 6] and the references therein) in what follows.

A mapping $\mathcal{T} : E \to E$ is called **isotone** or **nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called **nonincreasing** if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called **monotonic** or simply **monotone** if it is either nondecreasing or nonincreasing on E. A mapping $\mathcal{T} : E \to E$ is called **partially continuous** at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E and vice versa. A non-empty subset S of the partially ordered Banach space E is called **partially bounded** if every chain C in S is bounded. An operator $\mathcal{T}$ on a partially normed linear space E into itself is called **partially bounded** if $\mathcal{T}(E)$ is a partially bounded subset of E. $\mathcal{T}$ is called **uniformly partially bounded** if all chains C in $\mathcal{T}(E)$ are bounded by a unique constant. A non-empty subset S of the partially ordered Banach space E is called **partially compact** if every chain C in S is a relatively compact subset of E. A mapping $\mathcal{T} : E \to E$ is called **partially compact** if $\mathcal{T}(E)$ is a partially relatively compact subset of E. $\mathcal{T}$ is called **uniformly partially compact** if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called **partially totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a partially relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

Remark 2.1. *Suppose that $\mathcal{T}$ is a nondecreasing operator on E into itself. Then $\mathcal{T}$ is a partially bounded or partially compact if $\mathcal{T}(C)$ is a bounded or relatively compact subset of E for each chain C in E.*

Definition 2.2. *The order relation $\preceq$ and the metric d on a non-empty set E are said to be **$\mathcal{D}$-compatible** if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be $\mathcal{D}$-compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are $\mathcal{D}$-compatible. A subset S of E is called **Janhavi** if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are $\mathcal{D}$-compatible in it. In particular, if $S = E$, then E is called a **Janhavi metric** or **Janhavi Banach space**.*

Definition 2.3. *An upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a **$\mathcal{D}$-function** provided $\psi(0) = 0$. An operator $\mathcal{T} : E \to E$ is called partially **nonlinear $\mathcal{D}$-contraction** if there exists a $\mathcal{D}$-function ψ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi\big(\|x - y\|\big) \tag{4}$$

for all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, if $\psi(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a partial Lipschitz operator with a Lischitz constant k and moreover, if $0 < k < 1$, $\mathcal{T}$ is called a partial linear contraction on E with a contraction constant k.

The **Dhage iteration method** embodied in the following applicable hybrid fixed point principle of Dhage [6] in a partially ordered normed linear space is used as a key tool for our work contained in this paper. The details of a **Dhage iteration principle** and method are given in Dhage [6–8] and the references therein.

Theorem 2.4. *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that every compact chain C of E is Janhavi. Let $\mathcal{A}, \mathcal{B} : E \to E$ be two nondecreasing operators such that*

(a). $\mathcal{A}$ is a partially bounded and partially nonlinear $\mathcal{D}$-contraction,

(b). $\mathcal{B}$ is partially continuous and partially compact,

(c). there exists an element $\alpha_0 \in X$ such that $\alpha_0 \preceq \mathcal{A}\alpha_0 + \mathcal{B}\alpha_0$ or $\alpha_0 \succeq \mathcal{A}\alpha_0 + \mathcal{B}\alpha_0$.

Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* and the sequence $\{x_n\}$ of successive iterations defined by $x_0 = \alpha_0,\ x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n,\ n = 0, 1, \ldots;$ converges monotonically to x^*.

Remark 2.5. *The condition that every compact chain of E is Janhavi holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\preceq$ and the norm $\|\cdot\|$ in it. This simple fact is used to prove the main existence results of this paper.*

Remark 2.6. *The regularity of E in above Theorem 2.4 may be replaced with a stronger continuity condition of the operator $\mathcal{A}$ and $\mathcal{A}$ on E which is a result proved in Dhage [5].*

3. Main Results

In this section, we prove an existence and approximation result for the HFDE (1) on a closed and bounded interval $J = [a, b]$ under mixed partial Lipschitz and partial compactness type conditions on the nonlinearities involved in it. We place the HFDE (1) in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\le$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{5}$$

and

$$x \le y \iff x(t) \le y(t) \quad \text{for all} \quad t \in J. \tag{6}$$

Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\le$. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it. See Dhage [5–7] and the references therein. The following useful lemma concerning the Janhavi subsets of $C(J, \mathbb{R})$ follows immediately from the Arzelá-Ascoli theorem for compactness.

Lemma 3.1. *Let $\big(C(J, \mathbb{R}), \le, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\le$ defined by (5) and (6) respectively. Then every partially compact subset of $C(J, \mathbb{R})$ is Janhavi.*

Proof. The proof of the lemma is well-known and appears in the papers of Dhage [7], Dhage and Dhage [12] and so we omit the details. □

We introduce an order relation $\le_{\mathcal{C}}$ in $\mathcal{C}$ induced by the order relation $\le$ defined in $C(J, \mathbb{R})$. Thus, for any $x, y \in \mathcal{C}$, $x \le_{\mathcal{C}} y$ implies $x(\theta) \le y(\theta)$ for all $\theta \in I_0$. Moreover, if $x, y \in C(J, \mathbb{R})$ and $x \le y$, then $x_t \le_{\mathcal{C}} y_t$ for all $t \in I$. We need the following definition in what follows.

Definition 3.2. *A differentiable function $u \in C(J, \mathbb{R})$ is said to be a lower solution of the equation (3) if*

(1). $u_t \in \mathcal{C}$ for each $t \in I$, and

(2). the function $t \mapsto [u(t) - f(t, u(t))]$ is continuously differentiable on I and satisfies

$$\left.\begin{aligned} &\frac{d}{dt}\big[u(t) - f(t, u(t))\big] \le g(t, u_t),\ t \in I, \\ &u_0 \le_{\mathcal{C}} \phi. \end{aligned}\right\} \tag{$*$}$$

Similarly, a differentiable function $v \in C(J, \mathbb{R})$ is called an upper solution of the HFDE (3) if the above inequality is satisfied with reverse sign.

We consider the following set of assumptions in what follows:

(H_1) There exists a constant $M_f > 0$ such that $|f(t,x)| \leq M_f$ for all $t \in I$ and $x \in \mathcal{C}$;

(H_2) There exists $\mathcal{D}$-function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $0 \leq f(t,x) - f(t,y) \leq \varphi(x-y)$ for all $t \in I$ and $x, y \in \mathbb{R}$, $x \geq y$.

(H_3) The function g is bounded on $I \times \mathcal{C}$ with bound M_g.

(H_4) The function $g(t,x)$ is nondecreasing in x for each $t \in I$.

(H_5) HFDE (3) has a lower solution $u \in C(J,\mathbb{R})$.

Lemma 3.3. *Suppose that the hypothesis* (H_1) *holds. Then a function $x \in C(J,\mathbb{R})$ is a solution of the HFDE (3) if and only if it is a solution of the nonlinear integral equation*

$$x(t) = \begin{cases} \phi(0) - f(0,\phi(0)) + f(t,x(t)) + \displaystyle\int_0^t g(s,x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{7}$$

Theorem 3.4. *Suppose that hypotheses* (H_1)*-*(H_2) *and* (H_4) *hold. Then the HFDE (3) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_0 = u, \quad x_{n+1}(t) = \begin{cases} \phi(0) - f(0,\phi(0)) + f(t,x_n(t)) + \displaystyle\int_0^t g(s,x_s^n)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \tag{8}$$

where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$, converges monotonically to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then, in view of Lemma 3.1, every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ so that every compact chain C is Janhavi in E. Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = \begin{cases} -f(0,\phi(0)) + f(t,x(t)) & \text{if } t \in I, \\ 0, & \text{if } t \in I_0, \end{cases} \tag{9}$$

and

$$\mathcal{B}x(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t g(s,x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{10}$$

From the continuity of the functions f, g and the integral, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A},\mathcal{B} : E \to E$. Applying Lemma 3.3, the HFDE (3) is equivalent to the operator equation

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \ t \in J. \tag{11}$$

Now, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.4 in a series of following steps.

Step I: *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing on E.*

Let $x, y \in E$ be such that $x \geq y$. Then $x(t) \geq y(t)$ for all $t \in J$ and by hypothesis (H_2), we get

$$\mathcal{A}x(t) = \begin{cases} -f(0,\phi(0)) + f(t,x(t)), & \text{if } t \in I, \\ 0, & \text{if } t \in I_0, \end{cases}$$

$$\geq \begin{cases} -f(0,\phi(0)) + f(t,y(t)), & \text{if } t \in I, \\ 0, & \text{if } t \in I_0, \end{cases}$$

$$= \mathcal{A}y(t),$$

for all $t \in J$. This shows that the operator that the operator $\mathcal{A}$ is also nondecreasing on E. Next, let $x, y \in E$ be such that $x \geq y$. Then $x_t \geq y_t$ for all $t \in I$ and by hypothesis (H_2), we get

$$\mathcal{B}x(t) = \begin{cases} \phi(0) + \int_0^t g(s,x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases}$$

$$\geq \begin{cases} \phi(0) + \int_0^t g(s,y_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases}$$

$$= \mathcal{B}y(t),$$

for all $t \in J$. This shows that the operator that the operator $\mathcal{B}$ is also nondecreasing on E.

Step II: *$\mathcal{A}$ is a nonlinear $\mathcal{D}$-contraction on E.*

Let $x, y \in E$ be any two elements such that $x \geq y$. Then, by hypothesis (H_4),

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| \leq |f(t,x(t)) - f(t,y(t))| \leq \varphi(|x(t) - y(t)|) \leq \varphi(\|x - y\|) \tag{12}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{A}x - \mathcal{A}y\| \leq \psi(\|x-y\|)$ for all $x, y \in E$, $x \geq y$, where $\psi(r) = \varphi(r) < r$ for $r > 0$. As a result $\mathcal{A}$ is a partially nonlinear $\mathcal{D}$-contraction on E in view of Remark 2.6.

Step III: *$\mathcal{B}$ is partially continuous on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C such that $x_n \to x$ as $n \to \infty$. Then $x_s^n \to x_s$ as $n \to \infty$. Since the f is continuous, we have

$$\lim_{n\to\infty} \mathcal{B}x_n(t) = \begin{cases} \phi(0) + \int_0^t \left[\lim_{n\to\infty} g\big(s, x_s^n\big)\right] ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases}$$

$$= \begin{cases} \phi(0) + \int_0^t g(s,x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases}$$

$$= \mathcal{B}x(t)$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges to $\mathcal{B}x$ pointwise on J. Now we show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Now there are three cases:

Case I: Let $t_1, t_2 \in J$ with $t_1 > t_2 \geq 0$. Then we have

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \leq \Big| \int_0^{t_2} g\big(s, x_s^n\big)\,ds - \int_0^{t_1} g\big(s, x_s^n\big)\,ds \Big|$$

$$\leq \Big| \int_{t_1}^{t_2} \big|g\big(s, x_s^n\big)\big|\,ds \Big|$$

$$\leq M_g|t_2 - t_1|$$

$$\to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in \mathbb{N}$.

Case II: Let $t_1, t_2 \in J$ with $t_1 < t_2 \leq 0$. Then we have

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| = |\phi(t_2) - \phi(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in \mathbb{N}$.

Case III: Let $t_1, t_2 \in J$ with $t_1 < 0 < t_2$. Then we have

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \leq |\mathcal{B}x_n(t_2) - \mathcal{B}x_n(0)| + |\mathcal{B}x_n(0) - \mathcal{B}x_n(t_1)| \to 0 \text{ as } t_2 \to t_1.$$

Thus in all three cases, we obtain

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform and that $\mathcal{B}$ is a partially continuous operator on E into itself.

Step III: *$\mathcal{B}$ is partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathcal{T}x$. By hypothesis (H_2)

$$\begin{aligned}
|y(t)| &= |\mathcal{B}x(t)| \\
&\leq \begin{cases} |\phi(0)| + \displaystyle\int_0^t |g(s, x_s)|\, ds, & \text{if } t \in I, \\ |\phi(t)|, & \text{if } t \in I_0, \end{cases} \\
&\leq \|\phi\| + M_f T \\
&= r,
\end{aligned}$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\| \leq \|\mathcal{B}x\| \leq r$ for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is a uniformly bounded subset of E. Next we show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$, with $t_1 < t_2$. Then proceeding with the arguments that given in Step II it can be shown that

$$\big|y(t_2) - y(t_1)\big| = |\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $y \in \mathcal{B}(C)$. This shows that $\mathcal{B}(C)$ is an equicontinuous subset of E. Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous subset of functions in E and hence it is compact in view of Arzelá-Ascoli theorem. Consequently $\mathcal{B} : E \to E$ is a partially compact operator on E into itself.

Step IV: *u satisfies the operator inequality inequality $u \leq \mathcal{A}u + \mathcal{B}u$.*

By hypothesis (H$_4$), the HFDE (3) has a lower solution u defined on J. Then we have

$$\left.\begin{aligned}\frac{d}{dt}\big[u(t)-f(t,u(t))\big] \le g(t,u_t),\ t\in I,\\ u_0 \le_{\mathcal{C}} \phi.\end{aligned}\right\}$$

Integrating the above inequality from 0 to t, we get

$$u(t) \le \begin{cases}\phi(0)-f(0,\phi(0))+f(t,u(t))+\displaystyle\int_0^t g(s,u_s)\,ds, & \text{if } t\in I,\\ \phi(t), \quad \text{if } t\in I_0,\end{cases}$$
$$= \mathcal{A}u(t)+\mathcal{B}u(t)$$

for all $t\in J$. As a result we have that $u \le \mathcal{A}u+\mathcal{A}u$. Thus, $\mathcal{A}$ and $\mathcal{B}u$ satisfy all the conditions of Theorem 2.4 and so the operator equation $\mathcal{A}x+\mathcal{B}x = x$ has a solution. Consequently the integral equation and that the differential equation (3) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (9) converges monotonically to x^*. This completes the proof. □

Remark 3.5. *The conclusion of Theorems 3.4 also remains true if we replace the hypothesis (H$_4$) and (H$_7$) with the following ones:*

(H$_4'$) The HFDE (3) has an upper solution $v\in C(J,\mathbb{R})$.

The proof of Theorem 3.4 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

Example 3.6. *Given the closed and bounded intervals $I_0=[-1,0]$ and $I=[0,1]$, consider the HFDE*

$$\left.\begin{aligned}\frac{d}{dt}\big[x(t)-f_1(t,x(t))\big] = g_1(t,x_t),\ t\in I,\\ x_0 = \phi,\end{aligned}\right\} \tag{13}$$

where $\phi\in\mathcal{C}$ and $f_1: I\times\mathbb{R}\to\mathbb{R}$ and $g_1: I\times\mathcal{C}\to\mathbb{R}$ are continuous functions given by

$$\phi(t)=\sin t,\ \ t\in[-1,0],$$
$$f_1(t,x)=\begin{cases}\dfrac{|x|}{1+|x|}+1, & \text{if } x>0,\\ 1, & \text{if } x\le 0,\end{cases}$$

and

$$g_1(t,x)=\begin{cases}\tanh(\|x\|_{\mathcal{C}})+1, & \text{if } x>_{\mathcal{C}} 0,\ x\ne 0,\\ 1, & \text{if } x\le_{\mathcal{C}} 0,\end{cases}$$

for all $t\in I$. Clearly, f_1 is continuous and bounded on $I\times\mathbb{R}$ with bound $M_{f_1}=2$. We show that f_1 satisfies the hypothesis (H$_2$). Let $x,y\in\mathbb{R}$ be such that $x\ge y>0$. Then $|x|\ge|y|>0$ and therefore, we have

$$0\le f_1(t,x)-f_1(t,x)=\frac{|x|}{1+|x|}-\frac{|y|}{1+|y|}\le\varphi(|x-y|)=\varphi(x-y)$$

for all $t \in I$*, where* $\psi(r) = \dfrac{r}{1+r} < r$*,* $r > 0$*. Again, if* $x, y \in \mathbb{R}$ *be such that* $x \leq y \leq 0$*, then we obtain*

$$0 \leq f_1(t,x) - f_1(t,x) \leq \varphi(x-y)$$

for all $t \in I$*. This shows that the function* $f_1(t,x)$ *satisfies the hypothesis (*H_2*). Next,* g_1 *is bounded on* $I \times \mathcal{C}$ *with* $M_{f_1} = 2$*. Again, let* $x, y \in \mathcal{C}$ *be such that* $x \geq_{\mathcal{C}} y > 0$*. Then* $\|x\|_{\mathcal{C}} \geq \|y\|_{\mathcal{C}} > 0$ *and therefore, we have*

$$g_1(t,x) = \tanh(\|x\|_{\mathcal{C}}) + 1 \geq \tanh(\|y\|_{\mathcal{C}}) + 1 = g_1(t,y)$$

for all $t \in I$*. Again, if* $x, y \in \mathcal{C}$ *be such that* $x \leq_{\mathcal{C}} y \leq_{\mathcal{C}} 0$*, then we obtain*

$$g_1(t,x) = 1 = g_1(t,y)$$

for all $t \in I_0$*. This shows that the function* $g_1(t,x)$ *is nondecreasing in* x *for each* $t \in I$*. Finally,*

$$u(t) = \begin{cases} 1-t, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in [-1,0], \end{cases}$$

is a lower solution of the HFDE (13) *defined on* J*. Thus,* f_1 *satisfies the hypotheses (*H_1*), (*H_2*) and (*H_4*). Hence we apply Theorem 3.4 and conclude that the HFDE (13) has a solution* x^* *on* J *and the sequence* $\{x_n\}$ *of successive approximation defined by*

$$x_0(t) = \begin{cases} 1-t, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in [-1,0] \end{cases}$$

$$x_{n+1}(t) = \begin{cases} -1 + f_1(t, x_n(t)) + \displaystyle\int_0^t g_1(s, x_s^n)\, ds, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in [-1,0], \end{cases}$$

for $n = 0, 1, \ldots,$ *converges monotonically to* x^**.*

Remark 3.7. *We note that if the HFDEs* (3) *has a lower solution* u *as well as an upper solution* v *such that* $u \leq v$*, then under the given conditions of Theorem 3.4 it has corresponding solutions* x_* *and* x^* *and these solutions satisfy* $x_* \leq x^*$*. Hence they are the minimal and maximal solutions of the HFDE* (3) *respectively in the vector segment* $[u, v]$ *of the Banach space* $E = C(J, \mathbb{R})$*, where the vector segment* $[u, v]$ *is a set in* $C(J, \mathbb{R})$ *defined by* $[u, v] = \{x \in C(J, \mathbb{R}) \mid u \leq x \leq v\}$*. This is because the order relation* $\leq$ *defined by* (6) *is equivalent to the order relation defined by the order cone* $\mathcal{K} = \{x \in C(J, \mathbb{R}) \mid x \geq \theta\}$ *which is a closed set in* $C(J, \mathbb{R})$*.*

References

[1] D.D.Bainov, S. Hristova, *Differential Equations with Maxima*, Chapman & Hall/CRC Pure and Applied Mathematics, (2011).

[2] B.C.Dhage, *Local fixed point theory involving the sum of two operators in Banach spaces*, Fixed Point Theory, 4(1)(2003), 49-60.

[3] B.C.Dhage, *Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations*, Differ. Equ. Appl., 2(2010), 465-486.

[4] B.C.Dhage, *Basic results in the theory of hybrid differential equations with mixed perturbations of mixed type*, Funct. Diff. Equ., 19(1-2)(2012), 87-106.

[5] B.C.Dhage, *Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations*, Differ. Equ. Appl., 5(2013), 155-184.

[6] B.C.Dhage, *Partially condensing mappings in partially ordered normed linear spaces and applications to functional integral equations*, Tamkang J. Math., 45(4)(2014), 397-427.

[7] B.C.Dhage, *Nonlinear $\mathcal{D}$-set-contraction mappings in partially ordered normed linear spaces and applications to functional hybrid integral equations*, Malaya J. Mat., 3(1)(2015), 62-85.

[8] B.C.Dhage, *Some generalizations of a hybrid fixed point theorem in a partially ordered metric space and nonlinear functional integral equations*, Differ. Equ. Appl., 8(2016), 77-97.

[9] B.C.Dhage, *Approximating solutions of nonlinear periodic boundary value problems with maxima*, Cogent Mathematics, 2(2016), 12pages.

[10] B.C.Dhage, *Two general fixed point principles and applications*, J. of Nonlinear Anal. Appl., 2016(1)(2016), 23-27.

[11] Bapurao C.Dhage, Shyam B.Dhage and John R.Graef, *Dhage iteration method for initial value problems of nonlinear first order hybrid functional integrodifferential equations*, J. Fixed Point Theory Appl., 18(2)(2016), 309326.

[12] B.C.Dhage and S.B.Dhage, *Approximating solutions of nonlinear first order ordinary differential equations*, GJMS Special Issue for Recent Advances in Mathematical Sciences and Applications-13, 2(2)(2013), 25-35.

[13] S.B.Dhage and B.C.Dhage, *Dhage iteration method for Approximating positive solutions of PBVPs of nonlinear hybrid differential equations with maxima*, Intern. Jour. Anal. Appl., 10(2)(2016), 101-111.

[14] S.B.Dhage, B.C.Dhage and J.B.Graef, *Dhage iteration method for initial value problem for nonlinear first order integrodifferential equations*, J. Fixed Point Theory Appl., 18(2016), 309-325.

[15] B.C.Dhage, S.B.Dhage and S.K.Ntouyas, *Approximating solutions of nonlinear hybrid differential equations*, Appl. Math. Lett., 34(2014), 76-80.

[16] B.C.Dhage and N.S.Jadhav, *Basic results on hybrid differential equations with linear perturbation of second type*, Tamkang J. Math., 44(2)(2013), 171-186.

[17] D.Guo and V.Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, London, (1988).

[18] J.K.Hale, *Theory of Functional Differential equations*, Springer-Verlag, New York-Berlin, (1977).