



# Products of Toeplitz and Hankel Operators on the Harmonic Bergman Space

Research Article

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**Abstract:** In this paper, we discuss the product problems of Toeplitz operators, small Hankel and big Hankel operators with quasihomogeneous symbols on the harmonic Bergman space of the unit disk.

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**Keywords:** Toeplitz operator, small Hankel operator, big Hankel operator, quasihomogeneous symbols, harmonic Bergman space.

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## 1. Introduction

Let  $\mathbb{C}$  be the complex plane and  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  be the open unit disk in  $\mathbb{C}$ . Let  $dA$  denote Lebesgue area measure on  $\mathbb{D}$ , normalized so that the measure of  $\mathbb{D}$  is 1. In polar coordinate,  $dA = \frac{rdrd\theta}{\pi}$ .  $L^2(\mathbb{D}, dA)$  is the Hilbert space consisting of all Lebesgue square integrable functions on  $\mathbb{D}$  with the inner product

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} dA(z).$$

The Bergman space  $L_a^2$  is the closed subspace consisting of the analytic functions in  $L^2(\mathbb{D}, dA)$ . Let  $P$  be the orthogonal projection from  $L^2(\mathbb{D}, dA)$  onto  $L_a^2$ , then  $P$  can be expressed by

$$(Pf)(z) = \langle f, K_z \rangle = \int_{\mathbb{D}} f(w) \frac{1}{(1 - z\bar{w})^2} dA(w),$$

where  $K_z(w) = \frac{1}{(1 - z\bar{w})^2}$  is the Bergman reproducing kernel. The harmonic Bergman space, denoted by  $L_h^2$ , is the closed subspace of  $L^2(\mathbb{D}, dA)$  consisting of the harmonic functions on  $\mathbb{D}$ . It is well known that  $L_h^2$  is also a Hilbert space and the set  $\{\sqrt{n+1}z^n\}_{n=0}^{\infty} \cup \{\sqrt{n+1}\bar{z}^n\}_{n=1}^{\infty}$  is the orthonormal basis of  $L_h^2$ . We will write  $Q$  for the orthogonal projection from  $L^2(\mathbb{D}, dA)$  onto  $L_h^2$ . It is easy to verify that each point evaluation is a bounded linear functional on  $L_h^2$ . It follows that, for each  $z \in \mathbb{D}$ , there exists a unique function  $R_z$  (called the harmonic Bergman kernel) in  $L_h^2$  such that

$$f(z) = \langle f, R_z \rangle \text{ for every } f \in L_h^2.$$

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It is easy to get that  $R_z = K_z + \overline{K_z} - 1$ , then

$$Qf = Pf + \overline{P\overline{f}} - Pf(0).$$

For a function  $\phi \in L^\infty(\mathbb{D}, dA)$ , we define the Toeplitz operator  $T_\phi : L_h^2 \rightarrow L_h^2$  with symbol  $\phi$  by

$$T_\phi(f) = Q(\phi f).$$

Let  $U : L^2(\mathbb{D}, dA) \rightarrow L^2(\mathbb{D}, dA)$  be the unitary operator defined by  $Uf(z) = \widetilde{f(z)} = f(\overline{z})$ , where  $f$  belongs to  $L^2(\mathbb{D}, dA)$ . Let  $g$  be in  $L^\infty(\mathbb{D}, dA)$ , and let  $M_\phi$  be a multiplication linear operator on  $L^2(\mathbb{D}, dA)$  by defined  $M_\phi(f) = gf$ , then we can define the small Hankel operator  $h_g : L_h^2 \rightarrow L_h^2$  by

$$h_g(f) = QUM_\phi f,$$

and big Hankel operator  $H_g : L_h^2 \rightarrow L_h^2$  by

$$H_g(f) = QM_\phi Uf.$$

Let  $\phi \in L^1(\mathbb{D}, dA)$  be a radial function, i.e.,  $\phi(z) = \phi(|z|)$ ,  $z \in \mathbb{D}$ . A function  $f$  is said to be quasihomogeneous of degree  $k \in \mathbb{Z}$  if

$$f(re^{i\theta}) = e^{ik\theta} \phi(r),$$

where  $\phi$  is a radial function. In this case,  $T_f$  (or  $h_f$ , or  $H_f$ ) is also called quasihomogeneous Toeplitz (or small Hankel, or big Hankel) operator of degree  $k$ .

For product problem, on the Hardy space, Brown and Halmos [1] showed that if  $f$  and  $g$  are bounded functions on the unit circle, then  $T_f T_g$  is another Toeplitz operator if and only if either  $\overline{f}$  or  $g$  is homomorphic. From this, it is easy to deduced that if  $f, g \in L^\infty(\mathbb{T})$  such that  $T_f T_g = 0$ , then one of the symbols must be the zero function. In the setting of the Bergman space, the condition either  $\overline{f}$  or  $g$  is homomorphic is still sufficient, but it is no longer necessary. Ahern and Čučković [2] showed that a Brown-Halmos type result holds for Toeplitz operators with harmonic symbols on  $L_a^2$ .

In 2003, Čučković [3] proved that if  $f \in L^\infty(\mathbb{D}, dA)$  such that  $T_f T_{z^j - \overline{z}^l} = 0$ , where  $j, l$  are both positive integers, then  $f = 0$ . Later in [4], Louhichi, Rao and Yousef considered the zero product problem for  $f, g \in L^\infty(\mathbb{D}, dA)$  with  $g = \sum_{k=-\infty}^N e^{ik\theta} g_k$ , where  $g_k$  is a bounded radial function and  $N$  is a positive integer. Recently, Le [5] studied the finite rank product problem for  $f, g \in L^2(\mathbb{D}, dA)$ , where  $g$  is of the form  $\sum_{k=-\infty}^N e^{ik\theta} g_k$ . On the harmonic Bergman space, Dong and Zhou [6] characterized when the product of quasihomogeneous Toeplitz operators is a Toeplitz operator. In paper [7], Guan and Lu solved the product problem of quasihomogeneous Toeplitz operator and quasihomogeneous small Hankel operator. In 2016, Yang, Lu and Wang [8] investigated the finite rank product problem of several quasihomogeneous Toeplitz operators. Motivated by recent results on the unit disk in [6, 7] and [8], in this paper, on the pluriharmonic space of the unit disk, we characterize the product of Toeplitz operators, small Hankel and big Hankel operators with quasihomogeneous symbols.

## 2. Preliminaries

In order to get our main results, we shall often use Lemma 2.1 in [6] and Lemma 2.3 in [7], which can be stated as follows:

**Lemma 2.1.** *Let  $p \in \mathbb{Z}$  and  $\phi$  be a bounded radial function. Then for each  $k \in \mathbb{N}$ ,*

$$T_{e^{ip\theta}\phi}(z^k) = \begin{cases} 2(k+p+1)\widehat{\phi}(2k+p+2)z^{k+p} & \text{if } k \geq -p \\ 2(-k-p+1)\widehat{\phi}(-p+2)\bar{z}^{-k-p} & \text{if } k < -p; \end{cases}$$

$$T_{e^{ip\theta}\phi}(\bar{z}^k) = \begin{cases} 2(k-p+1)\widehat{\phi}(2k-p+2)\bar{z}^{k-p} & \text{if } k \geq p \\ 2(p-k+1)\widehat{\phi}(p+2)z^{p-k} & \text{if } k < p. \end{cases}$$

**Lemma 2.2.** *Let  $p \in \mathbb{Z}$  and  $\phi$  be a bounded radial function. Then for each  $k \in \mathbb{N}$ ,*

$$h_{e^{ip\theta}\phi}(z^k) = \begin{cases} 2(k+p+1)\widehat{\phi}(2k+p+2)\bar{z}^{k+p} & \text{if } k \geq -p \\ 2(-k-p+1)\widehat{\phi}(-p+2)z^{-k-p} & \text{if } k < -p; \end{cases}$$

$$h_{e^{ip\theta}\phi}(\bar{z}^k) = \begin{cases} 2(k-p+1)\widehat{\phi}(2k-p+2)z^{k-p} & \text{if } k \geq p \\ 2(p-k+1)\widehat{\phi}(p+2)\bar{z}^{p-k} & \text{if } k < p. \end{cases}$$

Using the definition of big Hankel operator, we get the following lemma by direct calculation.

**Lemma 2.3.** *Let  $p \in \mathbb{Z}$  and  $\phi$  be a bounded radial function. Then for each  $k \in \mathbb{N}$ ,*

$$H_{e^{ip\theta}\phi}(z^k) = \begin{cases} 2(k-p+1)\widehat{\phi}(2k-p+2)\bar{z}^{k-p} & \text{if } k \geq p \\ 2(p-k+1)\widehat{\phi}(p+2)z^{p-k} & \text{if } k < p; \end{cases}$$

$$H_{e^{ip\theta}\phi}(\bar{z}^k) = \begin{cases} 2(k+p+1)\widehat{\phi}(2k+p+2)z^{k-p} & \text{if } k \geq -p \\ 2(-p-k+1)\widehat{\phi}(-p+2)\bar{z}^{-p-k} & \text{if } k < -p. \end{cases}$$

**Remark 2.4.**

- (1). For  $f \in L^\infty(\mathbb{D}, dA)$  and  $k \in \mathbb{N}$ , Lemmas 2.1 and 2.2 imply that  $UT_f(z^k) = h_f(z^k)$  and  $UT_f(\bar{z}^k) = h_f(\bar{z}^k)$ , Lemmas 2.1 and 2.3 imply that  $T_f(z^k) = H_f(\bar{z}^k)$  and  $T_f(\bar{z}^k) = H_f(z^k)$ .
- (2). For  $k, p \in \mathbb{Z}$ , Lemmas 2.1-2.3 imply that the image of  $r^{|k|}e^{ik\theta}$  by a quasihomogeneous Toeplitz (or small Hankel, or big Hankel) operator of degree  $p$  is  $\lambda_{p,k}r^{|p+k|}e^{i(p+k)\theta}$  (or  $\delta_{p,k}r^{|p+k|}e^{-i(p+k)\theta}$ , or  $\gamma_{p,k}r^{|p-k|}e^{i(p-k)\theta}$ ) for some constant  $\lambda_{p,k}$  (or  $\delta_{p,k}$ , or  $\gamma_{p,k}$ ).

### 3. Main Results

In this section, we will study the product problems of Toeplitz operators, small Hankel and big Hankel operators with quasihomogeneous symbols on the harmonic Bergman space of the unit disk. We can deduce the following result immediately by Theorem 1.1 in [6] and Remark 2.4.

**Theorem 3.1.** *Let  $k \in \mathbb{Z}$  and  $f$  be a bounded function on  $\mathbb{D}$ . Then the following assertions are equivalent:*

- (a). For each  $n \in \mathbb{N}$ , there exists  $\lambda_n \in \mathbb{C}$  such that  $H_f(r^n e^{in\theta}) = \lambda_n r^{|k-n|} e^{i(k-n)\theta}$ .
- (b).  $f$  is a quasihomogeneous function of degree  $k$ .

Using Theorem 3.1, Theorem 1.1 in [6] and Theorem 3.1 in [7], we obtain that

**Theorem 3.2.** *Let  $\phi$  and  $\psi$  be two bounded radial functions on  $\mathbb{D}$  and  $p, q \in \mathbb{Z}$ . If there exists a bounded function  $m$  satisfying one of the following conditions:*

$$T_{e^{ip\theta}\phi}H_{e^{iq\theta}\psi} = H_m, \quad (1)$$

$$h_{e^{ip\theta}\phi}T_{e^{iq\theta}\psi} = h_m, \quad (2)$$

$$H_{e^{ip\theta}\phi}h_{e^{iq\theta}\psi} = T_m, \quad (3)$$

then  $m$  is a quasihomogeneous function of degree  $p + q$ .

*Proof.* Assume (1) holds. For any  $p, q, s, k \in \mathbb{Z}$ , it follows from Remark 2.4 that

$$T_{e^{ip\theta}\phi}(r^{|k|}e^{ik\theta}) = \lambda_{k,p}r^{|k+p|}e^{i(k+p)\theta}$$

and

$$H_{e^{iq\theta}\psi}(r^{|s|}e^{is\theta}) = \gamma_{s,q}r^{|q-s|}e^{i(q-s)\theta},$$

where  $\lambda_{k,p}$  and  $\delta_{s,q}$  are two constants depending on  $k, s$ . Then one can easily to show that

$$T_{e^{ip\theta}\phi}H_{e^{iq\theta}\psi}(r^{|s|}e^{is\theta}) = T_{e^{ip\theta}\phi}(\delta_{s,q}r^{|p-s|}e^{i(p-s)\theta}) = \lambda_{q-s,p}\delta_{s,q}r^{|q-s+p|}e^{i(q-s+p)\theta}$$

for some constants  $\lambda_{q-s,p}, \delta_{s,q}$ . Hence, if  $T_{e^{ip\theta}\phi}H_{e^{iq\theta}\psi} = H_m$ , by Theorem 3.1, we obtain that  $m$  is a quasihomogeneous function of degree  $p + q$ . Similarly, using Remark 2.4, we get that

$$h_{e^{ip\theta}\phi}T_{e^{iq\theta}\psi}(r^{|s|}e^{is\theta}) = \delta_{s+q,p}\lambda_{s,q}r^{s+q+p}e^{-i(s+q+p)\theta}$$

for some constants  $\delta_{s+q,p}, \lambda_{s,q}$  and

$$H_{e^{ip\theta}\phi}h_{e^{iq\theta}\psi}(r^{|s|}e^{is\theta}) = \gamma_{-(s+q),p}\delta_{s,q}r^{s+q+p}e^{i(s+q+p)\theta}$$

for some constants  $\gamma_{-(s+q),p}, \delta_{s,q}$ . It follows that if (2) or (3) holds,  $m$  is a quasihomogeneous function of degree  $p + q$ . This completes the proof.  $\square$

Using the similar discussion, we get the following results.

**Theorem 3.3.** *Let  $\phi$  and  $\psi$  be two bounded radial functions on  $\mathbb{D}$  and  $p, q \in \mathbb{Z}$ . If there exists a bounded function  $m$  satisfying one of the following conditions:*

$$T_{e^{ip\theta}\phi}H_{e^{iq\theta}\psi} = h_m,$$

$$h_{e^{ip\theta}\phi}T_{e^{iq\theta}\psi} = H_m,$$

$$h_{e^{ip\theta}\phi}H_{e^{iq\theta}\psi} = T_m,$$

then  $m$  is a quasihomogeneous function of degree  $-(p + q)$ .

**Theorem 3.4.** Let  $\phi$  and  $\psi$  be two bounded radial functions on  $\mathbb{D}$  and  $p, q \in \mathbb{Z}$ . If there exists a bounded function  $m$  satisfying one of the following conditions:

$$H_{e^{ip\theta}\phi}T_{e^{iq\theta}\psi} = H_m,$$

$$T_{e^{ip\theta}\phi}h_{e^{iq\theta}\psi} = H_m,$$

then  $m$  is a quasihomogeneous function of degree  $p - q$ .

**Theorem 3.5.** Let  $\phi$  and  $\psi$  be two bounded radial functions on  $\mathbb{D}$  and  $p, q \in \mathbb{Z}$ . If there exists a bounded function  $m$  satisfying one of the following conditions:

$$H_{e^{ip\theta}\phi}T_{e^{iq\theta}\psi} = h_m,$$

$$T_{e^{ip\theta}\phi}h_{e^{iq\theta}\psi} = h_m,$$

then  $m$  is a quasihomogeneous function of degree  $q - p$ .

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