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Some Characterization of Fuzzy Hazard Rate Order and Fuzzy Proportional Hazard Rate Order

Research Article

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Abstract: In this paper, we have recalled some of the known stochastic orders and the shifted version of them, so discussed their relations. Also, we obtained some applications of proportional hazard rate ordering.

Keywords: Fuzzy random variables, Fuzzy Hazard rate order, Shifted fuzzy Hazard rate order.

1. Introduction

Stochastic orders have been proven to be very useful in applied probability, statistics, reliability, operation research, economics and other fields. Various types of stochastic orders and associate properties have been developed rapidly over the years. A lot of research works have done on,hazard rate and reversed hazard rate orders due to their properties and applications in the various sciences, for example hazard rate order is a wellknown and useful tool in reliability theory and reversed hazard rate order is defined via stochastic comparison of inactivity time. We can refer reader to the papers such as, Chandra and Roy [5], Gupta and Nanda (2001), Nanda and Shaked (2001), Kochar et al. (2002), Kayid and Ahmad (2004), Ahmad et al. (2005) and Shaked and Shanthikumar (2007). Ramos-Romero and Sordo-Diaz (2001) introduced a new stochastic order between two absolutely continuous random variables and called it proportional Hazard Rate order (PHR) order, which is closely related to the usual Hazard Rate order. The proportional Hazard Rate order can be used to characterize random variables whose logarithms have log-concave (log-convex) densities. Many income random variables satisfy this property and they are said to have the increasing proportional Hazard Rate order ($IPHR$) and decreasing proportional Hazard Rate Order ($DPHR$) properties. As an application, they showed that the $IPHR$ and $DPHR$ properties are sufficient conditions for the Lorenz ordering of truncated distributions.

Jarrahiferiz et al. (2010) studied some other properties of the proportional Hazard Rate Order, then extended hazard rate and reversed hazard rate orders to proportional state similar to proportional Hazard Rate order called them proportional (reversed) hazard rate orders, and studied their properties and relations.

Shifted stochastic orders that are useful tools for establishing interesting inequalities that have been introduced and studied. Also, they have been touched upon in Belzunce et al. (2001). Lillo et al. (2001) have been studied in detail four shifted stochastic orders, namely the up likelihood ratio order, the down likelihood ratio order, the up hazard rate order and the

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down hazard rate order. They have compared them and obtained some basic and closure properties of them and have shown how those can be used for stochastic comparisons of order statistics. Recently, Aboukalam and Kayid (2007) obtained some new results about shifted hazard and shifted likelihood ratio orders.

In this paper we recall the proportional state of stochastic orders and the shifted version of them and so obtained some applications of proportional Hazard Rate order.

2. Preliminaries

Definition 2.1. *Let X be a universal set. Then a fuzzy set $\tilde{A} = \{(x, \mu_A(x))/x \in X\}$ of X is defined by its membership function $\mu_{\tilde{A}} : X \to [0,1]$.*

Definition 2.2. *For each $0 \leq \alpha \leq 1$, the α-cut of a set of $\tilde{A}$ is denoted by its, $\tilde{A}_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$.*

Definition 2.3. *A fuzzy number is a fuzzy set of R such that the following conditions are satisfies:*

(1). $\tilde{A}$ is normal if there exists $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$.

(2). $\tilde{A}$ is called convex if $\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda x_2)) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$.

(3). $\tilde{A}$ is called upper semi continuous with compact support: that is for every $\epsilon > 0$, there exists $d > 0$; $|x - y| < d \Rightarrow \mu_{\tilde{A}}(x) < \mu_{\tilde{A}}(y) + \epsilon$.

(4). The α-cut of fuzzy number is closed interval denoted by $A_\alpha = [A^L_\alpha, A^U_\alpha]$, where $A^L_\alpha = \inf\{x \in R; \mu_{\tilde{A}}(x) \geq \alpha\}$ and $A^U_\alpha = \sup\{x \in R; \mu_{\tilde{A}}(x) \geq \alpha\}$.

(5). If $\tilde{A}$ is closed and bounded fuzzy number with A^L_α, A^U_α and its membership function is strictly increasing on $[A^L_1, A^U_1]$ then $\tilde{A}$ is called canonical fuzzy number.

Definition 2.4. *A fuzzy random variable is a fuzzy set consisting of a membership function and a basic set of underlying variables. A fuzzy random variable X is a map $X : O \to F(R)$ satisfying the following conditions*

(1). For each $\alpha \in (0,1]$ both X^L_α and X^U_α defined as $X^L_\alpha(w)(x) = \inf\{x \in R : X(w)(x) \leq \alpha\}$ and $X^U_\alpha(w)(x) = \sup\{x \in R : X(w)(x) \leq \alpha\}$ are finite real valued random variables defined on such (O, A, P) that the mathematical expectations EX^L_α and EX^U_α exist.

(2). For each $w \in O$ and $\alpha \in (0,1]$, $X^L_\alpha(w)(x) \geq \alpha$ and $X^U_\alpha(w)(x) \geq \alpha$.

Definition 2.5. *A fuzzy valued mapping $X : O \to F^m_o(R) = F^m_o(R) \times \cdots \times F^m_o(R)$ represented by $X(w) = (x(1,w) \times \cdots \times x(m,w)$ is called the fuzzy random vectors if for each k, $1 \leq k \leq m$, $x(k,w)$ is a fuzzy random variable.*

Definition 2.6. *$X(w)$ is a fuzzy random variable if and only if , $X(w) = [X^L_\alpha(w), X^U_\alpha(w)]$, where $X^L_\alpha(w)$ and $X^U_\alpha(w)$ are both random variables for each $\alpha \in (0,1]$ and $X(w) = \bigcup_{\alpha \in (0,1]} \alpha X_\alpha(w)$.*

Definition 2.7. *The a- cut of distribution function F of fuzzy random variable $\tilde{X}$ is defined by*

$$D_a = \left[\min\left\{\min_{\alpha \leq \beta \leq 1} F\left(\tilde{x}^L_\beta\right), \min_{\alpha \leq \beta \leq 1} F\left(\tilde{x}^U_\beta\right)\right\}. \max\left\{\max_{a \leq \beta \leq 1} F\left(\tilde{x}^L_\beta\right), \max_{\alpha \leq \beta \leq 1} F\left(\tilde{x}^U_\beta\right)\right\}\right]$$

Definition 2.8. *The membership function of the probability distribution function of $\tilde{x}$, denoted as $\tilde{F}(\tilde{x})$. It is defined as*

$$\mu_{\tilde{F}(x)}(r) = \sup_{\alpha \in (0,1]} \alpha_1 D_\alpha(r).$$

Definition 2.9. *The α-cut of probability density functions $f(x)$, where $x \in [X_\alpha^L, X_\alpha^U]$ is given by*

$$A_a = \left[\min\left\{\min_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^L\right), \min_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^U\right)\right\}. \max\left\{\max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^U\right)\right\}\right]$$

Definition 2.10. *The membership function of the probability distribution function of $\tilde{x}$, denoted as $\tilde{f}(\tilde{x})$ is given by*

$$\mu_{\tilde{f}(x)}(r) = \sup_{\alpha\in(0,1]\bigcap Q} \alpha_1 A_\alpha(r)$$

Definition 2.11 (Fuzzy Hazard Rate Order)**.** *Let X and Y are two non negative fuzzy random variables with continuous distribution functions and with Hazard rate functions $\tilde{r}(\tilde{x})$ and $\widetilde{q}(\tilde{x})$ respectively, then X is smaller than Y in Hazard rate order. Denoted as $X \leq_{FHR} Y$. If*

$$\frac{\min\left\{\min_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^L\right), \min_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^L\right), \min_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^U\right)\right\}} \geq \frac{\min\left\{\min_{\alpha\leq\beta\leq1} g\left(\tilde{y}_\beta^L\right), \min_{\alpha\leq\beta\leq1} g\left(\tilde{y}_\beta^U\right)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{y}_\beta^L\right), \min_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{y}_\beta^U\right)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^U\right)\right\}} \geq \frac{\max\left\{\max_{\alpha\leq\beta\leq1} g\left(\tilde{y}_\beta^L\right), \max_{\alpha\leq\beta\leq1} g\left(\tilde{y}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{y}_\beta^L\right), \max_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{y}_\beta^U\right)\right\}}$$

For each $\alpha, \beta \in (0,1] \cap Q$, where $\overline{F}, f$ are the survival and density functions of X respectively and $\overline{G}$, g are the survival and density functions of Y respectively.

Definition 2.12 (Reversed Fuzzy Hazard Rate Order)**.** *Let X and Y are two non negative fuzzy random variables with continuous distribution functions and with Reversed Hazard rate functions $\tilde{r}(\tilde{x})$ and $\widetilde{q}(\tilde{x})$ respectively, then X is smaller than Y in Reversed Hazard rate order. Denoted as $X \geq_{RFHR} Y$. If*

$$\frac{\min\left\{\min_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^L\right), \min_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^L\right), \min_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^U\right)\right\}} \leq \frac{\min\left\{\min_{\alpha\leq\beta\leq1} g\left(\tilde{x}_\beta^L\right), \min_{\alpha\leq\beta\leq1} g\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{x}_\beta^L\right), \min_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{x}_\beta^U\right)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^U\right)\right\}} \leq \frac{\max\left\{\max_{\alpha\leq\beta\leq1} g\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} g\left(\tilde{x}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{x}_\beta^U\right)\right\}}$$

For each $\alpha, \beta \in (0,1] \cap Q$, where $\overline{F}, f$ are the survival and density functions of X respectively and $\overline{G}$, g are the survival and density functions of Y respectively.

3. Shifted Proportional Fuzzy Hazard Rate Order

Definition 3.1 (Up Proportional Fuzzy Hazard Rate Order)**.** *Let X and Y are two non negative fuzzy random variables with continuous distribution functions and with Up proportional Hazard rate functions $\tilde{r}(\tilde{x})$ and $\widetilde{q}(\tilde{x})$ respectively, then X is smaller than Y in Hazard rate order. Denoted as $X \leq_{upfhr} Y$. If*

$$\frac{min\left\{\min_{\alpha\leq\beta\leq1} [f(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geq t], \min_{\alpha\leq\beta\leq1} [f(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geq t]\right\}}{\min\left\{\min_{\alpha\leq\beta\leq1} [\overline{F}(\tilde{x}_\alpha^L - t)/\tilde{x}_\alpha^L \geq t], \min_{\alpha\leq\beta\leq1} [\overline{F}(\tilde{x}_\alpha^U - t)/\tilde{x}_\alpha^U \geq t]\right\}} \geq \frac{\min\left\{\min_{\alpha\leq\beta\leq1} g\left(\tilde{y}_\beta^L\right), \min_{\alpha\leq\beta\leq1} g\left(\tilde{y}\right)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{y}_\beta^L\right), \min_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{y}_\beta^U\right)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leq\beta\leq 1}[f(\tilde{x}^L_\alpha - t)/\tilde{x}^L_\alpha \geq t],\ \max_{\alpha\leq\beta\leq 1}[f(\tilde{x}^U_\alpha - t)/\tilde{x}^U_\alpha \geq t]\right\}}{\max\left\{\max_{\alpha\leq\beta\leq 1}[\overline{F}(\tilde{x}^L_\alpha - t)/\tilde{x}^L_\alpha \geq t],\ \max_{\alpha\leq\beta\leq 1}[\overline{F}(\tilde{x}^U_\alpha - t)/\tilde{x}^U_\alpha \geq t]\right\}} \geq \frac{\max\left\{\max_{\alpha\leq\beta\leq 1} g\left(\tilde{y}^L_\beta\right),\ \max_{\alpha\leq\beta\leq 1} g\left(\tilde{y}\right)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq 1} \overline{G}\left(\tilde{y}^L_\beta\right),\ \max_{\alpha\leq\beta\leq 1} \overline{G}\left(\tilde{y}^U_\beta\right)\right\}}$$

For each $\alpha,\beta \in (0,1]\cap Q$, where $\overline{F}$, f are the survival and density functions of X respectively and $\overline{G}$, g are the survival and density functions of Y respectively.

Definition 3.2 (Down Proportional Fuzzy Hazard Rate Order)**.** *Let X and Y are two non negative fuzzy random variables with continuous distribution functions and with Up proportional Hazard rate functions $\tilde{r}(\tilde{x})$ and $\widetilde{q}(\tilde{x})$ respectively, then X is smaller than Y in Hazard rate order. Denoted as $X \leq_{dpfhr} Y$. If*

$$\frac{\min\left\{\min_{\alpha\leq\beta\leq 1} f\left(\tilde{x}^L_\beta\right),\ \min_{\alpha\leq\beta\leq 1} f\left(\tilde{x}^U_\beta\right)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq 1} \overline{F}\left(\tilde{x}^L_\beta\right),\ \min_{\alpha\leq\beta\leq 1} \overline{F}\left(\tilde{x}^U_\beta\right)\right\}} \leq \frac{\min\left\{\min_{\alpha\leq\beta\leq 1}[g(\tilde{y}^L_\alpha - t)/\tilde{y}^L_\alpha \geq t],\ \min_{\alpha\leq\beta\leq 1}[g(\tilde{y}^U_\alpha - t)/\tilde{y}^U_\alpha \geq t]\right\}}{\min\left\{\min_{\alpha\leq\beta\leq 1}[\overline{G}(\tilde{y}^L_\alpha - t)/\tilde{y}^L_\alpha \geq t],\ \min_{\alpha\leq\beta\leq 1}[\overline{G}(\tilde{y}^U_\alpha - t)/\tilde{y}^U_\alpha \geq t]\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leq\beta\leq 1} f\left(\tilde{x}^L_\beta\right),\ \max_{\alpha\leq\beta\leq 1} f\left(\tilde{x}^U_\beta\right)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq 1} \overline{F}\left(\tilde{x}^L_\beta\right),\ \max_{\alpha\leq\beta\leq 1} \overline{F}\left(\tilde{x}^U_\beta\right)\right\}} \leq \frac{max\left\{\max_{\alpha\leq\beta\leq 1}[g(\tilde{y}^L_\alpha - t)/\tilde{y}^L_\alpha \geq t],\ \max_{\alpha\leq\beta\leq 1}[g(\tilde{y}^U_\alpha - t)/\tilde{y}^U_\alpha \geq t]\right\}}{\max\left\{\max_{\alpha\leq\beta\leq 1}[\overline{G}(\tilde{y}^L_\alpha - t)/\tilde{y}^L_\alpha \geq t],\ \max_{\alpha\leq\beta\leq 1}[\overline{G}(\tilde{y}^U_\alpha - t)/\tilde{y}^U_\alpha \geq t]\right\}}$$

For each $\alpha,\beta \in (0,1]\cap Q$, where $\overline{F}$, f are the survival and density functions of X respectively and $\overline{G}$, g are the survival and density functions of Y respectively.

Definition 3.3 (Shifted Fuzzy Hazard Rate Order)**.** *Let X and Y are two non negative fuzzy random variables with continuous distribution functions and with Shifted fuzzy Hazard rate functions $\tilde{r}(\tilde{x})$ and $\widetilde{q}(\tilde{x})$ respectively, then X is smaller than Y in Hazard rate order. Denoted as $X \leq_{SFHR\uparrow} Y$. If*

$$\frac{\min\left\{\min_{\alpha\leq\beta\leq 1} f\left(\alpha+\tilde{x}^L_\alpha\right) g\left(\tilde{x}^U_\alpha\right),\ \min_{\alpha\leq\beta\leq 1} f\left(\alpha+\tilde{x}^U_\beta\right) g(\tilde{x}^L_\alpha)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq 1} \overline{F}\left(\alpha+\tilde{x}^L_\alpha\right) \overline{G}\left(\tilde{x}^U_\alpha\right),\ \min_{\alpha\leq\beta\leq 1} \overline{F}\left(\alpha+\tilde{x}^U_\beta\right) \overline{G}(\tilde{x}^L_\alpha)\right\}} \geq \frac{\min\left\{\min_{\alpha\leq\beta\leq 1} g\left(\alpha+\tilde{x}^U_\alpha\right) f\left(\tilde{x}^L_\alpha\right),\ \min_{\alpha\leq\beta\leq 1} g\left(\alpha+\tilde{x}^L_\alpha\right) f(\tilde{x}^U_\alpha)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq 1} \overline{G}\left(\alpha+\tilde{x}^U_\alpha\right) \overline{F}\left(\tilde{x}^L_\alpha\right),\ \min_{\alpha\leq\beta\leq 1} \overline{G}\left(\alpha+\tilde{x}^L_\beta\right) \overline{F}(\tilde{x}^U_\alpha)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leq\beta\leq 1} f\left(\alpha+\tilde{x}^L_\alpha\right) g\left(\tilde{x}^U_\alpha\right),\ \max_{\alpha\leq\beta\leq 1} f\left(\alpha+\tilde{x}^U_\beta\right) g(\tilde{x}^L_\alpha)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq 1} \overline{F}\left(\alpha+\tilde{x}^L_\alpha\right) \overline{G}\left(\tilde{x}^U_\alpha\right),\ \max_{\alpha\leq\beta\leq 1} \overline{F}\left(\alpha+\tilde{x}^U_\beta\right) \overline{G}(\tilde{x}^L_\alpha)\right\}} \geq \frac{\max\left\{\max_{\alpha\leq\beta\leq 1} g\left(\alpha+\tilde{x}^U_\alpha\right) f\left(\tilde{x}^L_\alpha\right),\ \max_{\alpha\leq\beta\leq 1} g\left(\alpha+\tilde{x}^L_\alpha\right) f(\tilde{x}^U_\alpha)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq 1} \overline{G}\left(\alpha+\tilde{x}^U_\alpha\right) \overline{F}\left(\tilde{x}^L_\alpha\right),\ \max_{\alpha\leq\beta\leq 1} \overline{G}\left(\alpha+\tilde{x}^L_\alpha\right) \overline{F}(\tilde{x}^U_\alpha)\right\}}$$

for each $\alpha,\beta \in (0,1]\cap Q$, where $\overline{F}$, f are the survival and density functions of X respectively and $\overline{G}$, g are the survival and density functions of Y respectively.

Definition 3.4 (Shifted Proportional Fuzzy Hazard Rate Order)**.** *Let X and Y are two non negative fuzzy random variables with continuous distribution functions and with Shifted fuzzy Hazard rate functions $\tilde{r}(\tilde{x})$ and $\widetilde{q}(\tilde{x})$ respectively, then X is smaller than Y in Hazard rate order. Denoted as $X \leq_{PFHR\uparrow} Y$. If,*

$$\frac{\min\left\{\min_{\alpha\leq\beta\leq 1} f\left(\alpha+\tilde{x}^L_\alpha\right) g\left(\lambda\tilde{x}^U_\alpha\right),\ \min_{\alpha\leq\beta\leq 1} f\left(\alpha+\tilde{x}^U_\beta\right) g(\widetilde{\lambda x}^L_\alpha)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq 1} \overline{F}\left(\alpha+\tilde{x}^L_\alpha\right) \overline{G}\left(\lambda\tilde{x}^U_\alpha\right),\ \min_{\alpha\leq\beta\leq 1} \overline{F}\left(\alpha+\tilde{x}^U_\beta\right) \overline{G}(\widetilde{\lambda x}^L_\alpha)\right\}} \geq \frac{\min\left\{\min_{\alpha\leq\beta\leq 1} g\left(\alpha+\tilde{x}^U_\alpha\right) f\left(\lambda\tilde{x}^L_\alpha\right),\ \min_{\alpha\leq\beta\leq 1} g\left(\alpha+\tilde{x}^L_\alpha\right) f(\lambda\tilde{x}^U_\alpha)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq 1} \overline{G}\left(\alpha+\tilde{x}^U_\alpha\right) \overline{F}\left(\lambda\tilde{x}^L_\alpha\right),\ \min_{\alpha\leq\beta\leq 1} \overline{G}\left(\alpha+\tilde{x}^L_\beta\right) \overline{F}(\lambda\tilde{x}^U_\alpha)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leq\beta\leq 1} f\left(\alpha+\tilde{x}^L_\alpha\right) g\left(\widetilde{\lambda x}^U_\alpha\right),\ \max_{\alpha\leq\beta\leq 1} f\left(\alpha+\tilde{x}^U_\beta\right) g(\lambda\tilde{x}^L_\alpha)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq 1} \overline{F}\left(\alpha+\tilde{x}^L_\alpha\right) \overline{G}\left(\lambda\tilde{x}^U_\alpha\right),\ \max_{\alpha\leq\beta\leq 1} \overline{F}\left(\alpha+\tilde{x}^U_\beta\right) \overline{G}(\lambda\tilde{x}^L_\alpha)\right\}} \geq \frac{\max\left\{\max_{\alpha\leq\beta\leq 1} g\left(\alpha+\tilde{x}^U_\alpha\right) f\left(\lambda\tilde{x}^L_\alpha\right),\ \max_{\alpha\leq\beta\leq 1} g\left(\alpha+\tilde{x}^L_\alpha\right) f(\lambda\tilde{x}^U_\alpha)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq 1} \overline{G}\left(\alpha+\tilde{x}^U_\alpha\right) \overline{F}\left(\lambda\tilde{x}^L_\alpha\right),\ \max_{\alpha\leq\beta\leq 1} \overline{G}\left(\alpha+\tilde{x}^L_\beta\right) \overline{F}(\lambda\tilde{x}^U_\alpha)\right\}}$$

for each $\alpha,\beta \in (0,1]\cap Q$, where $\overline{F}$, f are the survival and density functions of X respectively and $\overline{G}$, g are the survival and density functions of Y respectively.

Definition 3.5 (Increasing Fuzzy Proportional Hazard Rate Order)**.** *Let X has the increasing fuzzy proportional Hazard rate order property, $X\epsilon_{IFPHR}$ if,*

$$\frac{\min\left\{\min_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^L\right), \min_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^L\right), \min_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^U\right)\right\}} \geq \frac{\min\left\{\min_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^L\right), \min_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^L\right), \min_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^U\right)\right\}} \geq \frac{\max\left\{\max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

Definition 3.6 (Up Increasing Fuzzy Proportional Hazard Rate Order)**.** *A continuous non negative fuzzy random variables X admits Up increasing fuzzy proportional Hazard rate order property denoted by $X\epsilon_{UIPHRO}$ if,*

$$\frac{\min\left\{\min_{\alpha\leq\beta\leq1} [f(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geq t], \min_{\alpha\leq\beta\leq1} [f(\tilde{x}_\alpha^U-t)/\tilde{x}_\alpha^U\geq t]\right\}}{\min\left\{\min_{\alpha\leq\beta\leq1} [\overline{F}(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geq t], \min_{\alpha\leq\beta\leq1} [\overline{F}(\tilde{x}_\alpha^U-t)/\tilde{x}_\alpha^U\geq t]\right\}} \geq \frac{\min\left\{\min_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^L\right), \min_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^U\right)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{y}_\beta^L\right), \min_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leq\beta\leq1} [f(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geq t], \max_{\alpha\leq\beta\leq1} [f(\tilde{x}_\alpha^U-t)/\tilde{x}_\alpha^U\geq t]\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} [\overline{F}(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geq t], \max_{\alpha\leq\beta\leq1} [\overline{F}(\tilde{x}_\alpha^U-t)/\tilde{x}_\alpha^U\geq t]\right\}} \geq \frac{\max\left\{\max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^U\right)\right\}}$$

Theorem 3.7. *Let X and Y be two absolutely continuous non negative fuzzy random variables then,*

(1). $X\leq_{UPFHR} Y$

$$\frac{\min\left\{\min_{\alpha\leq\beta\leq1} [f(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geq t], \min_{\alpha\leq\beta\leq1} [f(\tilde{x}_\alpha^U-t)/\tilde{x}_\alpha^U\geq t]\right\}}{\min\left\{\min_{\alpha\leq\beta\leq1} [\overline{F}(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geq t], \min_{\alpha\leq\beta\leq1} [\overline{F}(\tilde{x}_\alpha^U-t)/\tilde{x}_\alpha^U\geq t]\right\}} \geq \frac{\min\left\{\min_{\alpha\leq\beta\leq1} g\left(\tilde{y}_\beta^L\right), \min_{\alpha\leq\beta\leq1} g\left(\tilde{y}\right)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{y}_\beta^L\right), \min_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{y}_\beta^U\right)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leq\beta\leq1} [f(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geq t], \max_{\alpha\leq\beta\leq1} [f(\tilde{x}_\alpha^U-t)/\tilde{x}_\alpha^U\geq t]\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} [\overline{F}(\tilde{x}_\alpha^L-t)/\tilde{x}_\alpha^L\geq t], \max_{\alpha\leq\beta\leq1} [\overline{F}(\tilde{x}_\alpha^U-t)/\tilde{x}_\alpha^U\geq t]\right\}} \geq \frac{max\left\{\max_{\alpha\leq\beta\leq1} g\left(\tilde{y}_\beta^L\right), \max_{\alpha\leq\beta\leq1} g\left(\tilde{y}\right)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{y}_\beta^L\right), \max_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{y}_\beta^U\right)\right\}}$$

(2). If either X or Y are both have a log concave density then, $X\leq_{SFHR} Y$ then

$$\frac{\min\left\{\min_{\alpha\leq\beta\leq1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\tilde{x}_\alpha^U\right), \min_{\alpha\leq\beta\leq1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\tilde{x}_\alpha^L)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right) \overline{G}\left(\tilde{x}_\alpha^U\right), \min_{\alpha\leq\beta\leq1} \overline{F}\left(\alpha+\tilde{x}_\beta^U\right) \overline{G}(\tilde{x}_\alpha^L)\right\}} \geq \frac{\min\left\{\min_{\alpha\leq\beta\leq1} g\left(\alpha+\tilde{x}_\alpha^U\right) f\left(\tilde{x}_\alpha^L\right), \min_{\alpha\leq\beta\leq1} g\left(\alpha+\tilde{x}_\alpha^L\right) f(\tilde{x}_\alpha^U)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq1} \overline{G}\left(\alpha+\tilde{x}_\alpha^U\right) \overline{F}\left(\tilde{x}_\alpha^L\right), \min_{\alpha\leq\beta\leq1} \overline{G}\left(\alpha+\tilde{x}_\beta^L\right) \overline{F}(\tilde{x}_\alpha^U)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leq\beta\leq1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\tilde{x}_\alpha^U\right), \max_{\alpha\leq\beta\leq1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\tilde{x}_\alpha^L)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right) \overline{G}\left(\tilde{x}_\alpha^U\right), \max_{\alpha\leq\beta\leq1} \overline{F}\left(\alpha+\tilde{x}_\beta^U\right) \overline{G}(\tilde{x}_\alpha^L)\right\}} \geq \frac{\max\left\{\max_{\alpha\leq\beta\leq1} g\left(\alpha+\tilde{x}_\alpha^U\right) f\left(\tilde{x}_\alpha^L\right), \max_{\alpha\leq\beta\leq1} g\left(\alpha+\tilde{x}_\alpha^L\right) f(\tilde{x}_\alpha^U)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} \overline{G}\left(\alpha+\tilde{x}_\alpha^U\right) \overline{F}\left(\tilde{x}_\alpha^L\right), \max_{\alpha\leq\beta\leq1} \overline{G}\left(\alpha+\tilde{x}_\beta^L\right) \overline{F}(\tilde{x}_\alpha^U)\right\}}$$

implies

$$\frac{\max\left\{\max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^U\right)\right\}} \geq \frac{\max\left\{\max_{\alpha\leq\beta\leq1} g\left(\tilde{y}_\beta^L\right), \max_{\alpha\leq\beta\leq1} g\left(\tilde{y}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{y}_\beta^L\right), \max_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{y}_\beta^U\right)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} f\left(\tilde{x}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^L\right), \max_{\alpha\leq\beta\leq1} \overline{F}\left(\tilde{x}_\beta^U\right)\right\}} \geq \frac{\max\left\{\max_{\alpha\leq\beta\leq1} g\left(\tilde{y}_\beta^L\right), \max_{\alpha\leq\beta\leq1} g\left(\tilde{y}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{y}_\beta^L\right), \max_{\alpha\leq\beta\leq1} \overline{G}\left(\tilde{y}_\beta^U\right)\right\}}$$

(3). $X\leq_{SFHR} Y$ iff there exist non negative fuzzy random variables Z such that with log concave density such that $X\leq_{SFHR} Z\leq_{SFHR} Y$, where Z is arbitrary random variable between $\tilde{x}_\alpha^U$ and $\tilde{y}_\alpha^L$.

Theorem 3.8. *Let X and Y be two absolutely continuous non negative fuzzy random variables then,*

(1). $X \leq_{UPFHR} Y$, *if*

$$\frac{\min\left\{\min\limits_{\alpha\leq\beta\leq 1}[f(\lambda\tilde{x}^L_\alpha - t)/\lambda\tilde{x}^L_\alpha \geq t],\ \min\limits_{\alpha\leq\beta\leq 1}[f(\lambda\tilde{x}^U_\alpha - t)/\lambda\tilde{x}^U_\alpha \geq t]\right\}}{\min\left\{\min\limits_{\alpha\leq\beta\leq 1}[\overline{F}(\widetilde{\lambda x}^L_\alpha - t)/\widetilde{\lambda x}^L_\alpha \geq t],\ \min\limits_{\alpha\leq\beta\leq 1}[\overline{F}(\lambda\tilde{x}^U_\alpha - t)/\widetilde{\lambda x}^U_\alpha \geq t]\right\}} \geq \frac{\min\left\{\min\limits_{\alpha\leq\beta\leq 1} g\left(\tilde{y}^L_\beta\right),\ \min\limits_{\alpha\leq\beta\leq 1} g\left(\tilde{y}\right)\right\}}{\min\left\{\min\limits_{\alpha\leq\beta\leq 1}\overline{G}\left(\tilde{y}^L_\beta\right),\ \min\limits_{\alpha\leq\beta\leq 1}\overline{G}\left(\tilde{y}^U_\beta\right)\right\}}$$

and

$$\frac{\max\left\{\max\limits_{\alpha\leq\beta\leq 1}[f(\lambda\tilde{x}^L_\alpha - t)/\lambda\tilde{x}^L_\alpha \geq t],\ \max\limits_{\alpha\leq\beta\leq 1}[f(\lambda\tilde{x}^U_\alpha - t)/\lambda\tilde{x}^U_\alpha \geq t]\right\}}{\max\left\{\max\limits_{\alpha\leq\beta\leq 1}[\overline{F}(\widetilde{\lambda x}^L_\alpha - t)/\lambda\tilde{x}^L_\alpha \geq t],\ \max\limits_{\alpha\leq\beta\leq 1}[\overline{F}(\lambda\tilde{x}^U_\alpha - t)/\widetilde{\lambda x}^U_\alpha \geq t]\right\}} \geq \frac{\max\left\{\max\limits_{\alpha\leq\beta\leq 1} g\left(\tilde{y}^L_\beta\right),\ \max\limits_{\alpha\leq\beta\leq 1} g\left(\tilde{y}\right)\right\}}{\max\left\{\max\limits_{\alpha\leq\beta\leq 1}\overline{G}\left(\tilde{y}^L_\beta\right),\ \max\limits_{\alpha\leq\beta\leq 1}\overline{G}\left(\tilde{y}^U_\beta\right)\right\}}$$

For all $t \geq 0$.

(2). $X \leq_{DPFHR} Y$, *if*

$$\frac{\min\left\{\min\limits_{\alpha\leq\beta\leq 1} f\left(\tilde{x}^L_\beta\right),\ \min\limits_{\alpha\leq\beta\leq 1} f\left(\tilde{x}^U_\beta\right)\right\}}{\min\left\{\min\limits_{\alpha\leq\beta\leq 1}\overline{F}\left(\tilde{x}^L_\beta\right),\ \min\limits_{\alpha\leq\beta\leq 1}\overline{F}\left(\tilde{x}^U_\beta\right)\right\}} \leq \frac{\min\left\{\min\limits_{\alpha\leq\beta\leq 1}[g(\tilde{y}^L_\alpha - t)/\tilde{y}^L_\alpha \geq t],\ \min\limits_{\alpha\leq\beta\leq 1}[g(\tilde{y}^U_\alpha - t)/\tilde{y}^U_\alpha \geq t]\right\}}{\min\left\{\min\limits_{\alpha\leq\beta\leq 1}[\overline{G}(\tilde{y}^L_\alpha - t)/\tilde{y}^L_\alpha \geq t],\ \min\limits_{\alpha\leq\beta\leq 1}[\overline{G}(\tilde{y}^U_\alpha - t)/\tilde{y}^U_\alpha \geq t]\right\}}$$

and

$$\frac{\max\left\{\max\limits_{\alpha\leq\beta\leq 1} f\left(\tilde{x}^L_\beta\right),\ \max\limits_{\alpha\leq\beta\leq 1} f\left(\tilde{x}^U_\beta\right)\right\}}{\max\left\{\max\limits_{\alpha\leq\beta\leq 1}\overline{F}\left(\tilde{x}^L_\beta\right),\ \max\limits_{\alpha\leq\beta\leq 1}\overline{F}\left(\tilde{x}^U_\beta\right)\right\}} \leq \frac{\max\left\{\max\limits_{\alpha\leq\beta\leq 1}[g(\tilde{y}^L_\alpha - t)/\tilde{y}^L_\alpha \geq t],\ \max\limits_{\alpha\leq\beta\leq 1}[g(\tilde{y}^U_\alpha - t)/\tilde{y}^U_\alpha \geq t]\right\}}{\max\left\{\max\limits_{\alpha\leq\beta\leq 1}[\overline{G}(\tilde{y}^L_\alpha - t)/\tilde{y}^L_\alpha \geq t],\ \max\limits_{\alpha\leq\beta\leq 1}[\overline{G}(\tilde{y}^U_\alpha - t)/\tilde{y}^U_\alpha \geq t]\right\}}$$

For all $t \geq 0$.

Theorem 3.9. *Let X and Y be two absolutely continuous non negative fuzzy random variables then,*

(1). $X \leq_{UPFHR} Y$, *if*

$$\frac{\min\left\{\min\limits_{\alpha\leq\beta\leq 1} f\left(\alpha+\tilde{x}^L_\alpha\right) g\left(\lambda\tilde{x}^U_\alpha\right),\ \min\limits_{\alpha\leq\beta\leq 1} f\left(\alpha+\tilde{x}^U_\beta\right) g(\lambda\tilde{x}^L_\alpha)\right\}}{\min\left\{\min\limits_{\alpha\leq\beta\leq 1}\overline{F}\left(\alpha+\tilde{x}^L_\alpha\right)\overline{G}\left(\lambda\tilde{x}^U_\alpha\right),\ \min\limits_{\alpha\leq\beta\leq 1}\overline{F}\left(\alpha+\tilde{x}^U_\beta\right)\overline{G}(\widetilde{\lambda x}^L_\alpha)\right\}} \geq \frac{\min\left\{\min\limits_{\alpha\leq\beta\leq 1} g\left(\alpha+\tilde{x}^U_\alpha\right) f\left(\lambda\tilde{x}^L_\alpha\right),\ \min\limits_{\alpha\leq\beta\leq 1} g\left(\alpha+\tilde{x}^L_\alpha\right) f(\lambda\tilde{x}^U_\alpha)\right\}}{\min\left\{\min\limits_{\alpha\leq\beta\leq 1}\overline{G}\left(\alpha+\tilde{x}^U_\alpha\right)\overline{F}\left(\lambda\tilde{x}^L_\alpha\right),\ \min\limits_{\leq\beta\leq 1}\overline{G}\left(\alpha+\tilde{x}^L_\beta\right)\overline{F}(\lambda\tilde{x}^U_\alpha)\right\}}$$

and

$$\frac{\max\left\{\max\limits_{\alpha\leq\beta\leq 1} f\left(\alpha+\tilde{x}^L_\alpha\right) g\left(\widetilde{\lambda x}^U_\alpha\right),\ \max\limits_{\alpha\leq\beta\leq 1} f\left(\alpha+\tilde{x}^U_\beta\right) g(\lambda\tilde{x}^L_\alpha)\right\}}{\max\left\{\max\limits_{\alpha\leq\beta\leq 1}\overline{F}\left(\alpha+\tilde{x}^L_\alpha\right)\overline{G}\left(\lambda\tilde{x}^U_\alpha\right),\ \max\limits_{\alpha\leq\beta\leq 1}\overline{F}\left(\alpha+\tilde{x}^U_\beta\right)\overline{G}(\lambda\tilde{x}^L_\alpha)\right\}} \geq \frac{\max\left\{\max\limits_{\alpha\leq\beta\leq 1} g\left(\alpha+\tilde{x}^U_\alpha\right) f\left(\lambda\tilde{x}^L_\alpha\right),\ \max\limits_{\alpha\leq\beta\leq 1} g\left(\alpha+\tilde{x}^L_\alpha\right) f(\lambda\tilde{x}^U_\alpha)\right\}}{\max\left\{\max\limits_{\alpha\leq\beta\leq 1}\overline{G}\left(\alpha+\tilde{x}^U_\alpha\right)\overline{F}\left(\lambda\tilde{x}^L_\alpha\right),\ \max\limits_{\alpha\leq\beta\leq 1}\overline{G}\left(\alpha+\tilde{x}^L_\beta\right)\overline{F}(\lambda\tilde{x}^U_\alpha)\right\}}$$

$x \in [X^L_\alpha - a, X^U_\beta - a] \cup [\frac{Y^L_\alpha}{\lambda}, \frac{Y^U_\beta}{\lambda}]$ *for all* $a \geq 0$.

(2). $X \leq_{DPFHR} Y$, *if*

$$\frac{\min\left\{\min\limits_{\alpha\leq\beta\leq 1} g\left(\alpha+\lambda\tilde{x}^U_\alpha\right) f\left(\tilde{x}^L_\alpha\right),\ \min\limits_{\alpha\leq\beta\leq 1} g\left(\alpha+\lambda\tilde{x}^L_\alpha\right) f(\tilde{x}^U_\beta)\right\}}{\min\left\{\min\limits_{\alpha\leq\beta\leq 1}\overline{G}\left(\alpha+\lambda\tilde{x}^U_\alpha\right)\overline{F}\left(\tilde{x}^L_\alpha\right),\ \min\limits_{\alpha\leq\beta\leq 1}\overline{G}\left(\alpha+\lambda\tilde{x}^L_\alpha\right)\overline{F}(\tilde{x}^U_\beta)\right\}} \geq \frac{\min\left\{\min\limits_{\alpha\leq\beta\leq 1} f\left(\alpha+\lambda\tilde{x}^L_\alpha\right) g\left(\tilde{x}^U_\alpha\right),\ \min\limits_{\alpha\leq\beta\leq 1} f\left(\alpha+\lambda\tilde{x}^U_\beta\right) g(\tilde{x}^L_\alpha)\right\}}{\min\left\{\min\limits_{\alpha\leq\beta\leq 1}\overline{F}\left(\alpha+\lambda\tilde{x}^L_\alpha\right)\overline{G}\left(\tilde{x}^U_\alpha\right),\ \min\limits_{\alpha\leq\beta\leq 1}\overline{F}\left(\alpha+\widetilde{\lambda x}^U_\beta\right)\overline{G}(\tilde{x}^L_\alpha)\right\}}$$

and

$$\frac{\max\left\{\max\limits_{\alpha\leq\beta\leq 1} g\left(\alpha+\lambda\tilde{x}^U_\alpha\right) f\left(\tilde{x}^L_\alpha\right),\ \max\limits_{\alpha\leq\beta\leq 1} g\left(\alpha+\lambda\tilde{x}^L_\alpha\right) f(\tilde{x}^U_\beta)\right\}}{\max\left\{\max\limits_{\alpha\leq\beta\leq 1}\overline{G}\left(\alpha+\lambda\tilde{x}^U_\alpha\right)\overline{F}\left(\tilde{x}^L_\alpha\right),\ \max\limits_{\alpha\leq\beta\leq 1}\overline{G}\left(\alpha+\lambda\tilde{x}^L_\alpha\right)\overline{F}(\tilde{x}^U_\beta)\right\}} \geq \frac{\max\left\{\max\limits_{\alpha\leq\beta\leq 1} f\left(\alpha+\lambda\tilde{x}^L_\alpha\right) g\left(\tilde{x}^U_\alpha\right),\ \max\limits_{\alpha\leq\beta\leq 1} f\left(\alpha+\lambda\tilde{x}^U_\beta\right) g(\tilde{x}^L_\alpha)\right\}}{\max\left\{\max\limits_{\alpha\leq\beta\leq 1}\overline{F}\left(\alpha+\lambda\tilde{x}^L_\alpha\right)\overline{G}\left(\tilde{x}^U_\alpha\right),\ \max\limits_{\alpha\leq\beta\leq 1}\overline{F}\left(\alpha+\lambda\tilde{x}^U_\beta\right)\overline{G}(\tilde{x}^L_\alpha)\right\}}$$

Is increasing in $x \geq 0$ *for all* $a \geq 0$.

(3). $X \leq_{UPFHR} Y$ *and* $X \leq_{DPFHR} Y$,

$$\frac{\min\left\{\min\limits_{\alpha\leq\beta\leq 1} f\left(\alpha+\tilde{x}^L_\alpha\right) g\left(\tilde{x}^U_\alpha\right),\ \min\limits_{\alpha\leq\beta\leq 1} f\left(\alpha+\tilde{x}^U_\beta\right) g(\tilde{x}^L_\alpha)\right\}}{\min\left\{\min\limits_{\alpha\leq\beta\leq 1}\overline{F}\left(\alpha+\tilde{x}^L_\alpha\right)\overline{G}\left(\tilde{x}^U_\alpha\right),\ \min\limits_{\alpha\leq\beta\leq 1}\overline{F}\left(\alpha+\tilde{x}^U_\beta\right)\overline{G}(\tilde{x}^L_\alpha)\right\}} \geq \frac{\min\left\{\min\limits_{\alpha\leq\beta\leq 1} g\left(\alpha+\tilde{x}^U_\alpha\right) f\left(\tilde{x}^L_\alpha\right),\ \min\limits_{\alpha\leq\beta\leq 1} g\left(\alpha+\tilde{x}^L_\alpha\right) f(\tilde{x}^U_\alpha)\right\}}{\min\left\{\min\limits_{\alpha\leq\beta\leq 1}\overline{G}\left(\alpha+\tilde{x}^U_\alpha\right)\overline{F}\left(\tilde{x}^L_\alpha\right),\ \min\limits_{\alpha\leq\beta\leq 1}\overline{G}\left(\alpha+\tilde{x}^L_\beta\right)\overline{F}(\tilde{x}^U_\alpha)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\tilde{x}_\alpha^U\right), \max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\tilde{x}_\alpha^L)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right)\overline{G}\left(\tilde{x}_\alpha^U\right), \max_{\alpha\le\beta\le1}\overline{F}\left(\alpha+\tilde{x}_\beta^U\right)\overline{G}(\tilde{x}_\alpha^L)\right\}} \ge \frac{\max\left\{\max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^U\right) f\left(\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^L\right) f(\tilde{x}_\alpha^U)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\alpha^U\right)\overline{F}\left(\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1}\overline{G}\left(\alpha+\tilde{x}_\beta^L\right)\overline{F}(\tilde{x}_\alpha^U)\right\}}$$

and then

$$\frac{\min\left\{\min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\tilde{x}_\alpha^U\right), \min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\tilde{x}_\alpha^L)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right)\overline{G}\left(\tilde{x}_\alpha^U\right), \min_{\alpha\le\beta\le1}\overline{F}\left(\alpha+\tilde{x}_\beta^U\right)\overline{G}(\tilde{x}_\alpha^L)\right\}} \le \frac{\min\left\{\min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^U\right) f\left(\tilde{x}_\alpha^L\right), \min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^L\right) f(\tilde{x}_\alpha^U)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\alpha^U\right)\overline{F}\left(\tilde{x}_\alpha^L\right), \min_{\alpha\le\beta\le1}\overline{G}\left(\alpha+\tilde{x}_\beta^L\right)\overline{F}(\tilde{x}_\alpha^U)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\tilde{x}_\alpha^U\right), \max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\tilde{x}_\alpha^L)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right)\overline{G}\left(\tilde{x}_\alpha^U\right), \max_{\alpha\le\beta\le1}\overline{F}\left(\alpha+\tilde{x}_\beta^U\right)\overline{G}(\tilde{x}_\alpha^L)\right\}} \le \frac{\max\left\{\max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^U\right) f\left(\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^L\right) f(\tilde{x}_\alpha^U)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\alpha^U\right)\overline{F}\left(\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1}\overline{G}\left(\alpha+\tilde{x}_\beta^L\right)\overline{F}(\tilde{x}_\alpha^U)\right\}}$$

respectively.

Theorem 3.10. *Let X and Y be two absolutely continuous non negative fuzzy random variables then,*

(1). $X \le_{SFHR} Y$ and $X \in_{IFPHR}$,

$$\frac{\min\left\{\min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\lambda\tilde{x}_\alpha^U\right), \min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\widetilde{\lambda x}_\alpha^L)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right)\overline{G}\left(\lambda\tilde{x}_\alpha^U\right), \min_{\alpha\le\beta\le1}\overline{F}\left(\alpha+\tilde{x}_\beta^U\right)\overline{G}(\widetilde{\lambda x}_\alpha^L)\right\}} \ge \frac{\min\left\{\min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^U\right) f\left(\lambda\tilde{x}_\alpha^L\right), \min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^L\right) f(\lambda\tilde{x}_\alpha^U)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\alpha^U\right)\overline{F}\left(\lambda\tilde{x}_\alpha^L\right), \min_{\alpha\le\beta\le1}\overline{G}\left(\alpha+\tilde{x}_\beta^L\right)\overline{F}(\lambda\tilde{x}_\alpha^U)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\widetilde{\lambda x}_\alpha^U\right), \max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\lambda\tilde{x}_\alpha^L)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right)\overline{G}\left(\lambda\tilde{x}_\alpha^U\right), \max_{\alpha\le\beta\le1}\overline{F}\left(\alpha+\tilde{x}_\beta^U\right)\overline{G}(\lambda\tilde{x}_\alpha^L)\right\}} \ge \frac{\max\left\{\max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^U\right) f\left(\lambda\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^L\right) f(\lambda\tilde{x}_\alpha^U)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\alpha^U\right)\overline{F}\left(\lambda\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1}\overline{G}\left(\alpha+\tilde{x}_\beta^L\right)\overline{F}(\lambda\tilde{x}_\alpha^U)\right\}}$$

(2). If $X \le_{PFHR} Y$ and f is log concave then,

$$\frac{\min\left\{\min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\lambda\tilde{x}_\alpha^U\right), \min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\widetilde{\lambda x}_\alpha^L)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right)\overline{G}\left(\lambda\tilde{x}_\alpha^U\right), \min_{\alpha\le\beta\le1}\overline{F}\left(\alpha+\tilde{x}_\beta^U\right)\overline{G}(\widetilde{\lambda x}_\alpha^L)\right\}} \ge \frac{\min\left\{\min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^U\right) f\left(\lambda\tilde{x}_\alpha^L\right), \min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^L\right) f(\lambda\tilde{x}_\alpha^U)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\alpha^U\right)\overline{F}\left(\lambda\tilde{x}_\alpha^L\right), \min_{\alpha\le\beta\le1}\overline{G}\left(\alpha+\tilde{x}_\beta^L\right)\overline{F}(\lambda\tilde{x}_\alpha^U)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\widetilde{\lambda x}_\alpha^U\right), \max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\lambda\tilde{x}_\alpha^L)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right)\overline{G}\left(\lambda\tilde{x}^U\right), \max_{\alpha\le\beta\le1}\overline{F}\left(\alpha+\tilde{x}_\beta^U\right)\overline{G}(\lambda\tilde{x}_\alpha^L)\right\}} \ge \frac{\max\left\{\max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^U\right) f\left(\lambda\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^L\right) f(\lambda\tilde{x}_\alpha^U)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\alpha^U\right)\overline{F}\left(\lambda\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1}\overline{G}\left(\alpha+\tilde{x}_\beta^L\right)\overline{F}(\lambda\tilde{x}_\alpha^U)\right\}}$$

(3). $X \le_{FHR} Y$ and $X \in_{UIFPHR}$ then

$$\frac{\min\left\{\min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\lambda\tilde{x}_\alpha^U\right), \min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\widetilde{\lambda x}_\alpha^L)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right)\overline{G}\left(\lambda\tilde{x}_\alpha^U\right), \min_{\alpha\le\beta\le1}\overline{F}\left(\alpha+\tilde{x}_\beta^U\right)\overline{G}(\widetilde{\lambda x}_\alpha^L)\right\}} \ge \frac{\min\left\{\min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^U\right) f\left(\lambda\tilde{x}_\alpha^L\right), \min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^L\right) f(\lambda\tilde{x}_\alpha^U)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\alpha^U\right)\overline{F}\left(\lambda\tilde{x}_\alpha^L\right), \min_{\alpha\le\beta\le1}\overline{G}\left(\alpha+\tilde{x}_\beta^L\right)\overline{F}(\lambda\tilde{x}_\alpha^U)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\widetilde{\lambda x}_\alpha^U\right), \max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\lambda\tilde{x}_\alpha^L)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right)\overline{G}\left(\lambda\tilde{x}_\alpha^U\right), \max_{\alpha\le\beta\le1}\overline{F}\left(\alpha+\tilde{x}_\beta^U\right)\overline{G}(\lambda\tilde{x}_\alpha^L)\right\}} \ge \frac{\max\left\{\max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^U\right) f\left(\lambda\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^L\right) f(\lambda\tilde{x}_\alpha^U)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\alpha^U\right)\overline{F}\left(\lambda\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1}\overline{G}\left(\alpha+\tilde{x}_\beta^L\right)\overline{F}(\lambda\tilde{x}_\alpha^U)\right\}}$$

Proof. When $X \le_{FHR} Y$, then $X_\alpha^L \le Y_\alpha^L$ and $X_\beta^U \le Y_\beta^U$. So, $\frac{X_\alpha^L}{\lambda} \ge \frac{Y_\alpha^L}{\lambda}$ and $\frac{X_\beta^U}{\lambda} \ge \frac{Y_\beta^U}{\lambda}$. If $X_\beta^L - a \le \frac{Y_\alpha^L}{\lambda}$, the proof is complete.

Otherwise, if $\frac{Y_\alpha^L}{\lambda} < X_\beta^U - a$, then for each $t \in (\frac{Y_\alpha^L}{\lambda}, X_\beta^U - a)$ we get,

$$\frac{\min\left\{\min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\lambda\tilde{x}_\alpha^U\right), \min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\lambda\tilde{x}_\alpha^L)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right)\overline{G}\left(\lambda\tilde{x}_\alpha^U\right), \min_{\alpha\le\beta\le1}\overline{F}\left(\alpha+\tilde{x}_\beta^U\right)\overline{G}(\widetilde{\lambda x}_\alpha^L)\right\}} \ge \frac{\min\left\{\min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^U\right) f\left(\lambda\tilde{x}_\alpha^L\right), \min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^L\right) f(\lambda\tilde{x}_\alpha^U)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\alpha^U\right)\overline{F}\left(\lambda\tilde{x}_\alpha^L\right), \min_{\alpha\le\beta\le1}\overline{G}\left(\alpha+\tilde{x}_\beta^L\right)\overline{F}(\lambda\tilde{x}_\alpha^U)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\lambda\tilde{x}_\alpha^U\right), \max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\lambda\tilde{x}_\alpha^L)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right) \overline{G}\left(\lambda\tilde{x}_\alpha^U\right), \max_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\beta^U\right) \overline{G}(\lambda\tilde{x}_\alpha^L)\right\}} \ge \frac{\max\left\{\max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^U\right) f\left(\lambda\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^L\right) f(\lambda\tilde{x}_\alpha^U)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\alpha^U\right) \overline{F}\left(\lambda\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\beta^L\right) \overline{F}(\lambda\tilde{x}_\alpha^U)\right\}}$$

Which is equal to

$$\frac{\min\left\{\min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\lambda\tilde{x}_\alpha^U\right), \min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\beta^U\right) g(\widetilde{\lambda x}_\alpha^L)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\alpha^L\right) \overline{G}\left(\lambda\tilde{x}_\alpha^U\right), \min_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\beta^U\right) \overline{G}(\widetilde{\lambda x}_\alpha^L)\right\}} \ge \frac{\min\left\{\min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^U\right) g\left(\lambda\tilde{x}_\alpha^L\right), \min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^L\right) g(\lambda\tilde{x}_\alpha^U)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\alpha^U\right) \overline{F}\left(\lambda\tilde{x}_\alpha^L\right), \min_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\beta^L\right) \overline{F}(\lambda\tilde{x}_\alpha^U)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\lambda\tilde{x}_\alpha^U\right), \max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\lambda\tilde{x}_\alpha^L)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right) \overline{G}\left(\lambda\tilde{x}_\alpha^U\right), \max_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\beta^U\right) \overline{G}(\lambda\tilde{x}_\alpha^L)\right\}} \ge \frac{\max\left\{\max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^U\right) g\left(\lambda\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}_\alpha^L\right) g(\lambda\tilde{x}_\alpha^U)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\alpha^U\right) \overline{G}\left(\lambda\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}_\beta^L\right) \overline{G}(\lambda\tilde{x}_\alpha^U)\right\}}$$

Which is multiplied to

$$\frac{\min\left\{\min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\lambda\tilde{x}_\alpha^U+a\right), \min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\lambda\tilde{x}_\alpha^L+a)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right) \overline{G}\left(\lambda\tilde{x}_\alpha^U+a\right), \min_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\beta^U\right) \overline{G}(\tilde{x}_\alpha^L+a)\right\}} \ge \frac{\min\left\{\min_{\alpha\le\beta\le1} f\left(\alpha+\lambda\tilde{x}_\alpha^L\right) g\left(\tilde{x}_\alpha^U+a\right), \min_{\alpha\le\beta\le1} f\left(\alpha+\lambda\tilde{x}_\beta^U\right) g(\tilde{x}_\alpha^L+a)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\lambda\tilde{x}_\alpha^L\right) \overline{G}\left(\tilde{x}_\alpha^U+a\right), \min_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\lambda\tilde{x}_\beta^U\right) \overline{G}(\tilde{x}_\alpha^L+a)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\alpha^L\right) g\left(\lambda\tilde{x}_\alpha^U+a\right), \max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}_\beta^U\right) g(\lambda\tilde{x}_\alpha^L+a)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\alpha^L\right) \overline{G}\left(\lambda\tilde{x}_\alpha^U+a\right), \max_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}_\beta^U\right) \overline{G}(\tilde{x}_\alpha^L+a)\right\}} \ge \frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\alpha+\lambda\tilde{x}_\alpha^L\right) g\left(\tilde{x}_\alpha^U+a\right), \max_{\alpha\le\beta\le1} f\left(\alpha+\lambda\tilde{x}_\beta^U\right) g(\tilde{x}_\alpha^L+a)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\lambda\tilde{x}_\alpha^L\right) \overline{G}\left(\tilde{x}_\alpha^U+a\right), \max_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\lambda\tilde{x}_\beta^U\right) \overline{G}(\tilde{x}_\alpha^L+a)\right\}}$$

Which both fractions are increasing in $t \in \left(\frac{Y_\alpha^L}{\lambda}, X_\beta^U - a\right)$. Note that $x \in \left[\frac{Y_\alpha^L}{\lambda} - a, \frac{Y_\beta^U}{\lambda} - a\right]$, then $g(x+a) > 0$. Since, $X_\alpha^L - a \le \frac{Y_\alpha^L}{\lambda} < X_\beta^U - a \le \frac{Y_\beta^L}{\lambda}$ then by using the previous theorem, proof is complete. □

Theorem 3.11. *Let X and Y be two absolutely continuous non negative fuzzy random variables. If $X \le_{UPFHR} Y$ then there exist a random variable Z,*

$$\frac{\min\left\{\min_{\alpha\le\beta\le1} [f(\tilde{Z}_\alpha^L - t)/\tilde{Z}_\alpha^L \ge t], \min_{\alpha\le\beta\le1} [f(\tilde{Z}_\alpha^U - t)/\tilde{Z}_\alpha^U \ge t]\right\}}{\min\left\{\min_{\alpha\le\beta\le1} [\overline{F}(\tilde{Z}_\alpha^L - t)/\tilde{Z}_\alpha^L \ge t], \min_{\alpha\le\beta\le1} [\overline{F}(\tilde{Z}_\alpha^U - t)/\tilde{Z}_\alpha^U \ge t]\right\}} \ge \frac{\min\left\{\min_{\alpha\le\beta\le1} f\left(\tilde{Z}_\beta^L\right), \min_{\alpha\le\beta\le1} f\left(\tilde{Z}_\beta^U\right)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{F}\left(\tilde{Z}_\beta^L\right), \min_{\alpha\le\beta\le1} \overline{F}\left(\tilde{Z}_\beta^U\right)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\le\beta\le1} [f(\tilde{Z}_\alpha^L - t)/\tilde{Z}_\alpha^L \ge t], \max_{\alpha\le\beta\le1} [f(\tilde{Z}_\alpha^U - t)/\tilde{Z}_\alpha^U \ge t]\right\}}{\max\left\{\max_{\alpha\le\beta\le1} [\overline{F}(\tilde{Z}_\alpha^L - t)/\tilde{Z}_\alpha^L \ge t], \max_{\alpha\le\beta\le1} [\overline{F}(\tilde{Z}_\alpha^U - t)/\tilde{Z}_\alpha^U \ge t]\right\}} \ge \frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\tilde{Z}_\beta^L\right), \max_{\alpha\le\beta\le1} f\left(\tilde{Z}_\beta^U\right)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\tilde{Z}_\beta^L\right), \max_{\alpha\le\beta\le1} \overline{F}\left(\tilde{Z}_\beta^U\right)\right\}}$$

such that

$$\frac{\min\left\{\min_{\alpha\le\beta\le1} f\left(\tilde{x}_\alpha^L\right) g\left(\lambda\tilde{x}_\alpha^U\right), \min_{\alpha\le\beta\le1} f\left(\tilde{x}_\beta^U\right) g(\lambda\tilde{x}_\alpha^L)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{F}\left(\tilde{x}_\alpha^L\right) \overline{G}\left(\lambda\tilde{x}_\alpha^U\right), \min_{\alpha\le\beta\le1} \overline{F}\left(\tilde{x}_\beta^U\right) \overline{G}(\lambda\tilde{x}_\alpha^L)\right\}} \ge \frac{\min\left\{\min_{\alpha\le\beta\le1} g\left(\tilde{x}_\alpha^U\right) f\left(\lambda\tilde{x}_\alpha^L\right), \min_{\alpha\le\beta\le1} g\left(\tilde{x}_\alpha^L\right) f(\lambda\tilde{x}_\alpha^U)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{G}\left(\tilde{x}_\alpha^U\right) \overline{F}\left(\lambda\tilde{x}_\alpha^L\right), \min_{\alpha\le\beta\le1} \overline{G}\left(\tilde{x}_\beta^L\right) \overline{F}(\lambda\tilde{x}_\alpha^U)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\tilde{x}_\alpha^L\right) g\left(\lambda\tilde{x}_\alpha^U\right), \max_{\alpha\le\beta\le1} f\left(\tilde{x}_\beta^U\right) g(\lambda\tilde{x}_\alpha^L)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\tilde{x}_\alpha^L\right) \overline{G}\left(\lambda\tilde{x}_\alpha^U\right), \max_{\alpha\le\beta\le1} \overline{F}\left(\tilde{x}_\beta^U\right) \overline{G}(\lambda\tilde{x}_\alpha^L)\right\}} \ge \frac{\max\left\{\max_{\alpha\le\beta\le1} g\left(\tilde{x}_\alpha^U\right) f\left(\lambda\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1} g\left(\tilde{x}_\alpha^L\right) f(\lambda\tilde{x}_\alpha^U)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{G}\left(\tilde{x}_\alpha^U\right) \overline{F}\left(\lambda\tilde{x}_\alpha^L\right), \max_{\alpha\le\beta\le1} \overline{G}\left(\tilde{x}_\beta^L\right) \overline{F}(\lambda\tilde{x}_\alpha^U)\right\}}$$

where z is the arbitrary random variable between $\tilde{x}_\alpha^L$ and $\tilde{y}_\alpha^L$.

Proof. For $X^L_\alpha \le Y^L_\alpha$ take z as an arbitrary random variable with UIPFHR property taking values on $X^L_\alpha, Y^L_\alpha]$. Suppose that $\tilde{y}^L_\alpha \le \tilde{x}^L_\alpha$. Set $k_x = \frac{f(x+a)^U_\alpha}{f(x)}$ and $k_x = \frac{g(x+a)^U_\alpha}{g(x)}$ which are transformation of x and y respectively. We know that,

$$\frac{\min\left\{\min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}^L_\alpha\right) g\left(\lambda\tilde{x}^U_\alpha\right), \min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}^U_\beta\right) g(\lambda\tilde{x}^L_\alpha)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}^L_\alpha\right)\overline{G}\left(\lambda\tilde{x}^U_\alpha\right), \min_{\alpha\le\beta\le1}\overline{F}\left(\alpha+\tilde{x}^U_\beta\right)\overline{G}(\widetilde{\lambda x}^L_\alpha)\right\}} \ge \frac{\min\left\{\min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}^U_\alpha\right) f\left(\lambda\tilde{x}^L_\alpha\right), \min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}^L_\alpha\right) f(\lambda\tilde{x}^U_\alpha)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}^U_\alpha\right)\overline{F}\left(\lambda\tilde{x}^L_\alpha\right), \min_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}^L_\beta\right)\overline{F}(\lambda\tilde{x}^U_\alpha)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}^L_\alpha\right) g\left(\lambda\tilde{x}^U_\alpha\right), \max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}^U_\beta\right) g(\lambda\tilde{x}^L_\alpha)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}^L_\alpha\right)\overline{G}\left(\lambda\tilde{x}^U_\alpha\right), \max_{\alpha\le\beta\le1}\overline{F}\left(\alpha+\tilde{x}^U_\beta\right)\overline{G}(\lambda\tilde{x}^L_\alpha)\right\}} \ge \frac{\max\left\{\max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}^U_\alpha\right) f\left(\lambda\tilde{x}^L_\alpha\right), \max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}^L_\alpha\right) f(\lambda\tilde{x}^U_\alpha)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}^U_\alpha\right)\overline{F}\left(\lambda\tilde{x}^L_\alpha\right), \max_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}^L_\beta\right)\overline{F}(\lambda\tilde{x}^U_\alpha)\right\}}$$

Implies $k_x(x+a) \le k_y(\lambda x)$, for all $x \ge 0$, $t \in (\tilde{y}^L_\alpha, \tilde{x}^U_\alpha - a)$. Implies $k_x(x+\alpha) \le \lambda k_y(\lambda x)$, for all $\tilde{y}^L_\alpha \le a \le \alpha \le \tilde{y}^U_\alpha$. □

Example 3.12. *Let X and Y be two non negative independent fuzzy random variables, if*

$$\frac{\min\left\{\min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}^L_\alpha\right) g\left(\lambda\tilde{x}^U_\alpha\right), \min_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}^U_\beta\right) g(\lambda\tilde{x}^L_\alpha)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}^L_\alpha\right)\overline{G}\left(\lambda\tilde{x}^U_\alpha\right), \min_{\alpha\le\beta\le1}\overline{F}\left(\alpha+\tilde{x}^U_\beta\right)\overline{G}(\widetilde{\lambda x}^L_\alpha)\right\}} \ge \frac{\min\left\{\min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}^U_\alpha\right) f\left(\lambda\tilde{x}^L_\alpha\right), \min_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}^L_\alpha\right) f(\lambda\tilde{x}^U_\alpha)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}^U_\alpha\right)\overline{F}\left(\lambda\tilde{x}^L_\alpha\right), \min_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}^L_\beta\right)\overline{F}(\lambda\tilde{x}^U_\alpha)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}^L_\alpha\right) g\left(\lambda\tilde{x}^U_\alpha\right), \max_{\alpha\le\beta\le1} f\left(\alpha+\tilde{x}^U_\beta\right) g(\lambda\tilde{x}^L_\alpha)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\alpha+\tilde{x}^L_\alpha\right)\overline{G}\left(\lambda\tilde{x}^U_\alpha\right), \max_{\alpha\le\beta\le1}\overline{F}\left(\alpha+\tilde{x}^U_\beta\right)\overline{G}(\lambda\tilde{x}^L_\alpha)\right\}} \ge \frac{\max\left\{\max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}^U_\alpha\right) f\left(\lambda\tilde{x}^L_\alpha\right), \max_{\alpha\le\beta\le1} g\left(\alpha+\tilde{x}^L_\alpha\right) f(\lambda\tilde{x}^U_\alpha)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}^U_\alpha\right)\overline{F}\left(\lambda\tilde{x}^L_\alpha\right), \max_{\alpha\le\beta\le1} \overline{G}\left(\alpha+\tilde{x}^L_\beta\right)\overline{F}(\lambda\tilde{x}^U_\alpha)\right\}}$$

there exist a random variables,

$$\frac{\min\left\{\min_{\alpha\le\beta\le1}[f(\tilde{Z}^L_\alpha - t)/\tilde{Z}^L_\alpha \ge t], \min_{\alpha\le\beta\le1}[f(\tilde{Z}^U_\alpha - t)/\tilde{Z}^U_\alpha \ge t]\right\}}{\min\left\{\min_{\alpha\le\beta\le1}[\overline{F}(\tilde{Z}^L_\alpha - t)/\tilde{Z}^L_\alpha \ge t], \min_{\alpha\le\beta\le1}[\overline{F}(\tilde{Z}^U_\alpha - t)/\tilde{Z}^U_\alpha \ge t]\right\}} \ge \frac{\min\left\{\min_{\alpha\le\beta\le1} f\left(\tilde{Z}^L_\beta\right), \min_{\alpha\le\beta\le1} f\left(\tilde{Z}^U_\beta\right)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{F}\left(\tilde{Z}^L_\beta\right), \min_{\alpha\le\beta\le1} \overline{F}\left(\tilde{Z}^U_\beta\right)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\le\beta\le1}[f(\tilde{Z}^L_\alpha - t)/\tilde{Z}^L_\alpha \ge t], \max_{\alpha\le\beta\le1}[f(\tilde{Z}^U_\alpha - t)/\tilde{Z}^U_\alpha \ge t]\right\}}{\max\left\{\max_{\alpha\le\beta\le1}[\overline{F}(\tilde{Z}^L_\alpha - t)/\tilde{Z}^L_\alpha \ge t], \max_{\alpha\le\beta\le1}[\overline{F}(\tilde{Z}^U_\alpha - t)/\tilde{Z}^U_\alpha \ge t]\right\}} \ge \frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\tilde{Z}^L_\beta\right), \max_{\alpha\le\beta\le1} f\left(\tilde{Z}^U_\beta\right)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\tilde{Z}^L_\beta\right), \max_{\alpha\le\beta\le1} \overline{F}\left(\tilde{Z}^U_\beta\right)\right\}}$$

such that, define $k^*(a) = \max_{v\ge a} k_x(a)$, $a \in (\tilde{y}^L_\alpha, \tilde{x}^U_\alpha)$. By Lillo et al., (2001)
$f^*(a) = [\int_{\tilde{y}^L_\alpha}^{\tilde{x}^L_\alpha} e^{\int_a^s k^*(v)dv} ds]^{-1}$ *is density function on $[\tilde{y}^L_\alpha, \tilde{x}^U_\alpha]$ Which admits,*

$$\frac{\min\left\{\min_{\alpha\le\beta\le1}[f(\tilde{Z}^L_\alpha - t)/\tilde{Z}^L_\alpha \ge t], \min_{\alpha\le\beta\le1}[f(\tilde{Z}^U_\alpha - t)/\tilde{Z}^U_\alpha \ge t]\right\}}{\min\left\{\min_{\alpha\le\beta\le1}[\overline{F}(\tilde{Z}^L_\alpha - t)/\tilde{Z}^L_\alpha \ge t], \min_{\alpha\le\beta\le1}[\overline{F}(\tilde{Z}^U_\alpha - t)/\tilde{Z}^U_\alpha \ge t]\right\}} \ge \frac{\min\left\{\min_{\alpha\le\beta\le1} f\left(\tilde{Z}^L_\beta\right), \min_{\alpha\le\beta\le1} f\left(\tilde{Z}^U_\beta\right)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{F}\left(\tilde{Z}^L_\beta\right), \min_{\alpha\le\beta\le1} \overline{F}\left(\tilde{Z}^U_\beta\right)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\le\beta\le1}[f(\tilde{Z}^L_\alpha - t)/\tilde{Z}^L_\alpha \ge t], \max_{\alpha\le\beta\le1}[f(\tilde{Z}^U_\alpha - t)/\tilde{Z}^U_\alpha \ge t]\right\}}{\max\left\{\max_{\alpha\le\beta\le1}[\overline{F}(\tilde{Z}^L_\alpha - t)/\tilde{Z}^L_\alpha \ge t], \max_{\alpha\le\beta\le1}[\overline{F}(\tilde{Z}^U_\alpha - t)/\tilde{Z}^U_\alpha \ge t]\right\}} \ge \frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\tilde{Z}^L_\beta\right), \max_{\alpha\le\beta\le1} f\left(\tilde{Z}^U_\beta\right)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\tilde{Z}^L_\beta\right), \max_{\alpha\le\beta\le1} \overline{F}\left(\tilde{Z}^U_\beta\right)\right\}}$$

Hence $k_x(a) \le \lambda k^*_y(\lambda a)$, $\forall\ a \in (X^L_\alpha, Y^L_\alpha)$, thus

$$\frac{\min\left\{\min_{\alpha\le\beta\le1} f\left(\tilde{x}^L_\alpha\right) g\left(\lambda\tilde{x}^U_\alpha\right), \min_{\alpha\le\beta\le1} f\left(\tilde{x}^U_\beta\right) g(\lambda\tilde{x}^L_\alpha)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{F}\left(\tilde{x}^L_\alpha\right)\overline{G}\left(\lambda\tilde{x}^U_\alpha\right), \min_{\alpha\le\beta\le1}\overline{F}\left(\tilde{x}^U_\beta\right)\overline{G}(\lambda\tilde{x}^L_\alpha)\right\}} \ge \frac{\min\left\{\min_{\alpha\le\beta\le1} g\left(\tilde{x}^U_\alpha\right) f\left(\lambda\tilde{x}^L_\alpha\right), \min_{\alpha\le\beta\le1} g\left(\tilde{x}^L_\alpha\right) f(\lambda\tilde{x}^U_\alpha)\right\}}{\min\left\{\min_{\alpha\le\beta\le1} \overline{G}\left(\tilde{x}^U_\alpha\right)\overline{F}\left(\lambda\tilde{x}^L_\alpha\right), \min_{\alpha\le\beta\le1} \overline{G}\left(\tilde{x}^L_\beta\right)\overline{F}(\lambda\tilde{x}^U_\alpha)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\le\beta\le1} f\left(\tilde{x}^L_\alpha\right) g\left(\lambda\tilde{x}^U_\alpha\right), \max_{\alpha\le\beta\le1} f\left(\tilde{x}^U_\beta\right) g(\lambda\tilde{x}^L_\alpha)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{F}\left(\tilde{x}^L_\alpha\right)\overline{G}\left(\lambda\tilde{x}^U_\alpha\right), \max_{\alpha\le\beta\le1}\overline{F}\left(\tilde{x}^U_\beta\right)\overline{G}(\lambda\tilde{x}^L_\alpha)\right\}} \ge \frac{\max\left\{\max_{\alpha\le\beta\le1} g\left(\tilde{x}^U_\alpha\right) f\left(\lambda\tilde{x}^L_\alpha\right), \max_{\alpha\le\beta\le1} g\left(\tilde{x}^L_\alpha\right) f(\lambda\tilde{x}^U_\alpha)\right\}}{\max\left\{\max_{\alpha\le\beta\le1} \overline{G}\left(\tilde{x}^U_\alpha\right)\overline{F}\left(\lambda\tilde{x}^L_\alpha\right), \max_{\alpha\le\beta\le1} \overline{G}\left(\tilde{x}^L_\beta\right)\overline{F}(\lambda\tilde{x}^U_\alpha)\right\}}$$

Where Z is a arbitrary random variable between X^U_α *and* Y^L_α

$$\frac{\min\left\{\min_{\alpha\leq\beta\leq 1} f\left(\alpha+\widetilde{xy}^L_\alpha\right) g\left(\widetilde{xy}^U_\alpha\right), \min_{\alpha\leq\beta\leq 1} f\left(\alpha+\widetilde{yx}^U_\beta\right) g(\widetilde{yx}^L_\alpha)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq 1} \overline{F}\left(\alpha+\widetilde{yx}^L_\alpha\right) \overline{G}\left(\widetilde{yx}^U_\alpha\right), \min_{\alpha\leq\beta\leq 1} \overline{F}\left(\alpha+\widetilde{yx}^U_\beta\right) \overline{G}(\widetilde{yx}^L_\alpha)\right\}} \geq \frac{\min\left\{\min_{\alpha\leq\beta\leq 1} g\left(\alpha+\widetilde{yx}^U_\alpha\right) f\left(\widetilde{yx}^L_\alpha\right), \min_{\alpha\leq\beta\leq 1} g\left(\alpha+\widetilde{yx}^L_\alpha\right) f(\widetilde{yx}^U_\alpha)\right\}}{\min\left\{\min_{\alpha\leq\beta\leq 1} \overline{G}\left(\alpha+\widetilde{yx}^U_\alpha\right) \overline{F}\left(\widetilde{yx}^L_\alpha\right), \min_{\alpha\leq\beta\leq 1} \overline{G}\left(\alpha+\widetilde{yx}^L_\beta\right) \overline{F}(\widetilde{yx}^U_\alpha)\right\}}$$

and

$$\frac{\max\left\{\max_{\alpha\leq\beta\leq 1} f\left(\alpha+\widetilde{xy}^L_\alpha\right) g\left(\widetilde{xy}^U_\alpha\right), \max_{\alpha\leq\beta\leq 1} f\left(\alpha+\widetilde{yx}^U_\beta\right) g(\widetilde{yx}^L_\alpha)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq 1} \overline{F}\left(\alpha+\widetilde{yx}^L_\alpha\right) \overline{G}\left(\widetilde{yx}^U_\alpha\right), \max_{\alpha\leq\beta\leq 1} \overline{F}\left(\alpha+\widetilde{yx}^U_\beta\right) \overline{G}(\widetilde{yx}^L_\alpha)\right\}} \geq \frac{\max\left\{\max_{\alpha\leq\beta\leq 1} g\left(\alpha+\widetilde{yx}^U_\alpha\right) f\left(\widetilde{yx}^L_\alpha\right), \max_{\alpha\leq\beta\leq 1} g\left(\alpha+\widetilde{yx}^L_\alpha\right) f(\widetilde{yx}^U_\alpha)\right\}}{\max\left\{\max_{\alpha\leq\beta\leq 1} \overline{G}\left(\alpha+\widetilde{yx}^U_\alpha\right) \overline{F}\left(\widetilde{yx}^L_\alpha\right), \max_{\alpha\leq\beta\leq 1} \overline{G}\left(\alpha+\widetilde{yx}^L_\beta\right) \overline{F}(\widetilde{yx}^U_\alpha)\right\}}$$

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