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Fixed Point Theorems for ψ-contractions in S-metric Spaces and KKS-metric Spaces

Research Article

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Abstract: In this paper we introduce the notion of ψ-contractions for self maps on S-metric spaces and establish a fixed point theorem for such maps. We also introduce a new subclass of S-metric spaces called KKS metric spaces and prove a fixed point theorem for a ψ-contraction on a KKS metric space. An open problem is also given at the end of the paper.

MSC: 54H25, 47H10.

Keywords: S-metric space, KKS metric space, ψ-contraction.

1. Introduction

Banach Contraction Principle in Metric spaces is one of the most important results in fixed point theory. Banach contraction Principle is considered to be the initial result of the fixed point theory in metric spaces. Later on many generalizations of metric spaces were obtained by several authors. Banach Contraction Principle was extended to such spaces. In 2006, Z. Mustafa and B.I. Sims [15] introduced the concept of G-metric space which is a generalization of metric space and proved some fixed point theorems in G-metric spaces. Subsequently many authors proved fixed point theorems in G-metric spaces [5, 8, 14–17]. In 2007, S. Sedghi, N. Shobe introduced D-metric space [9]. In 2012 Sedghi, Shobe and Aliouche [10] introduced the notion of S-metric space as a generalisation of G-Metric space and proved some fixed point Theorems for a self mapping on a complete S-metric space. Later several authors [1, 3, 6, 10–12] continued to study fixed point theory in S-metric spaces. In this paper, we define ψ-contraction map and prove some unique fixed point theorems for ψ-contraction maps on complete S-metric spaces. We also introduce a new class of S-metric spaces, called KKS-metric spaces and prove fixed point theorems for ψ-contractions on a complete KKS-metric space. An open problem is also given at the end of the paper.

2. Preliminaries

In this section we present the necessary definitions and results which are used either tacitly or explicitly in the next section. Mustafa and B. Sims [15, 16] introduced the notion of G-metric spaces as a generalization of metric spaces.

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Definition 2.1 ([16]). *Let X be a non-empty set and $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions: for all $x, y, z, a \in X$*

(G1) $G(x, y, z) = 0$ *if* $x = y = z$

(G2) $0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$

(G3) $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $x \neq y$

(G4) $G(x, y, z) = G(p(x, z, y))$ *where* p *is a permutation in* $\{x, y, z\}$

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$ *(rectangle inequality).*

Definition 2.2 ([9]). *Let X be a non-empty set. A generalized metric (or D^* metric) on X is a function $D^* : X^3 \to R^+$ that satisfies the following conditions for each $x, y, z, a \in X$*

(1). $D^*(x, y, z) \geq 0$

(2). $D^*(x, y, z) = 0$ *if and only if* $x = y = z$

(3). $D^*(x, y, z) = D^*(p\{x, y, z\})$ *(symmetry) where* p *is a permutation function*

(4). $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D^*) is called generalized metric space.

Example 2.3.

(1). Let (X, d) be a metric space. Define $D^* : X^3 \to R^+$ as $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$.

(2). $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ *where (X, d) is a metric space.*

(3). If $X = R^n$, then we define $D^*(x, y, z) = \|x + y - 2z\| + \|x + z - 2y\| + \|y + z - 2x\|$ *where $\|.\|$ is a norm on X.*

(4). If $X = R^+$, then we define $D^*(x, y, z) = \begin{cases} 0, & \text{if } x = y = z \\ \max\{x, y, z\}, & \text{otherwise} \end{cases}$.

Remark 2.4. *It is easy to see that every G-Metric is a D^*-Metric, but in general the converse does not hold.*

Example 2.5. *If $X = R$, we define* $D^*(x, y, z) = |x + y - 2z| + |x + z - 2y| + |y + z - 2x|$, *then (R, D^*) is a D^*-Metric space, but it is not a G-Metric space. Set $x = 5$, $y = -5$ and $z = 0$, then G_3 does not hold.*

S. Sedghi et al. [10] introduced the notion of S-Metric spaces as follows.

Definition 2.6 ([10]). *Let X be a non-empty set. An S-Metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions for each $x, y, z, a \in X$.*

(1). $S(x, y, z) = 0$ *if and only if* $x = y = z$

(2). $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The function S is called as S-Metric on X and the pair (X, S) is called an S-Metric Space.

Remark 2.7. *The notion of S-Metric space is a generalization of a G-Metric space and a D^*-Metric space.*

Example 2.8 ([10]).

(1). Let $X = R^n$, $\|.\|$ a norm on X. Define $S(x,y,z) = \|y+z-2x\| + \|y-z\|$, then (X,S) is an S-Metric space.

(2). Let $X = R^n$, $\|.\|$ a norm in X. Define S by $S(x,y,z) = \|x-z\| + \|y-z\|$, then (X,S) is an S-Metric space.

(3). Let (X,d) be a Metric space. Define $S(x,y,z) = d(x,z) + d(y,z)$ for all $x,y,z,a \in X$, then S is an S-Metric on X.

Observations 2.9 ([10]). *Let (X,S) be an S-Metric space. Then*

(1). $S(x,x,y) = S(y,y,x)$

(2). $S(x,x,z) \leq 2S(x,x,y) + S(z,z,y)$ and

(3). $S(x,y,y) \leq S(x,x,y)$.

Definition 2.10 ([10]). *Let (X,S) be an S-Metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ converges to x if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. In this case we write $x_n \to x$. We observe that if $x_n \to x$ and $x_n \to y$, then $x = y$.*

Definition 2.11 ([10]). *A sequence $\{x_n\} \subseteq X$ is called a Cauchy sequence, if $S(x_n,x_n,x_m) \to 0$ as $n,m \to \infty$. That is for each $\epsilon > 0$, there exists $n_0 \in N$ such that for all $n,m \geq n_0$, we have $S(x_n, x_n, x_m) < \epsilon$.*

Definition 2.12 ([10]). *An S-Metric space (X,S) is said to be complete if every Cauchy sequence in X is a convergent sequence.*

3. Main Results

In this section, we introduce a class of functions Ψ, define a ψ-contraction on an S-Metric space and prove fixed point theorems for ψ-contractions. Let $\Psi = \{\psi | \psi : [0,\infty) \to [0,\infty)$ where ψ is continuous, increasing, $\psi(t) = 0$ if $t = 0$ and $\psi(t) < t$ if $t > 0\}$.

Definition 3.1. *Let (X,S) be an S-Metric space, T be a self map on X and $\psi \in \Psi$. Suppose*

$$S(Tx,Ty,Tz) \leq \psi\left(\max\left\{S(x,y,z), S(x,Tx,Tx), S(y,Ty,Ty), S(z,Tz,Tz)\right\}\right) \;\; \forall\, x,y,z \in X.$$

Then T is called a ψ-contraction on X.

Now we prove the following fixed point theorem for a ψ-contraction on a complete S-Metric space.

Theorem 3.2. *Let (X,S) be a complete S-Metric space and $\psi \in \Psi$. Suppose $T : X \to X$ is a ψ-contraction. That is*

$$S(Tx,Ty,Tz) \leq \psi\left(\max\left\{S(x,y,z), S(x,Tx,Tx), S(y,Ty,Ty), S(z,Tz,Tz)\right\}\right) \; \forall\, x,y,z \in X. \tag{1}$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$. Write $x_{n+1} = Tx_n$ for $n = 0,1,2,\ldots$. If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of T. Hence we may suppose that $x_n \neq x_{n+1}$ for $n = 0,1,2,\ldots$. Then from (1) we get

$$\begin{aligned} S(x_{n+1},x_{n+1},x_{n+2}) &= S(Tx_n,Tx_n,Tx_{n+1}) \\ &\leq \psi\left(\max\left\{\begin{array}{c} S(x_n,x_n,x_{n+1}), S(x_n,Tx_n,Tx_n), \\ S(x_n,Tx_n,Tx_n), S(x_{n+1},Tx_{n+1},Tx_{n+1}) \end{array}\right\}\right) \end{aligned}$$

$$= \psi\left(max\left\{\begin{array}{c} S\left(x_n, x_n, x_{n+1}\right), S\left(x_n, x_{n+1}, x_{n+1}\right), \\ S\left(x_{n+1}, x_{n+2}, x_{n+2}\right)\end{array}\right\}\right)$$

$$\leq \psi\left(max\left\{\begin{array}{c} S\left(x_n, x_n, x_{n+1}\right), S\left(x_n, x_n, x_{n+1}\right), \\ S\left(x_{n+1}, x_{n+1}, x_{n+2}\right)\end{array}\right\}\right) \text{ (From Observation 2.9 (3))}$$

$$\leq \psi\left(\max\left\{S\left(x_n, x_n, x_{n+1}\right), S\left(x_{n+1}, x_{n+1}, x_{n+2}\right)\right\}\right)$$

If $S\left(x_{n+1}, x_{n+1}, x_{n+2}\right)$ is maximum, then

$$S\left(x_{n+1}, x_{n+1}, x_{n+2}\right) \leq \psi\left(S\left(x_{n+1}, x_{n+1}, x_{n+2}\right)\right)$$

$$< S\left(x_{n+1}, x_{n+1}, x_{n+2}\right) \quad (\because x_n \neq x_{n+1}), \text{ a contradiction}$$

$$S\left(x_{n+1}, x_{n+1}, x_{n+2}\right) \leq \psi\left(S\left(x_n, x_n, x_{n+1}\right)\right)$$

$$< S\left(x_n,\ x_n,\ x_{n+1}\right) \tag{2}$$

Therefore $\{S\left(x_n, x_n, x_{n+1}\right)\} \downarrow$ strictly. Let $\{S\left(x_n, x_n, x_{n+1}\right)\} \downarrow a$, say. On letting $n \to \infty$ in (2), we get $\alpha \leq \psi(\alpha)$, therefore $\alpha = 0$. Therefore

$$\{S\left(x_n, x_n, x_{n+1}\right)\} \downarrow 0 \tag{3}$$

Now $S\left(x_n, x_{n+1}, x_{n+1}\right) \leq S\left(x_n, x_n, x_{n+1}\right)$ (from Observation 2.9 (3)) $\to 0$ (by (3))

$$S\left(x_n, x_{n+1}, x_{n+1}\right) \to 0 \text{ as } n \to \infty \tag{4}$$

Now we show that $\{x_n\}$ is Cauchy. Suppose $\{x_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ and subsequences $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k > k$ and

$$S\left(x_{m_k}, x_{m_k}, x_{n_k}\right) \geq \epsilon \tag{5}$$

and

$$S\left(x_{m_k}, x_{m_k}, x_{n_k-1}\right) < \epsilon \tag{6}$$

Now

$$\epsilon \leq S\left(x_{m_k}, x_{m_k}, x_{n_k}\right) = S\left(x_{n_k}, x_{n_k}, x_{m_k}\right) \quad \text{(by (1))}$$

$$\leq 2S\left(x_{n_k},\ x_{n_k}, x_{n_k-1}\right) + S\left(x_{m_k}, x_{m_k}, x_{n_k-1}\right) \quad \text{(by Observation 2.9 (2))}$$

$$< 2S\left(x_{n_k}, x_{n_k}, x_{n_k-1}\right) + \epsilon \quad \text{(by (6))}$$

On letting $k \to \infty$, we get

$$\lim_{k\to\infty} S\left(x_{m_k}, x_{m_k}, x_{n_k}\right) = \epsilon \tag{7}$$

Consider

$$S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}\right) \leq 2S\left(x_{m_k-1}, x_{m_k-1}, x_{m_k}\right) + S\left(x_{n_k-1}, x_{n_k-1}, x_{m_k}\right)$$

$$\leq 2S\left(x_{m_k-1}, x_{m_k-1}, x_{m_k}\right) + 2S\left(x_{n_k-1}, x_{n_k-1}, x_{n_k}\right) + S\left(x_{m_k}, x_{m_k}, x_{n_k}\right)$$

$$\lim_{k\to\infty} \sup S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}\right) \leq \lim_{k\to\infty} S\left(x_{m_k}, x_{m_k}, x_{n_k}\right) = \epsilon \tag{8}$$

Consider

$$\begin{aligned} S\left(x_{m_k}, x_{m_k}, x_{n_k}\right) &\leq 2S\left(x_{m_k}, x_{m_k}, x_{m_k-1}\right) + S\left(x_{n_k}, x_{n_k}, x_{m_k-1}\right) \\ &= 2S\left(x_{m_k}, x_{m_k}, x_{m_k-1}\right) + 2S\left(x_{n_k}, x_{n_k}, x_{n_k-1}\right) + S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}\right) \end{aligned}$$

On letting $k \to \infty$, we get

$$\epsilon \leq \lim_{k\to\infty} \inf S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}\right) \tag{9}$$

From (8) and (9), we get

$$\lim_{k\to\infty} S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}\right) = \epsilon \tag{10}$$

Now from (5)

$$\begin{aligned} \epsilon &\leq S\left(x_{m_k}, x_{m_k}, x_{n_k}\right) = S\left(Tx_{m_k-1}, Tx_{m_k-1}, Tx_{n_k-1}\right) \\ &\leq \psi\left(max\left\{\begin{array}{c} S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}\right), S\left(x_{m_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right), \\ S\left(x_{m_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right), S\left(x_{n_k-1}, Tx_{n_k-1}, Tx_{n_k-1}\right) \end{array}\right\}\right) \\ &= \psi\left(max\left\{S\left(x_{m_k-1},\ x_{m_k-1}, x_{n_k-1}\right), S\left(x_{m_k-1}, x_{m_k}, x_{m_k}\right), S\left(x_{n_k-1}, x_{n_k}, x_{n_k}\right)\right\}\right) \end{aligned}$$

On letting $k \to \infty$, using (3) and (4) we get $\epsilon \leq \psi(\epsilon)$ ($\because \psi$ continuous), a contradiction. Therefore $\{x_n\}$ is Cauchy.

We show that T has a fixed point. Suppose $x_m \to l$ ($\because X$ is complete). Now

$$\begin{aligned} S\left(Tl, Tl, x_{m+1}\right) &= S(Tl, Tl, Tx_m) \\ &\leq \psi\left(\max\left\{S\left(l, l, x_m\right), S\left(l, Tl, Tl\right), S\left(l, Tl, Tl\right), S(x_m, Tx_m, Tx_m)\right\}\right) \\ &= \psi\left(\max\left\{S\left(l, l, x_m\right), S\left(l, Tl, Tl\right), S(x_m, x_{m+1}, x_{m+1})\right\}\right) \end{aligned}$$

Letting $m \to \infty$, we get

$$\begin{aligned} S\left(Tl, Tl, l\right) &\leq \psi\left(\max\left\{S\left(l, l, l\right), S\left(l, Tl, Tl\right), S(l, l, l)\right\}\right) \\ &= \psi\left(S\left(l, Tl, Tl\right)\right) \\ &\leq \psi\left(S\left(Tl, Tl, l\right)\right) \quad \text{(by Observation 2.9 (3))} \\ S\left(Tl, Tl, l\right) &\leq \psi\left(S\left(Tl, Tl, l\right)\right) \\ S\left(Tl, Tl, l\right) &= 0 \\ Tl &= l \end{aligned}$$

Therefore l is a fixed point of T. Suppose x and y are fixed points of T. Then $Tx = x$, $Ty = y$. Then from (1), we have

$$\begin{aligned} S\left(x, x, y\right) &= S\left(Tx, Tx, Ty\right) \\ &\leq \psi\left(\max\left\{S\left(x, x, y\right), S\left(x, Tx, Tx\right), S\left(x, Tx, Tx\right), S(y, Ty, Ty)\right\}\right) \\ &= \psi\left(\max\left\{S\left(x, x, y\right), S\left(x, x, x\right), S\left(x, x, x\right), S(y, y, y)\right\}\right) \\ &= \psi\left(S\left(x, x, y\right)\right) \\ S\left(x, x, y\right) &\leq \psi\left(S\left(x, x, y\right)\right) \\ S\left(x, x, y\right) &= 0 \end{aligned}$$

Therefore $x = y$. Thus T has unique fixed point. □

Lemma 3.3. *Let (X,S) be a complete S-Metric space, $\psi\in\Psi$ and let $T:X\to X$ be such that*

$$S(Tx,Ty,Tz)\le\psi\left(max\left(\begin{array}{c} max\left\{\begin{array}{c} S(x,y,z),S(x,Tx,Tx),\\ S(y,Ty,Ty),S(z,Tz,Tz)\end{array}\right\},\\ \frac{1}{3}max\left\{\begin{array}{c} S(x,Ty,Ty),S(x,Tz,Tz),S(y,Tx,Tx),\\ S(y,Tz,Tz),S(z,Tx,Tx),S(z,Ty,Ty)\end{array}\right\}\end{array}\right)\right)\ \forall\ x,y,z\in X. \tag{11}$$

Then $S(Tx,Tx,T(Tx))<S(x,x,Tx)$ if $x\neq Tx$.

Proof. For any $x\in X$, with $Tx\neq x$, we have from (11)

$$\begin{aligned}
S(Tx,Tx,T(Tx))&\le\psi\left(\max\left(\begin{array}{c}\max\left\{\begin{array}{c} S(x,x,Tx),S(x,Tx,Tx),\\ S(x,Tx,Tx),S(Tx,T(Tx),T(Tx))\end{array}\right\},\\ \frac{1}{3}\max\left\{\begin{array}{c} S(x,Tx,Tx),S(x,T(Tx),T(Tx)),S(x,Tx,Tx),\\ S(x,T(Tx),T(Tx)),S(Tx,Tx,Tx),S(Tx,Tx,Tx)\end{array}\right\}\end{array}\right)\right)\\
&\le\psi\left(\max\left(\begin{array}{c}\max\left\{S(x,x,Tx),S(x,Tx,Tx),S(Tx,T(Tx),T(Tx))\right\},\\ \frac{1}{3}\left\{S(x,T(Tx),T(Tx))\right\}\end{array}\right)\right)\\
&\le\psi\left(\max\left(\begin{array}{c}\max\left\{S(x,x,Tx),S(x,x,Tx),S(Tx,T(Tx),T(Tx))\right\},\\ \frac{1}{3}\left\{S(x,x,T(Tx))\right\}\end{array}\right)\right)\quad\text{(from Observation 2.9 (3))}\\
&\le\psi\left(\max\left(\begin{array}{c}\max\left\{S(x,x,Tx),S(Tx,T(Tx),T(Tx))\right\},\\ \frac{1}{3}\left[2S(x,x,Tx)+S(Tx,Tx,T(Tx))\right]\end{array}\right)\right)
\end{aligned}\tag{12}$$

$$S(Tx,Tx,T(Tx))\le\psi\left(\{\max S(x,x,Tx),S(Tx,Tx,T(Tx))\}\right)$$

$$\begin{aligned}S(Tx,Tx,T(Tx))&\le\psi\left(S(x,x,Tx)\right)\quad\text{since } x\neq Tx\\ &<S(x,x,Tx)\end{aligned}$$

$$S(Tx,Tx,T(Tx))<S(x,x,Tx)\tag{13}$$

□

Theorem 3.4. *Let (X,S) be a complete S-Metric space, $\psi\in\Psi$ and let $T:X\to X$ be such that*

$$S(Tx,Ty,Tz)\le\psi\left(\max\left(\begin{array}{c} \max\left\{\begin{array}{c} S(x,y,z),S(x,Tx,Tx),\\ S(y,Ty,Ty),S(z,Tz,Tz)\end{array}\right\},\\ \frac{1}{3}\max\left\{\begin{array}{c} S(x,Ty,Ty),S(x,Tz,Tz),S(y,Tx,Tx),\\ S(y,Tz,Tz),S(z,Tx,Tx),S(z,Ty,Ty)\end{array}\right\}\end{array}\right)\right)\ \forall\ x,y,z\in X. \tag{14}$$

Then T has a unique fixed point.

Proof. Let $x_0\in X$, and define inductively $x_{n+1}=Tx_n$, $n=0,1,2,\ldots.$ If $x_{n+1}=x_n$ for some n, then x_n is a fixed point of T. Hence we may suppose that $x_{n+1}\neq x_n$ for $n=0,1,2,\ldots.$ Then from Lemma 3.3, since $x_{n+1}\neq x_n$ for $n=0,1,2,\ldots,$ we get

$$\begin{aligned}S(x_{n+1},x_{n+1},x_{n+2})&\le\psi\left(S(x_n,x_n,x_{n+1})\right)\\ &<S(x_n,x_n,x_{n+1})\end{aligned}\tag{15}$$

Hence $\{S(x_n, x_n, x_{n+1})\}$ is a strictly decreasing sequence. Suppose $\{S(x_n, x_n, x_{n+1})\} \downarrow \alpha$. On letting $n \to \infty$ in (15) we get

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = \alpha \leq \psi(\alpha) \tag{16}$$

Hence $\alpha = 0$. Therefore

$$S(x_n, x_n, x_{n+1}) \to 0 \text{ as } n \to \infty \tag{17}$$

Further $S(x_n, x_n, x_{n-1}) = S(x_{n-1}, x_{n-1}, x_n) \to 0$ (from Observation 2.9 (1)) so that

$$S(x_n, x_n, x_{n-1}) \to 0 \text{ as } n \to \infty \tag{18}$$

Now we show that $\{x_n\}$ is Cauchy. Suppose $\{x_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ and subsequences $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k > k$ and

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \epsilon \tag{19}$$

and

$$S(x_{m_k}, x_{m_k}, x_{n_k-1}) < \epsilon \tag{20}$$

Now from (18)

$$\begin{aligned}
\epsilon \leq S(x_{m_k}, x_{m_k}, x_{n_k}) &= S(x_{n_k}, x_{n_k}, x_{m_k}) \quad \text{(from Observation 2.9 (1))} \\
&\leq 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + S(x_{m_k}, x_{m_k}, x_{n_k-1}) \quad \text{(from Observation 2.9 (2))} \\
&< 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + \epsilon \quad \text{(by (20))}
\end{aligned}$$

On letting $k \to \infty$, from (17), we get

$$\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \epsilon \tag{21}$$

Consider

$$\begin{aligned}
S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) &\leq 2S(x_{m_k-1}, x_{m_k-1}, x_{m_k}) + S(x_{n_k-1}, x_{n_k-1}, x_{n_k}) \\
&\leq 2S(x_{m_k-1}, x_{m_k-1}, x_{m_k}) + 2S(x_{n_k-1}, x_{n_k-1}, x_{m_k}) + S(x_{m_k}, x_{m_k}, x_{n_k})
\end{aligned}$$

$$\lim_{k\to\infty} \sup S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) \leq \lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \epsilon \tag{22}$$

Consider

$$\begin{aligned}
S(x_{m_k}, x_{m_k}, x_{n_k}) &\leq 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{n_k}, x_{n_k}, x_{m_k-1}) \\
&= 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1})
\end{aligned}$$

On letting $k \to \infty$, we get

$$\epsilon \leq \lim_{k\to\infty} \inf S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) \tag{23}$$

from (21) and (22), we get

$$\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) = \epsilon \tag{24}$$

Now

$$\begin{aligned}
\epsilon \leq S\left(x_{m_k}, x_{m_k}, x_{n_k}\right) &= S\left(Tx_{m_k-1}, Tx_{m_k-1}, Tx_{n_k-1}\right) \\
&\leq \psi\left(\max\left(\begin{array}{c} \max\left\{\begin{array}{c} S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}\right), S\left(x_{m_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right), \\ S\left(x_{m_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right), S\left(x_{n_k-1}, Tx_{n_k-1}, Tx_{n_k-1}\right) \end{array}\right\}, \\ \frac{1}{3}\max\left\{\begin{array}{c} S\left(x_{m_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right), S\left(x_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}\right), \\ S\left(x_{m_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right), S\left(x_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}\right), \\ S\left(x_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right), S\left(x_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right) \end{array}\right\} \end{array}\right)\right) \\
&= \psi\left(\max\left(\begin{array}{c} \max\left\{\begin{array}{c} S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}\right), S\left(x_{m_k-1}, x_{m_k}, x_{m_k}\right), \\ S\left(x_{n_k-1}, x_{n_k}, x_{n_k}\right), \end{array}\right\}, \\ \frac{1}{3}\max\left\{\begin{array}{c} S\left(x_{m_k-1}, x_{m_k}, x_{m_k}\right), S\left(x_{m_k-1}, x_{n_k}, x_{n_k}\right), \\ S\left(x_{n_k-1}, x_{m_k}, x_{m_k}\right), \end{array}\right\} \end{array}\right)\right)
\end{aligned}$$

On letting $k \to \infty$, we get

$$\begin{aligned}
\epsilon &\leq \psi\left(\max\left\{\epsilon, \frac{1}{3}\max\{0, \epsilon, \epsilon\}\right\}\right) \quad \text{(from (22) and (24))} \\
&= \psi\left(\max\left(\epsilon, \frac{1}{3}\epsilon\right)\right) \\
&= \psi(\epsilon)
\end{aligned}$$

Therefore $\epsilon \leq \psi(\epsilon)$, a contradiction. Therefore $\{x_n\}$ is Cauchy. Suppose $x_m \to p$.

We show that p is a fixed point of T.

Consider

$$\begin{aligned}
S\left(Tp, Tp, x_{m+1}\right) &= S\left(Tp, Tp, Tx_m\right) \\
&\leq \psi\left(\max\left(\begin{array}{c} \max\left\{\begin{array}{c} S\left(p, p, x_m\right), S\left(p, Tp, Tp\right), \\ S\left(p, Tp, Tp\right), S\left(x_m, Tx_m, Tx_m\right) \end{array}\right\}, \\ \frac{1}{3}\max\left\{\begin{array}{c} S\left(p, Tp, Tp\right), S\left(p, Tx_m, Tx_m\right), S\left(p, Tp, Tp\right), \\ S\left(p, Tx_m, Tx_m\right), S\left(x_m, Tp, Tp\right), S\left(x_m, Tp, Tp\right) \end{array}\right\} \end{array}\right)\right)
\end{aligned}$$

On letting $m \to \infty$

$$\begin{aligned}
S\left(Tp, Tp, p\right) &\leq \psi\left(\max\left(\begin{array}{c} \max\left\{\begin{array}{c} S\left(p, p, p\right), S\left(p, Tp, Tp\right), \\ S\left(p, Tp, Tp\right), S\left(p, p, p\right) \end{array}\right\}, \\ \frac{1}{3}max\left\{\begin{array}{c} S\left(p, Tp, Tp\right), S\left(p, p, p\right), S\left(p, Tp, Tp\right), S\left(p, p, p\right) \\ S\left(p, Tp, Tp\right), S\left(p, Tp, Tp\right) \end{array}\right\} \end{array}\right)\right) \\
&= \psi\left(\max\left(\max\{0, S(p, Tp, Tp)\}, \frac{1}{3}\max\{S(p, Tp, Tp), 0\}\right)\right) \\
&= \psi\left(S\left(p, Tp, Tp\right)\right) \\
&\leq \psi\left(S\left(p, p, Tp\right)\right) \\
&= \psi\left(S\left(Tp, Tp, p\right)\right)
\end{aligned}$$

Therefore $S(Tp, Tp, p) \leq \psi(S(Tp, Tp, p)) < S(Tp, Tp, p)$ if $p \neq T(p)$, a contradiction. Therefore $Tp = p$. Therefore p is a fixed point of T.

We show that fixed point of T is unique

Let p and q be fixed points of T. Then $Tp = p$ and $Tq = q$. Consider

$$\begin{aligned}
S(p,p,q) &= S(Tp,Tp,Tq) \\
&= \psi\left(\max\left(\begin{array}{c} \max\left\{\begin{array}{c} S(p,p,q), S(p,Tp,Tp), \\ S(p,Tp,Tp), S(q,Tq,Tq) \end{array}\right\}, \\ \frac{1}{3}\max\left\{\begin{array}{c} S(p,Tp,Tp), S(p,Tq,Tq), S(p,Tp,Tp), \\ S(p,Tq,Tq), S(q,Tp,Tp), S(q,Tp,Tp), \end{array}\right\} \end{array}\right)\right) \\
&= \psi\left(\max\left(\begin{array}{c} \max\left\{\begin{array}{c} S(p,p,q), S(p,p,p), \\ S(p,p,p), S(q,q,q) \end{array}\right\}, \\ \frac{1}{3}\max\left\{\begin{array}{c} S(p,p,p), S(p,q,q), S(p,p,p), \\ S(p,q,q), S(q,p,p), S(q,p,p), \end{array}\right\} \end{array}\right)\right) \\
&= \psi\left(\max\{S(p,p,q)\}, \frac{1}{3}\max\{S(p,q,q), S(q,p,p)\}\right) \\
&\leq \psi\left(\max\{S(p,p,q)\}, \frac{1}{3}\max\{S(p,p,q), S(q,q,p)\}\right) \\
&\leq \psi\left(\max\{S(p,p,q)\}, \frac{1}{3}\max\{S(p,p,q), S(p,p,q)\}\right) \quad \text{(by Observation 2.9 (3))} \\
&= \psi(S(p,p,q))
\end{aligned}$$

Therefore $S(p,p,q) \leq \psi(S(p,p,q)) < S(p,p,q)$ if $p \neq q$, a contradiction. Therefore $p = q$. Therefore Fixed point of T is unique. □

Note: It may be observed that Theorem 3.2 follows from Theorem 3.4 since (1) ⇒ (14). Now we obtain Banach Contraction Principle in S-Metric spaces, as a corollary.

Theorem 3.5. *Let (X, S) be a complete S-Metric space, $T : X \to X$, and $0 \leq \lambda < 1$. Suppose*

$$S(Tx, Ty, Tz) \leq \lambda S(x, y, z) \quad \forall\, x, y, z \in X. \tag{25}$$

Then T has a unique fixed point.

Proof. Define $\psi(t) = \lambda t$ for $t \geq 0$. Then $\psi \in \Psi$ and from (25) we have

$$\begin{aligned}
S(Tx, Ty, Tz) &\leq \lambda S(x, y, z) = \psi(S(x, y, z)) \\
&\leq \psi(\max\{S(x,y,z), S(x,Tx,Tx), S(y,Ty,Ty)\, S(z,Tz,Tz)\})
\end{aligned}$$

Now the result follows from Theorem 3.2 □

Now we introduce a new class of S-Metric spaces called KKS-Metric spaces and obtain a fixed point theorem for such spaces.

Definition 3.6 (KKS-Metric space). *Let X be a non empty set and $S : X^3 \to [0, \infty)$ be a function that satisfies the following conditions for each $x, y, z, a \in X$*

(1). $S(x,y,z)=0$ *if and only if* $x=y=z$

(2). $S(x,y,z)\le S(x,x,a)+S(y,y,a)+S(z,z,a)$

(3). $S(x,x,y)\le 2\max\{S(x,x,a),S(a,a,y)\}$

Then (X,S) *is called a KKS-Metric space.*

We observe that every KKS-Metric space is a S-Metric space. Now we prove a fixed point theorem for a self map on a KKS-Metric space.

Theorem 3.7. *Let* (X,S) *be a complete KKS-Metric space,* $\psi\in\Psi$ *and let* $T:X\to X$ *be such that*

$$S(Tx,Ty,Tz)\le\psi\left(\max\left(\begin{array}{c}\max\left\{\begin{array}{c}S(x,y,z),S(x,Tx,Tx),\\ S(y,Ty,Ty),S(z,Tz,Tz)\end{array}\right\},\\ \frac{1}{2}\max\left\{\begin{array}{c}S(x,Ty,Ty),S(x,Tz,Tz),S(y,Tx,Tx),\\ S(y,Tz,Tz),S(z,Tx,Tx),S(z,Ty,Ty)\end{array}\right\}\end{array}\right)\right)\ \forall\ x,y,z\in X. \tag{26}$$

Then T *has a unique fixed point in* X.

Proof. Suppose $x\in X$. Then from (26), we have

$$\begin{aligned}S(Tx,Tx,T(Tx))&\le\psi\left(\max\left(\begin{array}{c}\max\left\{\begin{array}{c}S(x,x,Tx),S(x,Tx,Tx),\\ S(x,Tx,Tx),S(Tx,T(Tx),T(Tx))\end{array}\right\},\\ \frac{1}{2}\max\left\{\begin{array}{c}S(x,Tx,Tx),S(x,T(Tx),T(Tx)),S(x,Tx,Tx),\\ S(x,T(Tx),T(Tx)),S(Tx,Tx,Tx),S(Tx,Tx,Tx)\end{array}\right\}\end{array}\right)\right)\\
&\le\psi\left(\max\left(\begin{array}{c}\max\{S(x,x,Tx),S(x,x,Tx),S(Tx,T(Tx),T(Tx))\},\\ \frac{1}{2}max\{S(x,x,Tx),S(x,T(Tx),T(Tx))\}\end{array}\right)\right)\quad\text{(by Observation 2.9 (3))}\\
&\le\psi\left(\max\left(\begin{array}{c}\max\{S(x,x,Tx),S(Tx,T(Tx),T(Tx))\},\\ \frac{1}{2}\{S(x,T(Tx),T(Tx))\}\end{array}\right)\right)\\
&\le\psi\left(\max\left(\begin{array}{c}\max\{S(x,x,Tx),S(Tx,T(Tx),T(Tx))\},\\ \frac{1}{2}\{S(x,x,T(Tx))\}\end{array}\right)\right)\\
&\le\psi\left(\max\left\{S(x,x,Tx),S(Tx,Tx,T(Tx)),\frac{1}{2}S(x,x,T(Tx))\right\}\right)\\
&\le\psi\left(max\left\{\begin{array}{c}S(x,x,Tx),S(Tx,Tx,T(Tx)),\\ \frac{1}{2}(2\max\{S(x,x,Tx),S(Tx,Tx,T(Tx))\})\end{array}\right\}\right)\\
&=\psi(\max\{S(x,x,Tx),S(Tx,Tx,T(Tx))\})\end{aligned}$$

If $Tx\ne T(Tx)$. then

$$\begin{aligned}S(Tx,Tx,T(Tx)&\le\psi(S(x,x,Tx))\\ &<S(x,x,Tx)\end{aligned}\tag{27}$$

Let $x_0\in X$ and $x_{n+1}=Tx_n$ for $n=0,1,2,\ldots$. Take $x=x_n$ in (27), we get (assuming $x_n\ne x_{n+1}\ \forall\ n$)

$$S(Tx_n,Tx_n,T(Tx_n))=S(x_{n+1},x_{n+1},x_{n+2})\downarrow\ \textit{strictly}\tag{28}$$

$$< S(x_n, x_n, x_{n+1}) \downarrow \alpha \ \ (say) \tag{29}$$

Now on letting $n \to \infty$ from (28), we get $\alpha \leq \psi(\alpha)$ ($\because \psi$ is continuous). Therefore $\alpha = 0$. Therefore

$$S(x_n, x_n, x_{n+1}) \downarrow 0 \tag{30}$$

Hence $S(x_n, x_{n+1}, x_{n+1}) \leq S(x_n, x_n, x_{n+1}) \to 0$. Therefore

$$S(x_n, x_n, x_{n+1}) \to 0 \tag{31}$$

Now we show that $\{x_n\}$ is Cauchy. Suppose $\{x_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ and subsequences $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k > k$ and

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \epsilon \tag{32}$$

and

$$S(x_{m_k}, x_{m_k}, x_{n_k - 1}) < \epsilon \tag{33}$$

Now

$$\begin{aligned}
\epsilon \leq S(x_{m_k}, x_{m_k}, x_{n_k}) &= S(x_{n_k}, x_{n_k}, x_{m_k}) \quad \text{(from Observation 2.9 (1))}\\
&\leq 2S(x_{n_k}, x_{n_k}, x_{n_k - 1}) + S(x_{m_k}, x_{m_k}, x_{n_k - 1}) \quad \text{(by Observation 2.9 (2))}\\
&< 2S(x_{n_k}, x_{n_k}, x_{n_k - 1}) + \epsilon
\end{aligned}$$

On letting $k \to \infty$, from (31), we get

$$\lim_{k \to \infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \epsilon \tag{34}$$

Consider

$$\begin{aligned}
S(x_{m_k - 1}, x_{m_k - 1}, x_{n_k - 1}) &\leq 2S(x_{m_k - 1}, x_{m_k - 1}, x_{m_k}) + S(x_{n_k - 1}, x_{n_k - 1}, x_{m_k}) \quad \text{(From Observation 2.9 (2))}\\
&\leq 2S(x_{m_k - 1}, x_{m_k - 1}, x_{m_k}) + 2S(x_{n_k - 1}, x_{n_k - 1}, x_{n_k}) + S(x_{m_k}, x_{m_k}, x_{n_k})\\
\lim_{k \to \infty} S(x_{m_k - 1}, x_{m_k - 1}, x_{n_k - 1}) &\leq \lim_{k \to \infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \epsilon
\end{aligned} \tag{35}$$

Consider

$$\begin{aligned}
S(x_{m_k}, x_{m_k}, x_{n_k}) &\leq 2S(x_{m_k}, x_{m_k}, x_{m_k - 1}) + S(x_{n_k}, x_{n_k}, x_{m_k - 1})\\
&= 2S(x_{m_k}, x_{m_k}, x_{m_k - 1}) + 2S(x_{n_k}, x_{n_k}, x_{n_k - 1}) + S(x_{m_k - 1}, x_{m_k - 1}, x_{n_k - 1})
\end{aligned}$$

On letting $k \to \infty$, we get

$$\epsilon \leq \lim_{k \to \infty} \inf S(x_{m_k - 1}, x_{m_k - 1}, x_{n_k - 1}) \tag{36}$$

From (**??**) and (36), we get

$$\lim_{k \to \infty} S(x_{m_k - 1}, x_{m_k - 1}, x_{n_k - 1}) = \epsilon \tag{37}$$

Now

$$
\begin{aligned}
&\epsilon \leq S\left(x_{m_k}, x_{m_k}, x_{n_k}\right) \\
&= S\left(Tx_{m_k-1}, Tx_{m_k-1}, Tx_{n_k-1}\right) \\
&\leq \psi\left(\max\left(\begin{array}{c}\max\left\{\begin{array}{c}S\left(x_{m_k-1}, x_{m_k-1}, Tx_{n_k-1}\right), S\left(x_{m_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right), \\ S\left(x_{m_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right), S\left(x_{n_k-1}, Tx_{n_k-1}, Tx_{n_k-1}\right)\end{array}\right\}, \\ \frac{1}{2}\max\left\{\begin{array}{c}S\left(x_{m_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right), S\left(x_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}\right), \\ S\left(x_{m_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right), S\left(x_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}\right), \\ S\left(x_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right), S\left(x_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right)\end{array}\right\}\end{array}\right)\right) \\
&= \psi\left(\max\left(\begin{array}{c}\max\left\{\begin{array}{c}S\left(x_{m_k-1}, x_{m_k-1}, Tx_{n_k-1}\right), S\left(x_{m_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right), \\ S\left(Tx_{n_k-1}, T\left(Tx_{n_k-1}\right), T\left(Tx_{n_k-1}\right)\right),\end{array}\right\}, \\ \frac{1}{2}\max\left\{\begin{array}{c}S\left(x_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}\right), S\left(x_{m_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right), \\ S\left(Tx_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right),\end{array}\right\}\end{array}\right)\right)
\end{aligned}
$$

On letting $k \to \infty$ we get

$$
\begin{aligned}
\epsilon &\leq \psi\left(\max\left\{S\left(x_{m_k-1},\ x_{m_k-1}, T\ x_{n_k-1}\right)\right\}, \frac{1}{2}\max\left\{\begin{array}{c}S\left(x_{m_k-1},\ Tx_{n_k-1}, Tx_{n_k-1}\right), \\ S\left(Tx_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1}\right)\end{array}\right\}\right) \\
&\leq \psi\left(\max\left\{S\left(x_{m_k-1}, x_{m_k-1}, Tx_{n_k-1}\right)\right\}, \frac{1}{2}\max\left\{\begin{array}{c}S\left(x_{m_k-1}, x_{m_k-1}, Tx_{n_k-1}\right), \\ S\left(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{m_k-1}\right)\end{array}\right\}\right) \\
&\leq \psi\left(\begin{array}{c}\max\left\{S\left(x_{m_k-1}, x_{m_k-1}, Tx_{n_k-1}\right)\right\}, \\ \frac{1}{2}\max\left\{\begin{array}{c}2S\left(x_{m_k-1}, x_{m_k-1}, Tx_{n_k}\right) + S\left(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{n_k}\right), \\ 2S\left(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{n_k}\right) + S\left(Tx_{m_k-1}, Tx_{m_k-1}, Tx_{n_k}\right)\end{array}\right\}\end{array}\right) \\
&\leq \psi\left(\max\left(\epsilon, \frac{1}{2}\left(2\epsilon\right)\right)\right) \\
&= \psi\left(\epsilon\right)
\end{aligned}
$$

Therefore $\epsilon \leq \psi\left(\epsilon\right)$,a contradiction. Therefore $\{x_n\}$ is Cauchy. Suppose $x_m \to p$. We show that p is a fixed point of T. Consider

$$
\begin{aligned}
S\left(Tp, Tp, x_{m+1}\right) &= S\left(Tp, Tp, Tx_m\right) \\
&\leq \psi\left(\max\left(\begin{array}{c}\max\left\{\begin{array}{c}S\left(p, p, x_m\right), S\left(p, Tp, Tp\right), \\ S\left(p, Tp, Tp\right), S\left(x_m, Tx_m, Tx_m\right)\end{array}\right\}, \\ \frac{1}{2}\max\left\{\begin{array}{c}S\left(p, Tp, Tp\right), S\left(p, Tx_m, Tx_m\right), S\left(p, Tp, Tp\right), \\ S\left(p, Tx_m, Tx_m\right), S\left(x_m, Tp, Tp\right), S\left(x_m, Tp, Tp\right)\end{array}\right\}\end{array}\right)\right)
\end{aligned}
$$

On letting $m \to \infty$

$$
S\left(Tp, Tp, p\right) \leq \psi\left(\max\left(\begin{array}{c}\max\left\{\begin{array}{c}S\left(p, p, p\right), S\left(p, Tp, Tp\right), \\ S\left(p, Tp, Tp\right), S\left(p, p, p\right)\end{array}\right\}, \\ \frac{1}{2}\max\left\{\begin{array}{c}S\left(p, Tp, Tp\right), S\left(p, p, p\right), S\left(p, Tp, Tp\right), S\left(p, p, p\right) \\ S\left(p, Tp, Tp\right), , S\left(p, Tp, Tp\right)\end{array}\right\}\end{array}\right)\right)
$$

$$
\begin{aligned}
&= \psi\left(\max\left(\max\{0, S(p,Tp,Tp)\}, \frac{1}{2}\max\{S(p,Tp,Tp), 0\}\right)\right)\\
&= \psi(S(p,Tp,Tp))\\
&\le \psi(S(p,p,Tp))\\
&= \psi(S(Tp,Tp,p))
\end{aligned}
$$

Therefore $S(Tp,Tp,p) \le \psi(S(Tp,Tp,p)) < S(Tp,Tp,p)$ if $\neq T(p)$, a contradiction. Therefore $Tp = p$. Therefore p is a fixed point of T. We show that fixed point of T is unique. Let p and q be fixed points of T. Then $Tp = p$ and $Tq = q$. Consider

$$
\begin{aligned}
S(p,p,q) &= S(Tp,Tp,Tq)\\
&= \psi\left(\max\left(\begin{array}{c} \max\left\{\begin{array}{c} S(p,p,q), S(p,Tp,Tp), \\ S(p,Tp,Tp), S(q,Tq,Tq) \end{array}\right\}, \\ \frac{1}{2}\max\left\{\begin{array}{c} S(p,Tp,Tp), S(p,Tq,Tq), S(p,Tp,Tp), \\ S(p,Tq,Tq), S(q,Tp,Tp), S(q,Tp,Tp), \end{array}\right\} \end{array}\right)\right)\\
&= \psi\left(\max\left(\begin{array}{c} \max\left\{\begin{array}{c} S(p,p,q), S(p,p,p), \\ S(p,p,p), S(q,q,q) \end{array}\right\}, \\ \frac{1}{2}\max\left\{\begin{array}{c} S(p,p,p), S(p,q,q), S(p,p,p), \\ S(p,q,q), S(q,p,p), S(q,p,p), \end{array}\right\} \end{array}\right)\right)\\
&= \psi\left(\max\{S(p,p,q)\}, \frac{1}{2}\max\{S(p,q,q), S(q,p,p)\}\right)\\
&\le \psi\left(\max\{S(p,p,q)\}, \frac{1}{2}\max\{S(p,p,q), S(q,q,p)\}\right)\\
&\le \psi\left(\max\{S(p,p,q)\}, \frac{1}{2}\max\{S(p,p,q), S(p,p,q)\}\right) \quad \text{(by Observation 2.9 (3))}\\
&= \psi(S(p,p,q))
\end{aligned}
$$

Therefore $S(p,p,q) \le \psi(S(p,p,q)) < S(p,p,q)$ if $\neq q$, a contradiction. Therefore $p = q$. Therefore Fixed point of T is unique. □

Open Problem

Is every KKS- Metric space a S-Metric space?

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