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On Domatic Energy of a Graph

Research Article

K.B.Murthy[1]* and B.V.Dhananjayamurthy[2]

1 Department of Agricultural Statistics, Applied Mathematics and Computer Science, University of Agricultural Sciences, GKVK, Bengaluru, India.

2 Department of Mathematics, Nitte Meenakshi Institute of Technology, Yelahanka, Bengaluru, India.

Abstract: In this paper, Motivated by the partition energy and the minimum covering energy of a graph, we introduce domatic energy $E_d(G)$, of a graph G.

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Keywords: Domatic Partition, domatic matrix, domatic eigenvalues, domatic energy of a graph.

1. Introduction

In this paper, we are concerned only with simple graphs, i.e., a finite, having no loops, no multiple and directed edges. Let $G = (V, E)$ be such a graph with vertex set $V(G)$ and edges set $E(G)$. As usual, we denote by $n = |V|$ and $m = |E|$ to the number of vertices and edges in a graph G, respectively. The degree of a vertex v in a graph$G$, denoted by $d(v)$, is the number of vertices adjacent to v. For any vertex v of a graph G, the open neighborhood of v is the set $N(v) = \{u \in V\ : uv \in E(G)$. For a subset $S \subseteq V(G)$ the degree of a vertex $v \in V(G)$ with respect to a subset S is $d_s(v) = \{N(v) \cap S\}$. Let G be a simple graph with vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and edge set $E(G)$. A sub set C of V is called a covering set of G if every edge of G is incident to at least one vertex of C. Any covering set with minimum cardinality is called a minimum covering set of G. For graph theoretic terminology we refer to Harray Book [7]. A set $D \subseteq V(G)$ is called a dominating set of a graph G if every vertex $v \in V(G) - D$ adjacent to some vertex in D.

The minimum cardinality of such set is called the domination number of G and denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called a γ-set for G. A thorough treatment of domination in graphs can be found in the book by Haynes at el. [8]. The domatic number $d(G)$ of a graph G is the maximum positive integer k such that $V(G)$ can partitioned into k pair wise disjoint dominating sets. A partition V into pair wise disjoint dominating sets is called a domatic partition. The concept of a domatic number was introduced by E.J.Cockayne at el. [4]. Let $P_k^d = \{D_1, D_2, \ldots, D_k\}$ be the domatic partition sets of $V(G)$. Then, in present paper, we consider that $\gamma(G) = |D_1| \leq |D_2| \leq \cdots \leq |D_k|$, where $|D_i|$ is the cardinality of domatic partite set D_i for $i = 1, 2, \ldots, k$ of a graph G.

The concept energy of a graph introduced by I. Gutman [5] in the year 1978. Let G be a graph with n vertices and m edges and let $A(G) = (a_{ij})$ be the adjacency matrix of G. The eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a matrix $A(G)$, assumed in

* E-mail: kbmurthy2005@gmail.com

non increasing order, are the eigen values of the graph G. Let $\lambda_1, \lambda_2, \ldots, \lambda_t$ for $t \leq n$ be the distinct eigen values of G with multiplicity $m_1, m_2, \ldots, m_t$, respectively, the multiset of eigen values of $A(G)$ is called the spectrum of G and denoted by

$$Spec\,(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_t \\ m_1 & m_2 & \cdots & m_t \end{pmatrix}$$

As A is real symmetric, the eigen values of G are real with sum equal to zero. The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G, i.e, $E\,(G) = \sum\limits_{i=1}^{n} |\lambda_i|$. For more details on the Mathematical aspects of the theory of graph energy we refer to [2, 6, 10]. The minimum covering energy, $E_C(G)$ of a graph, which depends on its particular minimum cover set C, introduced by C. Adiga et al. [1] in (2012). Where the minimum covering matrix of G is the $n \times n$ matrix, denoted $A_C\,(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_iv_j \in E; \\ 1, & \text{if } i = j \text{ and } v_i \in C; \\ 0, & otherwise. \end{cases}$$

Recently, the partition energy of a graph was introduced by E.Sampathkumar et al. [11] in (2015). Let G be a graph and $P_k = \{V_1, V_2, \ldots, V_k\}$ be a partition of its vertex set $V(G)$, the partition matrix of G is the $n \times n$ matrix, denoted $A_P\,(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 2, & \text{if } v_i \text{ and } v_j \text{ are adjacent where } v_i, v_j \in V_r; \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent where } v_i, v_j \in V_r; \\ 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent between the sets } v_r \text{ and } V_s \text{ for } r \neq s, \text{ where } v_i \in V_r \text{ and where } v_i \in V_s \\ 0, & \text{otherwise.} \end{cases}$$

Motivated by these papers, we introduce domatic energy of a graph, denoted by $E_d\,(G)$. The domatic energies of some standard graphs are computed. Some properties of the domatic energy and also Upper and lower bounds for $E_d(G)$ are established. It is possible that the minimum monopoly energy that we are considering in this paper may be having some applications in chemistry as well as in other areas. The following results appear in paper [4].

Proposition 1.1.

(a). For any graph G, $d\,(G) \leq d + 1$.

(b). $d\,(G) \geq 2$, *if and only if G has no isolated vertices.*

(c). For any tree T with $n \geq 2$ *vertices,* $d\,(T) = 2$.

Lemma 1.2. *Let G be a graph with n vertices and let* $P_k^d = \{D_1, D_2, \ldots, D_k\}$ *be the domatic partition sets of* $V(G)$. *If* $|D_1| = |D_2| = \ldots |D_k| = \gamma(G)$, *then* $d\,(G) = \frac{n}{\gamma(G)}$.

Proposition 1.3. *Let G be a graph with n vertices and let* $P_k^d = \{D_1, D_2, \ldots, D_k\}$ *be the domatic partition sets of* $V(G)$. *Then* $\sum\limits_{i=1}^{d(G)} |D_i|^2 > \frac{n^2}{d(G)}$ *the bound attains on the complete graphs* K_n, $n \geq 2$ *and on the cycle* C_n, *for* $n \equiv 0$ *mod* 3.

Corollary 1.4. *Let G be a graph with n vertices and let* $P_k^d = \{D_1, D_2, \ldots, D_k\}$ *be the domatic partition sets of* $V(G)$. *Then* $\sum\limits_{i=1}^{d(G)} |D_i|^2 > \frac{n^2}{d+1}$ *the bound attains on the complete graphs* K_n, $n \geq 2$ *and on the cycle* C_n, *for* $n \equiv 0$ *mod* 3.

2. The Domatic Energy of Graphs

Let G be a graph with n vertices and let $P_k^d = \{D_1, D_2, \ldots, D_k\}$ be the domatic partition sets of $V(G)$, such that $\gamma(G) = |D_1| \leq |D_2| \leq \cdots \leq |D_k|$. Then a domatic matrix of G is the $n \times n$ matrix, denoted $A_d(G) = (d_{ij})$, where

$$d_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E; \\ |D_i|, & \text{if } i = j \text{ and } v_i \in D_i; \\ 0, & otherwise. \end{cases}$$

The Characteristic polynomial of $A_d(G)$, denoted by $f_n(G, \lambda)$, is defined as $f_n(G, \lambda) = \det(\lambda I - A_d(G))$. The domatic eigenvalues of G are the eigenvalues of $A_d(G)$. Since $A_d(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n$. The domatic energy of G is defined as $E_d(G) = \sum_{i=1}^{n} |\lambda_i|$. We first compute the domatic energy of a graph G in Figure 1

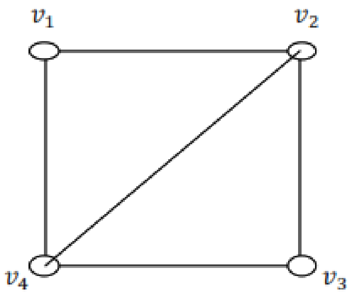

Figure 1. **Graph G**

Example 2.1. *Let G be a graph in Figure 1, with vertices v_1, v_2, v_3, v_4. Then the domatic partition P_k^d of G is equal to $\{v_2\}, \{v_4\}, \{v_1, v_3\}$ and hence the domatic matrix of G is*

$$A_d(G) = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The Characteristic polynomial $A_d(G)$ is$f_n(G, \lambda) = \lambda^4 - 6\lambda^3 + 8\lambda^2$. Then the domatic eigenvalues of G are $\lambda_1 = 4$, $\lambda_2 = 2$ and $\lambda_3 = \lambda_4 = 0$. Therefore, the domatic energy of G is $E_d(G) = 6$.

3. Some Properties of Domatic Energy of Graphs

In this section, we introduce some properties of Characteristic polynomials of domatic matrix and some properties of domatic eigenvalues of a graph G.

Theorem 3.1. *Let G be a graph with n vertices, m edges and domatic number d(G) and let $f_n(G, \lambda) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n$ be the Characteristic polynomial of the domatic matrix of G . Then*

(1). $c_0 = 1$.

(2). $c_1 = -\sum_{i=1}^{d(G)} |D_i|^2$.

(3). $c_2 = \sum\limits_{1 \le i \le j \le d(G)} |D_i|\,|D_j| - m.$

Proof.

(1). From the definition of the Characteristic polynomial $f_n(G, \lambda)$.

(2). Since the sum of the determinants of all 1×1 principal sub-matrices of $A_d(G)$ is the trace of $A_d(G)$, which evidently is equal to $\sum i = 1^n |D_i|$, where D_i is a domatic partite set which contain a corresponding vertex v_i in G for every $i = 1, 2, \ldots, n$. Hence, if D_i, for every $i = 1, 2, \ldots, d(G)$ contains k vertices, i.e., $|D_i| = k$, then $\sum i = 1^n |D_i| = \sum i = 1^{d(G)} k\,|D_i| = \sum i = 1^n |D_i|^2$. Thus, $(-1)^1 c_1 = \sum\limits_{i=1}^{d(G)} |D_i|^2$.

(3). $(-1)^2 c_2$ is equal to the sum of determinants of all 2×2 principal sub-matrices of $A_d(G)$, that is

$$\begin{aligned} c_2 &= \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = \sum_{1 \le i < j \le n} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= \sum_{1 \le i < j \le n} a_{ii}a_{jj} - \sum_{1 \le i < j \le n} {a_{ij}}^2 \\ &= \sum_{1 \le i < j \le d(G)} |D_i|\,|D_j| - m. \end{aligned}$$

□

Theorem 3.2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A_d(G)$. Then*

(1). $\sum\limits_{i}^{n} \lambda_i = \sum\limits_{i=1}^{d(G)} |D_i|^2.$

(2). $\sum\limits_{i}^{n} {\lambda_i}^2 = \sum\limits_{i=1}^{d(G)} |D_i|^3 + 2m.$

Proof.

(1). Since the sum of the eigenvalues of $A_d(G)$ is the trace of $A_d(G)$, which evidently equal to $\sum i = 1^n |D_i|$, where D_i is a domatic partite set which contain a corresponding vertex v_i in G for every$i = 1, 2, \ldots, n$. Hence, if D_i, for every $i = 1, 2, \ldots, d(G)$ contains k vertices, i.e., $|D_i| = k$, then

$$\sum i = 1^n |D_i| = \sum i = 1^{d(G)} k\,|D_i| = \sum i = 1^n |D_i|^2.$$

Therefore, $\sum\limits_{i=1}^{n} \lambda_i = \sum\limits_{i=1}^{n} a_{ii} = \sum\limits_{i=1}^{d(G)} |D_i|^2.$

(2). Similarly the sum of squares of the eigenvalues of $A_d(G)$ is the trace of $(A_d(G))^2$. Then

$$\sum_{i}^{n} {\lambda_i}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} = \sum_{i=1}^{n} {a_{ii}}^2 + \sum_{i \ne j}^{n} a_{ij}a_{ji} = \sum_{i=1}^{n} |D_i|^2 + 2\sum_{i<j}^{n} {a_{ij}}^2 = \sum_{i=1}^{d(G)} |D_i|^3 + 2m.$$

□

Theorem 3.3. *Let G be a graph with n vertices and m edges and let $\lambda_1(G)$ be the largest eigenvalues of $A_d(G)$. Then* $\lambda_1(G) \ge \frac{n}{d+1} + \frac{2m}{n}$.

Proof. Let G be a graph with n vertices and let λ_1 be the largest eigenvalues of $A_d(G)$. Then from [2] we have $\lambda_1 = \max\limits_{X \neq 0}\left\{\frac{X^t A\ X}{X^t X}\right\}$, where X is any nonzero vector and $X^t$is its transpose and A is a matrix. If we take $X = J = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$. If $P_k^d(G) = \{D_1, D_2, \ldots, D_k\}$, then we have

$$\lambda_1 \geq \frac{J^t A_d(G) J}{J^t J} = \frac{\sum_{i=1}^{n} a_{ii} + \sum_{i=j}^{n} a_{ij} a_{ji}}{n} = \frac{\sum_{i=1}^{n} |D_i|^2 + 2m}{n} \geq \frac{n}{d+1} + \frac{2m}{n}.$$

□

Theorem 3.4. *Let G be a graph with n vertices. If the domatic energy $E_d(G)$ of G is a rational number, then $E_d(G) = \sum_{i=1}^{d(G)} |D_i|^2$ mod 2.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be domatic eigenvalues of G. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive and the rest are non-positive, then

$$\sum_{i}^{n} |\lambda_i| = (\lambda_1 + \lambda_2 + \cdots + \lambda_r) - (\lambda_r + 1 + \cdots + \lambda_n) = 2(\lambda_1 + \lambda_2 + \cdots + \lambda_r) - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)$$

Hence by Theorem 3.2, we have $E_d(G) = 2(\lambda_1 + \lambda_2 + \cdots + \lambda_r) - \sum_{i=1}^{d(G)} |D_i|^2$. Since $\lambda_1, \lambda_2, \ldots, \lambda_r$ are algebraic integers, so is their sum. Therefore, $(\lambda_1 + \lambda_2 + \cdots + \lambda_r)$ must be integer if $E_M(G)$ is rational. Hence, the Theorem holds. □

4. The Domatic Energies of Some standard Graphs

In this section, we compute the exact values of the minimum monopoly energy of some standard graphs.

Theorem 4.1. *For the complete graph K_n, for $n \geq 2$, the minimum monopoly energy is $E_d(K_n) = n$.*

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Since every vertex v of K_n is a dominating set of K_n. Hence, $P_k^d(G) = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$. Then,

$$A_M(K_n) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{n \times n}$$

The respective characteristic polynomial is

$$f_n(K_n, \lambda) = \begin{vmatrix} \lambda - 1 & -1 & \cdots & -1 \\ -1 & \lambda - 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda - 1 \end{vmatrix}_{n \times n} = \lambda^{(n-1)}(\lambda - n).$$

The domatic spectrum of K_n will be written as

$$D\ Spec(K_n) = \begin{pmatrix} 0 & n \\ n-1 & 1 \end{pmatrix}.$$

Hence, the domatic energy of a complete graph is $E_d(K_n) = n$. □

Theorem 4.2. *For the complete bipartite graph $K_{r,r}$, for $r \leq 2$ with $n = 2r$ vertices, the domatic energy is $E_d(K_{r,r}) = 3n-4$.*

Proof. For the complete bipartite graph $K_{r,r}$, for $r \leq 2$ with vertex set

$$V = \{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_r\}.$$

The domatic partition is $P_k^d(K_{r,r}) = \{\{v_i, u_i\} : \text{ for every } 1 \leq i \leq r\}$. Then

$$A_d(K_{r,r}) = \begin{pmatrix} 2 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 2 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 2 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 2 \end{pmatrix}_{2r \times 2r}$$

The Characteristic polynomial of $A_d(K_{r,r})$ is

$$f_n(K_{r,r}, \lambda) = \begin{pmatrix} \lambda-2 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & \lambda-2 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda-2 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 2 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & \lambda-2 \end{pmatrix}_{2r \times 2r}$$

$$= (\lambda + (r-2))(\lambda - (r+2))(\lambda - 2)^{(2r-2)}$$

and $D\,Spec(K_{r,r}) = \begin{pmatrix} 2 & r+2 & -(r-2) \\ 2r-2 & 1 & 1 \end{pmatrix}$. Hence, $E_d(K_{r,r}) = 2(2r-2) + r + 2 + r - 2 = 6r - 4 = 3n - 4$. □

Theorem 4.3. *For a star graph $K_{1,n}$, $n \geq 2$ the domatic energy $A_d(K_{1,n}) = n^2 + 1$.*

Proof. Let $K_{1,n}$ be a star graph with vertex set $V = \{v_0, v_1, v_2, \ldots, v_n\}$, where v_0 is the center vertex. The domatic partition of $K_{1,n}$ is $P_k^d(K_{1,n}) = \big\{\{v_0\}, \{v_1, v_2, \ldots, v_n\}\big\}$. Then

$$A_d(K_{1,n}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & n & 0 & \cdots & 0 \\ 1 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & n \end{pmatrix}_{(n+1) \times (n+1)}$$

The Characteristic polynomial of $A_d\left(K_{1,n}\right)$ is

$$f_n\left(K_{1,n}\right,\lambda) = \begin{pmatrix} \lambda-1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda-n & 0 & \cdots & 0 \\ -1 & 0 & \lambda-n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda-n \end{pmatrix}_{(n+1)\times(n+1)}$$

$$= \lambda(\lambda-(n+1)(\lambda-n)^{(n-1)}.$$

and

$$D\ Spec\left(K_{1,n}\right) = \begin{pmatrix} 0 & n+1 & n \\ 1 & 1 & n-1 \end{pmatrix}$$

Therefore, the domatic energy of a star graph is

$$E_d\left(K_{1,n}\right) = (n+1) + n\left(n-1\right) = n^2+1.$$

$\square$

Definition 4.4. *The double star graph $S_{n,m}$ is the graph constructed from union $K_{1,n}$ and $K_{1,m}$ by join whose centers v_0 with u_0. Then $V\left(S_{n,m}\right) = V\left(K_{1,n}\right) \cup V\left(K_{1,m}\right) = \{v_0, v_1, \ldots, v_n, u_0, u_1, \ldots u_m\}$ and edge set $E\left(S_{n,m}\right) = \{v_0u_0, v_0v_i, u_0u_j/1 \leq i \leq n, 1 \leq j \leq m\}$.*

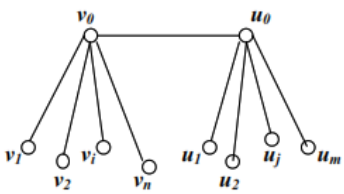

Figure 2. **Double Star Graph**

Theorem 4.5. *For the double star graph $S_{t,t}$ with $t \geq 1$, the domatic energy is $E_d\left(S_{t,t}\right) = 4t^2 + 2t + 4$.*

Proof. For the Double star graph $S_{t,t}$ with $V = \{v_0, v_1, \ldots, v_t, u_0, u_1, \ldots, u_t\}$ the minimum dominating set is $M = \{v_0, u_0\}$. Then the domatic partition of $S_{t,t}$ is $P_k^d\left(S_{t,t}\right) = \{\{u_0, v_0\}, \{u_1, u_2, \ldots, u_t, v_2, \ldots, v_t\}$. Hence

$$A_d\left(S_{t,t}\right) = \begin{pmatrix} 2 & 1 & 1 & \ldots & 1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 2t & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \ldots & 2t & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 & 2 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 2t & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & \ldots & 2t \end{pmatrix}_{2t\times 2t}$$

The Characteristic polynomial of $A_M(S_{t,t})$ is

$$f_n(S_{t,t},\lambda)=\begin{vmatrix} \lambda-2 & -1 & -1 & \ldots & -1 & -1 & 0 & 0 & \ldots. & 0 \\ -1 & \lambda-2t & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots. & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \ldots & \lambda-2t & 0 & 0 & 0 & \ldots. & 0 \\ -1 & 0 & 0 & \ldots & 0 & \lambda-2 & -1 & -1 & \ldots. & -1 \\ 0 & 0 & 0 & \ldots & 0 & -1 & \lambda-2t & 0 & \ldots. & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -1 & 0 & 0 & \ldots & \lambda-2t \end{vmatrix}_{2t\times 2t}$$

$$=\left(\lambda^2-(2t+3)\lambda+5t\right)\left(\lambda^2-(2t+1)\lambda+t\right)(\lambda-2t)^{(2t-2)}$$

Then the domatic spectrum of $S_{t,t}$ is

$$D\ Spec(S_{t,t})=\begin{pmatrix} 2t & \frac{(2t+3)+\sqrt{4t^2-8t+9}}{2} & \frac{(2t+3)-\sqrt{4t^2-8t+9}}{2} & \frac{(2t+1)+\sqrt{4t^2+1}}{2} & \frac{(2t+1)+\sqrt{4t^2+1}}{2} \\ 2t-2 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence, the domatic energy of $S_{t,t}$ is $E_d(S_{t,t})=4t^2+2t+4$. □

Definition 4.6. *The crown graph S_n^0 for an integer $n\geq 3$ is the graph with vertex set $\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\}$ and edge set $\{u_iv_i : 1\leq i,\ j\leq n,\ i\neq j\}$. Therefore S_n^0 coincides with the complete bipartite graph $K_{n,n}$ with the horizontal edges removed.*

Theorem 4.7. *For $n\geq 3$, the domatic energy of the crown graph S_n^0 is $E_d\left(S_n^0\right)=4n$.*

Proof. For the crown graph S_n^0 with vertex set $V=\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\}$, the subset $D=\{u_iv_i : 1\leq i\leq n\}$ is a minimum dominating set of S_n^0. Then $P_k^d\left(S_n^0\right)=\{\{u_i,v_i\} : 1\leq i\leq n\}$. Hence,

$$A_d\left(S_n^0\right)=\begin{pmatrix} 2 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 2 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 2 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 2 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 2 \end{pmatrix}_{2n\times 2n}$$

The Characteristic polynomial of $A_M\left(S_n^0\right)$ is

$$f_n\left(S_n^0\right,\lambda) = \begin{vmatrix} \lambda-2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & \lambda-2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda-2 & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & \lambda-2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda-2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & \lambda-2 \end{vmatrix}_{2n\times 2n}$$

$$= \lambda^{n-1}(\lambda-2)^n(\lambda-2n).$$

Then, the domatic spectrum of S_n^0 is $D\ Spec\left(S_n^0\right) = \begin{pmatrix} 0 & 2 & 2n \\ n-1 & n & 1 \end{pmatrix}$. Therefore, the domatic energy of S_n^0 is $E_d\left(S_n^0\right) = 4n$. □

5. Bounds on Domatic Energy of Graphs

In this section we shall investigate with some bounds for minimum monopoly energy of graphs.

Theorem 5.1. *Let G be a connected graph with n vertices, m edges and domatic number $d(G)$. Then*

$$\sqrt{2m + \sum_{i=1}^{d(G)} |D_i|^3} \leq E_d(G) \leq \sqrt{n\left(2m + \sum_{i=1}^{d(G)} |D_i|^3\right)}.$$

Proof. Let G be a connected graph with n vertices, m edges and let $P_k^d = \{D_1, D_2, \dots, D_k\}$ be the domatic partition sets of $V(G)$, such that $\lambda(G) = |D_1| \leq |D_2| \leq \cdots \leq |D_K|$. Consider the Cauchy-Schwartiz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

By choose $a_i = 1$ and $b_i = |\lambda_i|$ and by Theorem 3.2, we get

$$\left(E_d(G)\right)^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} {\lambda_i}^2\right) \leq n\left(2m + \sum_{i=1}^{d(G)} |D_i|^3\right).$$

Therefore, the upper bound holds. Now, since $\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \geq \sum_{i=1}^{n} {\lambda_i}^2$, it follows by Theorem 3.2, that

$$\left(E_d(G)\right)^2 \geq 2m + \sum_{i=1}^{d(G)} |D_i|^3.$$

Therefore, the lower bound holds. □

Theorem 5.2. *For a connected graph G with n vertices, m edges and domatic number $d(G)$. $E_M(G) \geq \sqrt{2m + d(G)\lambda(G)^3}$.*

Proof. Let G be a connected graph with n vertices and m edges and let $P_k^d = \{D_1, D_2, \ldots, D_k\}$ be the domatic partition sets of $V(G)$, such that $\lambda(G) = |D_1| \leq |D_2| \leq \ldots |D_K|$. By Theorem 5.1, we have

$$\begin{aligned} E_d(G) &\geq \sqrt{2m + \sum_{i=1}^{d(G)} |D_i|^3} \geq \sqrt{2m + \sum_{i=1}^{d(G)} \lambda(G)^3} \\ &= \sqrt{2m + d(G)\lambda(G)^3}. \end{aligned}$$

$\square$

Similar to Koolen and Moulton's [9] upper bound for energy of a graph, upper bound for $E_M(G)$ is given in the following theorem.

Theorem 5.3. *Let G be a graph of order n and size m. Then*

$$E_d(G) \leq \left(\frac{2m}{n} + \frac{n}{d+1}\right) + \sqrt{(n-1)\left[2m + \sum_{i=1}^{d(G)} |D_i|^3 - \left(\frac{2m}{n} + \frac{n}{d+1}\right)^2\right]}.$$

Proof. Consider the Cauchy-Schwartiz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

By choose $a_i = 1$ and $b_i = |\lambda_i|$, we have $\left(\sum\limits_{i=2}^{n} |\lambda_i|\right)^2 \leq \left(\sum\limits_{i=2}^{n} 1\right)\left(\sum\limits_{i=2}^{n} {\lambda_i}^2\right)$. Hence, by Theorem 3.2, we have

$$(E_d(G) - |\lambda_1|)^2 \leq (n-1)\left(2m + \sum_{i=1}^{d(G)} |D_i|^3 - {\lambda_1}^2\right).$$

Therefore,

$$E_d(G) \leq \lambda_1 + \sqrt{(n-1)\left(2m + \sum_{i=1}^{d(G)} |D_i|^3 - {\lambda_1}^2\right)}.$$

From Theorem 3.3, we have $\lambda_1 \geq \frac{2m}{n} + \frac{n}{d+1}$. Since $f(x) = x + \sqrt{(n-1)\left(2m + \sum\limits_{i=1}^{d(G)} |D_i|^3 - x^2\right)}$ is a decreasing function, we have $f(\lambda_1) \leq f\left(\frac{2m}{n} + \frac{n}{d+1}\right)$. Thus, $E_d(G) \leq f(\lambda_1) \leq f\left(\frac{2m}{n} + \frac{n}{d+1}\right)$. Therefore,

$$E_d(G) \leq \left(\frac{2m}{n} + \frac{n}{d+1}\right) + \sqrt{(n-1)\left[2m + \sum_{i=1}^{d(G)} |D_i|^3 - \left(\frac{2m}{n} + \frac{n}{d+1}\right)^2\right]}.$$

$\square$

Theorem 5.4. *Let G be a connected graph with n vertices and m edges. If $D = det(A_d(G)$, then*

$$E_d(G) \geq \sqrt{2m + \sum_{i=1}^{d(G)} |D_i|^3 + n(n-1)D^{\frac{2}{n}}}.$$

Proof. Since $(E_d(G))^2 = \left(\sum\limits_{i=1}^{n} |\lambda_i|\right)^2 = \left(\sum\limits_{i=1}^{n} |\lambda_i|\right)\left(\sum\limits_{i=1}^{n} |\lambda_i|\right) = \sum\limits_{i=1}^{n} |\lambda_i|^2 + 2\sum\limits_{i<j} |\lambda_i||\lambda_j|$. Using the inequality between the arithmetic and geometric means, we get

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{1/[n(n-1)]}.$$

Hence, by this and Theorem 3.2 we get

$$\begin{aligned}(E_d(G))^2 &\geq \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{1/[n(n-1)]}\\ &\geq \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\left(\prod_{i=j}|\lambda_i|\right)^{2(n-1)^{1/[n(n-1)]}}\\ &= \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\left|\prod_{i=j}\lambda_i\right|^{2/n}\\ &= 2m + \sum_{i=1}^{d(G)}|D_i|^3 + n(n-1)D^{\frac{2}{n}}.\end{aligned}$$

□

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