Queuing System of Bulk Arrival Model With Optional Services in Third Stage and Two Different Vacation Policies

Research Article

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Abstract: This paper presents an introduction into the most important queueing theory concepts used for modeling computer performance. In this paper, customers arrive in batches to the queuing system. Arrival of customers follows a poisson process. The service to the customers is given in three stages in which the first two stages are compulsory and the third stage is optional. Server can avail vacation of short or long period after the completion of service. Moreover, the server may breakdown at random which leads to a repair process immediately. By means of supplementary variable technique and generating function approach the probability generating function of the cloud computing user queue size is determined. The other performance measures of the model are derived by means of Little's law. A trade-off between the Quality of service and queue length is observed.

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1. Introduction

Numerous authors have carried out enormous works in this vacation models. Doshi [3] has analyzed a single server vacation models. An M[x]/G/1 queueing system with N-policy and multiple vacations was studied by Lee et al. [6]. Madan [11] analyzed a single server two stage heterogeneous service and binomial schedule server vacations. Madan [9] has given a detailed work on two stage heterogeneous service with deterministic server vacations. Vignesh and Maragathasundari [23] studied a queuing system of compulsory three stages of service with optional fourth stage of service. A M/G/1 queue with Second optional service and Bernoulli schedule vacations was analyzed by Madan et al [8]. Arumuganathan and Ramaswamy [1] analyzed a Bulk Queue with State Dependent Arrivals and Multiple Vacations. Haridass and Arumuganathan [5] analyzed a batch arrival general bulk service queueing system with variant threshold policy for secondary jobs.Maragathasundari [14] made an analysis on bulk arrival queuein model with service interruption and delay in repair. Khalaf et.al [4] studied a batch arrival queuing system with random breakdown, delay time and standby server. Srinivasan and Maragathasundari [16]. studied multi stages of service in non markovian queuing model. Ranjitham [20] analysed the batch arrival queuing model of two stages of service. Ayappan et.al [2] studied M[x]/G/1 queue with two types of service and multiple server vacations with restricted admissibility. Saravanarajan and Chandrasekaran [21] analysed M/G/1 feedback queue with two types of service, Bernoulli schedule vacation and random breakdowns. Rakesh Kumar and Sumeet Kumar [19] studied two heterogeneous

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server Markovian model and retention of reneged customers. Srinivasan and Maragathasundari [15] made an analysis on non markovian queue with multiple server vacation. Rajadurai et al., [18] investigated M[x]/G/1 retrial queue with two phase service Bernoulli server vacation and random breakdowns. Madan [12] has given M[x]/G/1 queue with third stage optional service and deterministic server vacations. Optional service in Non Markovian queue has been discussed in detail by Maragathasundari and Srinivasan [13]. Little J.D.C [7] gave out the proof for queueing performance measures. Madan [10] analyzed the queueing model with short and long vacations.Miriam cathy joy and Maragathasundari [17] investigated a study on the optional services in web hosting queueing service.A queuing system with general service distribution and extended vacation was analysed by Sowmiyah [22]. Vignesh and Maragathasundari [24] studied a batch arrival queueing system of restricted admissibility. All the above works are done in obligatory vacation, discretionary get-away, multi server get-away and so on. Vacation; by it methods for support work are not talked about in detail in view of the need of the system. In this work, short and long vacations are presented as a standard technique which bolsters the system in limiting the framework breakdown and runs easily. General support work can be worked out in short vacation and the significant upkeep work should be possible in long excursion.

1.1.   We have the following assumptions

- Customers arrive at a poison process with arrival rate λ.

- The first essential stage of service follows a general distributions with distribution function $M_1(v)$, and density function $m_1(v)$.

Let $\mu_1(x)$ dx be the conditional probability of completion of the first essential stage of service during $(x, x + dx)$ given that elapsed time is x. So

$$\mu_1(x) = \frac{m_1(x)}{1 - M_1(x)}$$
$$m_1(v) = \mu_1(v)e^{-\int\limits_0^v \mu_1(x)dx}$$

- The second essential stage of service follows a general distributions with distribution function $M_2(v)$, and density function $m_2(v)$.

Let $\mu_2(x)$ dx be the conditional probability of completion of the second essential stage of service during $(x, x + dx)$ given that elapsed time is x. So

$$\mu_2(x) = \frac{m_2(x)}{1 - M_2(x)}$$
$$m_2(v) = \mu_2(v)e^{-\int\limits_0^v \mu_2(x)dx}$$

- Service time of the third stage of optional service follows a general distributions with distribution function $M_i(v)$, and density function $m_i(v)$, $j = 1$ to n

Let $\mu_i(x)$ dx be the conditional probability of completion of the ith stage of service during $(x, x + dx)$ given that elapsed time is x. So

$$\mu_i(x) = \frac{m_i(v)}{1 - M_i(v)}, \; i = 1 \text{ to } n$$
$$m_i(v) = \mu_i(v)e^{-\int\limits_0^v \mu_i(x)dx}, \; i = 1 \text{ to } n$$

- Vacation time of short period V^e follows general (arbitrary) distribution with distribution function $D_1(s)$ and the density function $d_1(s)$. The conditional probability distribution $\gamma_1(x)dx$ of a completion of a vacation time during $(x, x + dx)$ given that the elapsed vacation time is x, is given by

$$\gamma_1(x) = \frac{d_1(v)}{1 - D_1(v)},$$

$$d_1(v) = \gamma_1(v)e^{-\int_0^s \gamma_1(x)dx}$$

- Vacation time of long period V^f follows general (arbitrary) distribution with distribution function $D_2(s)$ and the density function $d_2(s)$. The conditional probability distribution $\gamma_2(x)dx$ of a completion of a vacation time during $(x, x + dx)$ given that the elapsed vacation time is x, is given by

$$\gamma_2(x) = \frac{d_2(v)}{1 - D_2(v)},$$

$$d_2(v) = \gamma_2(v)e^{-\int_0^s \gamma_2(x)dx}$$

1.2. Notations

We define $P_{n,1}^{(1)}(x,t)$ = probability that at time t, there are n customers in the queue excluding the one customer being served, the server is active providing the first stage of service and the elapsed service time for this customer is x.

$P_{n,1}^{(1)}(x,t) = \int_0^\infty P_{n,1}^{(1)}(x,t)dx$ denotes the probability that at time t there are n customer in the queue excluding the one customer in the first stage of service irrespective of the value of x.

$P_{n,1}^{(2)}(x,t)$ = probability that at time t, the server is active providing the second stage of service and there are $n(\geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x.

$P_{n,1}^{(2)}(x,t) = \int_0^\infty P_{n,1}^{(2)}(x,t)dx$ denotes the probability that at time t there are n customer in the queue excluding the one customer in the second stage of service irrespective of the value of x.

$P_{n,1}^{(i)}(x,t)$ = probability that at time t, the server is active providing the optional third stage of service and there are $n(\geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x.

$P_{n,1}^{(i)}(x,t) = \int_0^\infty P_{n,1}^{(3)}(x,t)dx$ denotes the probability that at time t there are n customer in the queue excluding the one customer in the optional third stage of service irrespective of the value of x; $i = 1$ to n.

$V_n^1(x,t)$ = probability that at time t the server is under short vacation with elapsed vacation time x and there are $n(\geq 0)$ customers waiting in the queue for service.

$V_n^1(t) = \int_0^\infty V_n^e(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under short vacation irrespective of the value of x.

$V_n^2(x,t)$ = probability that at time t the server is under long vacation with elapsed vacation time x and there are $n(\geq 0)$customers waiting in the queue for service.

$V_n^2(t) = \int_0^\infty V_n^f(x,t)dx$ denotes the probability that at time $t$there are n customers in the queue and the server is under long vacation irrespective of the value of x.

$Q(t)$ = probability that at time t there are no customers in the system and the server is idle but available in the system.

1.3. Equations Governing the System

The steady state equations based on the assumptions of our model are given as follows:

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t) + \frac{\partial}{\partial x}P_n^{(1)}(x,t) + (\lambda + \mu_1(x)P_n^{(1)}(x,t)) = \sum_{j=1}^{n-1} C_j P_{n-j}^{(i)}(x,t) \tag{1}$$

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t) + \frac{\partial}{\partial x}P_0^{(1)}(x,t) + (\lambda + \mu_1(x)P_0^{(1)}(x,t)) = 0 \tag{2}$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t) + \frac{\partial}{\partial x}P_n^{(2)}(x,t) + (\lambda + \mu_2(x)P_n^{(2)}(x,t)) = \sum_{j=1}^{n-1}C_jP_{n-j}^{(2)}(x,t) \tag{3}$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t) + \frac{\partial}{\partial x}P_0^{(2)}(x,t) + (\lambda + \mu_2(x)P_0^{(2)}(x,t)) = 0 \tag{4}$$

$$\frac{\partial}{\partial x}P_n^{(i)}(x,t) + \frac{\partial}{\partial x}P_n^{(i)}(x,t) + (\lambda + \mu_i(x)P_n^{(i)}(x,t)) = \sum_{j=1}^{n-1}C_jP_{n-j}^{(i)}(x,t) \tag{5}$$

$$\frac{\partial}{\partial x}P_0^{(i)}(x,t) + \frac{\partial}{\partial x}P_0^{(i)}(x,t) + (\lambda + \mu_i(x)P_n^{(i)}(x,t)) = 0 \tag{6}$$

$$\frac{d}{dt}Q(t) + \lambda Q(t) = \int_0^\infty V_0^1(x,t)\gamma_1(x)dx + \int_0^\infty V_0^2(x,t)\gamma_2(x)dx + \gamma_3\int_0^\infty P_0^{(i)}(x,t)\mu_i(x)dx \tag{7}$$

$$\frac{\partial}{\partial x}V_n^{(1)}(x,t) + \frac{\partial}{\partial x}V_n^{(1)}(x,t) + (\lambda + \gamma_1(x)V_n^{(1)}(x,t)) = \lambda\sum_{j=1}^{n-1}C_jV_{n-j}^{(1)}(x,t) \tag{8}$$

$$\frac{\partial}{\partial x}V_0^{(1)}(x,t) + \frac{\partial}{\partial x}V_0^{(1)}(x,t) + (\lambda + \gamma_1(x)V_0^{(1)}(x,t)) = 0 \tag{9}$$

$$\frac{\partial}{\partial x}V_n^{(2)}(x,t) + \frac{\partial}{\partial x}V_n^{(2)}(x,t) + (\lambda + \gamma_2(x)V_n^{(2)}(x,t)) = \lambda\sum_{j=1}^{n-1}C_jV_{n-j}^{(2)}(x,t) \tag{10}$$

$$\frac{\partial}{\partial x}V_0^{(2)}(x,t) + \frac{\partial}{\partial x}V_0^{(2)}(x,t) + (\lambda + \gamma_2(x)V_0^{(2)}(x,t)) = 0 \tag{11}$$

The boundary conditions are given by

$$P_0^{(i)}(x,t) = r_3\int_0^\infty P_{n+1}^{(i)}(x,t)\mu_i(x)dx + \int_0^\infty V_{n+1}^{(1)}(x,t)\gamma_1(x)dx + \int_0^\infty V_{n+1}^{(2)}(x,t)\gamma_2(x)dx + \lambda C_{n+1}Q(t) \tag{12}$$

$$P_n^{(2)}(0,t) = \int_0^\infty P_{n+1}^{(1)}(x,t)\mu_1(x)dx \tag{13}$$

$$P_n^{(i)}(0,t) = m\int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)dx \tag{14}$$

$$V_n^{(1)}(0,t) = \gamma_1\int_0^\infty P_n^{(i)}(x,t)\mu_i(x)dx \tag{15}$$

$$V_n^{(2)}(0,t) = \gamma_2\int_0^\infty P_n^{(i)}(x,t)\mu_i(x)dx \tag{16}$$

Initial Conditions are $Q(0) = 1$, $P_n^{(1)} = 0$, $V_n^{(1)} = 0$, ${}_n^{(2)} = 0$ for $n \geq 0$.

2.   Probability Generating Function of the Queue Size

We define the Probability Generating Function as

$$\left.\begin{aligned}P_n(x,z,t) &= \sum_{n=0}^\infty z^nP_n(x,t)\\ V^{(1)}(x,z,t) &= \sum_{n=0}^\infty z^nV_n^{(1)}(x,t)\\ V^{(2)}(x,z,t) &= \sum_{n=0}^\infty z^nV_n^{(2)}(x,t)\\ P_Q(x,t) &= \sum_{n=0}^\infty z^nP_n(t), C(z) = \sum_{n=0}^\infty C_iz^i, |z| \leq 1\end{aligned}\right\} \tag{17}$$

Similarly Laplace transform of PGF are to be defined, we take Laplace transform of Equation (1) to (16)

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s) + (s + \lambda + \mu_1(x))\bar{P}_n^{(1)}(x,s) = \lambda\sum_{j=1}^{n-1}C_j\bar{P}_{n-j}^{(1)}(x,s) \tag{18}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(0,s) + (s + \lambda + \mu_1(x))\bar{P}_n^{(1)}(0,s) = 0 \tag{19}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s) + (s + \lambda + \mu_2(x))\bar{P}_n^{(2)}(x,s) = \lambda\sum_{j=1}^{n-1}C_j\bar{P}_{n-j}^{(2)}(x,s) \tag{20}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(0,s) + (s + \lambda + \mu_2(x))\bar{P}_n^{(2)}(0,s) = 0 \tag{21}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(i)}(x,s) + (s + \lambda + \mu_i(x))\bar{P}_n^{(i)}(x,s) = \lambda\sum_{j=1}^{n-1}C_j\bar{P}_{n-j}^{(i)}(x,s) \tag{22}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(i)}(0,s) + (s + \lambda + \mu_i(x))\bar{P}_n^{(i)}(0,s) = 0 \tag{23}$$

$$(s+\lambda)\bar{Q}(s) = 1 + \int_0^\infty \bar{V}_0^{(1)}(x,s)\gamma_1(x)dx + \int_0^\infty \bar{V}_0^{(2)}(x,s)\gamma_2(x)dx + \gamma_3\int_0^\infty \bar{P}_0^{(i)}(x,s)\gamma_i(x)dx \tag{24}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(1)}(x,s) + (s + \lambda + \gamma_1(x))\bar{V}_n^{(1)}(x,s) = \lambda\sum_{j=1}^{n-1}C_j\bar{V}_{n-j}^{(1)}(x,s) \tag{25}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(1)}(0,s) + (s + \lambda + \gamma_1(x))\bar{V}_0^{(1)}(0,s) = 0 \tag{26}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(2)}(x,s) + (s + \lambda + \gamma_2(x))\bar{V}_n^{(2)}(x,s) = \lambda\sum_{j=1}^{n-1}C_j\bar{V}_{n-j}^{(2)}(x,s) \tag{27}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(2)}(0,s) + (s + \lambda + \gamma_2(x))\bar{V}_0^{(2)}(0,s) = 0 \tag{28}$$

$$\bar{P}_n^{(1)}(0,s) = r_3\int_0^\infty \bar{P}_{n+1}^{(1)}(x,z,s)\mu_i(x)dx + \int_0^\infty \bar{V}_{n+1}^{(1)}(x,s)\gamma_1(x)dx + \int_0^\infty \bar{V}_{n+1}^{(2)}(x,s)\gamma_2(x)dx$$
$$+ r_3(1-m)\int_0^\infty \bar{P}_{n+1}^{(2)}(x,s)\mu_2(x)dx + \lambda C_{n+1}\bar{Q}(s) \tag{29}$$

$$\bar{P}_n^{(2)}(0,s) = \int_0^\infty \bar{P}_n^{(1)}(x,s)\mu_i(x)dx \tag{30}$$

$$\bar{P}_n^{(i)}(0,s) = m\int_0^\infty \bar{P}_n^{(2)}(x,s)\mu_2(x)dx \tag{31}$$

$$\bar{V}_n^{(1)}(0,s) = r_1\int_0^\infty \bar{P}_n^{(i)}(x,s)\mu_i(x)dx \tag{32}$$

$$\bar{V}_n^{(2)}(0,s) = r_2\int_0^\infty \bar{P}_n^{(i)}(x,s)\mu_i(x)dx \tag{33}$$

Multiply Equation (18) by Z^n, taking summation $n = 1 to \infty$ then adding to Equation (19) and using (17) we get

$$\frac{\partial}{\partial x}\bar{P}^{(1)}(x,z,s) + (s + \lambda - \lambda C(z) + \mu_1(x))\bar{P}^{(1)}(x,z,s) = 0 \tag{34}$$

Similarly,

$$\frac{\partial}{\partial x}\bar{P}^{(2)}(x,z,s) + (s + \lambda - \lambda C(z) + \mu_2(x))\bar{P}^{(2)}(x,z,s) = 0 \tag{35}$$

$$\frac{\partial}{\partial x}\bar{P}^{(i)}(x,z,s) + (s + \lambda - \lambda C(z) + \mu_i(x))\bar{P}^{(i)}(x,z,s) = 0 \tag{36}$$

$$\frac{\partial}{\partial x}\bar{V}^{(1)}(x,z,s) + (s + \lambda - \lambda C(z) + \gamma_1(x))\bar{V}^{(1)}(x,z,s) = 0 \tag{37}$$

$$\frac{\partial}{\partial x}\bar{V}^{(2)}(x,z,s) + (s + \lambda - \lambda C(z) + \gamma_2(x))\bar{V}^{(2)}(x,z,s) = 0 \tag{38}$$

$$z\bar{P}^{(1)}(0,z,s) = r_3\int_0^\infty \bar{P}^{(i)}(x,z,s)\mu_i(x)dx + \int_0^\infty \bar{V}^{(1)}(x,z,s)\gamma_1(x)dx + \int_0^\infty \bar{V}^{(2)}(x,z,s)\gamma_2(x)dx$$

$$-\left[r_3\int_0^\infty \bar{P}_0^{(i)}(x,s)\mu_i(x)dx + \int_0^\infty \bar{V}_0^{(1)}(x,s)\gamma_1(x)dx + \int_0^\infty \bar{V}_0^{(1)}(x,s)\gamma_1(x)dx\right] + \lambda C(z)\bar{Q}(s) \qquad (39)$$

Using (24), Equation (39) becomes,

$$Z\bar{P}^{(1)}(0,z,s) = r_3\int_0^\infty \bar{P}^{(i)}(x,z,s)\mu_i(x)dx + \int_0^\infty \bar{V}^{(1)}(x,z,s)\gamma_1(x)dx + \int_0^\infty \bar{V}^{(2)}(x,z,s)\gamma_2(x)dx$$

$$+ 1 - (s + \lambda - \lambda C(z))\bar{Q}(s) \qquad (40)$$

$$\bar{P}^{(2)}(0,z,s) = \int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx \qquad (41)$$

$$\bar{P}^{(i)}(0,z,s) = m\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx \qquad (42)$$

$$\bar{V}^{(1)}(0,z,s) = r_1\int_0^\infty \bar{P}^{(i)}(x,z,s)\gamma_1(x)dx \qquad (43)$$

$$\bar{V}^{(2)}(x,z,s) = r_2\int_0^\infty \bar{P}^{(i)}(x,z,s)\gamma_2(x)dx \qquad (44)$$

Integrating equations (34) to (38) between 0 & x, we get

$$\bar{P}^{(1)}(x,z,s) = \bar{P}^{(1)}(0,z,s)e^{-s+\lambda-\lambda C(z)x - \int_0^x \mu_1(t)dt} \qquad (45)$$

$$\bar{P}^{(2)}(x,z,s) = \bar{P}^{(2)}(0,z,s)e^{-(s+\lambda-\lambda C(z))x - \int_0^x \mu_2(t)dt} \qquad (46)$$

$$\bar{P}^{(i)}(x,z,s) = \bar{P}^{(i)}(0,z,s)e^{-s+\lambda-\lambda C(z)x - \int_0^x \mu_i(t)dt} \qquad (47)$$

$$\bar{V}^{(1)}(x,z,s) = \bar{V}^{(1)}(0,z,s)e^{-(s+\lambda-\lambda C(z))x - \int_0^x \gamma_1(t)dt} \qquad (48)$$

$$\bar{V}^{(2)}(x,z,s) = \bar{V}^{(2)}(0,z,s)e^{-(s+\lambda-\lambda C(z))x - \int_0^x \gamma_2(t)dt} \qquad (49)$$

Again integrating the above by parts, we get

$$\bar{P}^{(1)}(z,s) = \bar{P}^{(1)}(0,z,s)\left[\frac{1 - \bar{M}_1[s+\lambda-\lambda C(z)]}{s+\lambda-\lambda C(z)}\right] \qquad (50)$$

Where $\bar{M}_1[s+\lambda-\lambda C(z)] = e^{-(s+\lambda-\lambda C(z))x}dM_1(x)$ is the Laplace Stieltjes transform of the service time S_1. Similarly,

$$\bar{P}^{(2)}(z,s) = \bar{P}^{(2)}(0,z,s)\left[\frac{1 - \bar{M}_2[s+\lambda-\lambda C(z)]}{s+\lambda-\lambda C(z)}\right] \qquad (51)$$

$$\bar{P}^{(i)}(z,s) = \bar{P}^{(i)}(0,z,s)\left[\frac{1 - \bar{M}_i[s+\lambda-\lambda C(z)]}{s+\lambda-\lambda C(z)}\right] \qquad (52)$$

$$\bar{V}^{(1)}(z,s) = \bar{V}^{(1)}(0,z,s)\left[\frac{1 - \bar{D}_1[s+\lambda-\lambda C(z)]}{s+\lambda-\lambda C(z)}\right] \qquad (53)$$

$$\bar{V}^{(2)}(z,s) = \bar{V}^{(2)}(0,z,s)\left[\frac{1 - \bar{D}_2[s+\lambda-\lambda C(z)]}{s+\lambda-\lambda C(z)}\right] \qquad (54)$$

Next multiply Equation (45), (46), (47), (48) and (49) by $\mu_1(x), \mu_2(x), \mu_i(x), \gamma_1(x)$ and $\gamma_2(x)$ respectively. And by the usuage of equations (1) to (6), integrating we get,

$$\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}^{(1)}(0,z,s)\bar{M}_1[s+\lambda-\lambda C(z)] \qquad (55)$$

$$\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}^{(2)}(0,z,s)\bar{M}_2[s+\lambda-\lambda C(z)] \qquad (56)$$

$$\int_0^\infty \bar{P}^{(i)}(x,z,s)\mu_i(x)dx = \bar{P}^{(i)}(0,z,s)\bar{M}_i[s+\lambda-\lambda C(z)] \qquad (57)$$

$$\int_0^\infty \bar{V}^{(1)}(x,z,s)\gamma_1(x)dx = \bar{V}^{(1)}(0,z,s)\bar{D}_1[s+\lambda-\lambda C(z)] \qquad (58)$$

$$\int_0^\infty \bar{V}^{(2)}(x,z,s)\gamma_2(x)dx = \bar{V}^{(2)}(0,z,s)\bar{D}_2[s+\lambda-\lambda C(z)] \tag{59}$$

Now the Equations (40) to (44) becomes,

$$\bar{P}^{(1)}(0,z,s) = \frac{(1-(s+\lambda-\lambda C(z))\bar{Q}(s))}{Z-r_3\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]} \tag{60}$$

$$-(\bar{D}_1\left[s+\lambda-\lambda C(z)\right]r_1+\bar{D}_2\left[s+\lambda-\lambda C(z)\right]r_2)m$$

$$\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]$$

$$\bar{P}^{(2)}(0,z,s) = \frac{\bar{M}_1\left[s+\lambda-\lambda C(z)\right](1-(s+\lambda-\lambda C(z))\bar{Q}(s))}{Z-r_3\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]} \tag{61}$$

$$-(\bar{D}_1\left[s+\lambda-\lambda C(z)\right]r_1+\bar{D}_2\left[s+\lambda-\lambda C(z)\right]r_2)m$$

$$\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]$$

$$\bar{P}^{(i)}(0,z,s) = \frac{m\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right](1-(s+\lambda-\lambda C(z))\bar{Q}(s))}{Z-r_3\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]} \tag{62}$$

$$-(\bar{D}_1\left[s+\lambda-\lambda C(z)\right]r_1+\bar{D}_2\left[s+\lambda-\lambda C(z)\right]r_2)m$$

$$\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]$$

$$\bar{V}^1(0,z,s) = \frac{r_1m\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right](1-(s+\lambda-\lambda C(z))\bar{Q}(s))}{Z-r_3\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]} \tag{63}$$

$$-(\bar{D}_1\left[s+\lambda-\lambda C(z)\right]r_1+\bar{D}_2\left[s+\lambda-\lambda C(z)\right]r_2)m$$

$$\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]$$

$$\bar{V}^{(2)}(0,z,s) = \frac{r_2m\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right](1-(s+\lambda-\lambda C(z))\bar{Q}(s))}{Z-r_3\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]} \tag{64}$$

$$-(\bar{D}_1\left[s+\lambda-\lambda C(z)\right]r_1+\bar{D}_2\left[s+\lambda-\lambda C(z)\right]r_2)m$$

$$\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]$$

Using equations (61)-(65) we get,

$$\bar{P}^{(1)}(z,s) = \frac{(1-(s+\lambda-\lambda C(z))\bar{Q}(s))\left[1-\frac{\bar{M}_1[s+\lambda-\lambda C(z)]}{s+\lambda-\lambda C(z)}\right]}{Z-r_3\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]} \tag{65}$$

$$-(\bar{D}_1\left[s+\lambda-\lambda C(z)\right]r_1+\bar{D}_2\left[s+\lambda-\lambda C(z)\right]r_2)m$$

$$\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]$$

$$\bar{P}^{(2)}(z,s) = \frac{(1-(s+\lambda-\lambda C(z))\bar{Q}(s))\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\left[1-\frac{\bar{M}_2[s+\lambda-\lambda C(z)]}{s+\lambda-\lambda C(z)}\right]}{Z-r_3\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]} \tag{66}$$

$$-(\bar{D}_1\left[s+\lambda-\lambda C(z)\right]r_1+\bar{D}_2\left[s+\lambda-\lambda C(z)\right]r_2)m$$

$$\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]$$

$$(1-(s+\lambda-\lambda C(z))\bar{Q}(s))$$

$$\bar{P}^{(i)}(z,s) = \frac{\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\left[1-\frac{\bar{M}_2\left[s+\lambda-\lambda C(z)\right]}{s+\lambda-\lambda C(z)}\right]}{Z-r_3\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]} \tag{67}$$

$$-(\bar{D}_1\left[s+\lambda-\lambda C(z)\right]r_1+\bar{D}_2\left[s+\lambda-\lambda C(z)\right]r_2)m$$

$$\bar{M}_1\left[s+\lambda-\lambda C(z)\right]\bar{M}_2\left[s+\lambda-\lambda C(z)\right]\bar{M}_i\left[s+\lambda-\lambda C(z)\right]$$

$$\bar{V}^{(1)}(z,s) = \cfrac{\begin{array}{c}(1 - (s + \lambda - \lambda C(z))\bar{Q}(s))r_1\bar{M}_1\left[s + \lambda - \lambda C(z)\right] \\[2mm] \bar{M}_2\left[s + \lambda - \lambda C(z)\right]\bar{M}_i\left[s + \lambda - \lambda C(z)\right]\left[1 - \cfrac{\bar{M}_2\left[s + \lambda - \lambda C(z)\right]}{s + \lambda - \lambda C(z)}\right]\end{array}}{\begin{array}{c}Z - r_3\bar{M}_1\left[s + \lambda - \lambda C(z)\right]\bar{M}_2\left[s + \lambda - \lambda C(z)\right]\bar{M}_i\left[s + \lambda - \lambda C(z)\right] \\[2mm] -(\bar{D}_1\left[s + \lambda - \lambda C(z)\right]r_1 + \bar{D}_2\left[s + \lambda - \lambda C(z)\right]r_2)m \\[2mm] \bar{M}_1\left[s + \lambda - \lambda C(z)\right]\bar{M}_2\left[s + \lambda - \lambda C(z)\right]\bar{M}_i\left[s + \lambda - \lambda C(z)\right]\end{array}} \tag{68}$$

$$\bar{V}^{(2)}(z,s) = \cfrac{\begin{array}{c}r_2\bar{M}_1\left[s + \lambda - \lambda C(z)\right]\bar{M}_2\left[s + \lambda - \lambda C(z)\right]\bar{M}_i\left[s + \lambda - \lambda C(z)\right] \\[2mm] (1 - (s + \lambda - \lambda C(z))\bar{Q}(s)) \\[2mm] \left[1 - \cfrac{(\bar{D}_2\left[s + \lambda - \lambda C(z)\right]^2)}{s + \lambda - \lambda C(z)}\right]\end{array}}{\begin{array}{c}Z - r_3\bar{M}_1\left[s + \lambda - \lambda C(z)\right]\bar{M}_2\left[s + \lambda - \lambda C(z)\right]\bar{M}_i\left[s + \lambda - \lambda C(z)\right] \\[2mm] -(\bar{D}_1\left[s + \lambda - \lambda C(z)\right]r_1 + \bar{D}_2\left[s + \lambda - \lambda C(z)\right]r_2)m \\[2mm] \bar{M}_1\left[s + \lambda - \lambda C(z)\right]\bar{M}_2\left[s + \lambda - \lambda C(z)\right]\bar{M}_i\left[s + \lambda - \lambda C(z)\right]\end{array}} \tag{69}$$

3. Steady State Solution

The Corresponding steady state solution can be obtained by the usage of Tauberian Property. Now multiply the above equations (66)-(69) by s, taking limit as $s \to 0$ and applying the property we have,

$$P^{(1)}(z,s) = \cfrac{[-1 + \bar{M}_1[s + \lambda - \lambda C(z)]]Q}{\begin{array}{c}Z - r_3\bar{M}_1[s + \lambda - \lambda C(z)]\bar{M}_2[s + \lambda - \lambda C(z)]\bar{M}_i[s + \lambda - \lambda C(z)] \\[2mm] -(\bar{D}_1[s + \lambda - \lambda C(z)]r_1 + \bar{D}_2[s + \lambda - \lambda C(z)]r_2)m \\[2mm] \bar{M}_1[s + \lambda - \lambda C(z)]\bar{M}_2[s + \lambda - \lambda C(z)]\bar{M}_i[s + \lambda - \lambda C(z)]\end{array}} \tag{70}$$

$$P^{(2)}(z,s) = \cfrac{\bar{M}_1[s + \lambda - \lambda C(z)][-1 + \bar{M}_2[s + \lambda - \lambda C(z)]]Q}{\begin{array}{c}Z - r_3\bar{M}_1[s + \lambda - \lambda C(z)]\bar{M}_2[s + \lambda - \lambda C(z)]\bar{M}_i[s + \lambda - \lambda C(z)] \\[2mm] -(\bar{D}_1[s + \lambda - \lambda C(z)]r_1 + \bar{D}_2[s + \lambda - \lambda C(z)]r_2)m \\[2mm] \bar{M}_1[s + \lambda - \lambda C(z)]\bar{M}_2[s + \lambda - \lambda C(z)]\bar{M}_i[s + \lambda - \lambda C(z)]\end{array}} \tag{71}$$

$$P^{(i)}(z,s) = \cfrac{m\bar{M}_1[s + \lambda - \lambda C(z)]\bar{M}_2[s + \lambda - \lambda C(z)][-1 + \bar{M}_1[s + \lambda - \lambda C(z)]]Q}{\begin{array}{c}Z - r_3\bar{M}_1[s + \lambda - \lambda C(z)]\bar{M}_2[s + \lambda - \lambda C(z)]\bar{M}_i[s + \lambda - \lambda C(z)] \\[2mm] -(\bar{D}_1[s + \lambda - \lambda C(z)]r_1 + \bar{D}_2[s + \lambda - \lambda C(z)]r_2)m \\[2mm] \bar{M}_1[s + \lambda - \lambda C(z)]\bar{M}_2[s + \lambda - \lambda C(z)]\bar{M}_i[s + \lambda - \lambda C(z)]\end{array}} \tag{72}$$

$$V^{(1)}(z,s) = \cfrac{r_1 m\bar{M}_1[s + \lambda - \lambda C(z)]\bar{M}_2[s + \lambda - \lambda C(z)][-1 + \bar{D}_1[s + \lambda - \lambda C(z)]]Q}{\begin{array}{c}Z - r_3\bar{M}_1[s + \lambda - \lambda C(z)]\bar{M}_2[s + \lambda - \lambda C(z)]\bar{M}_i[s + \lambda - \lambda C(z)] \\[2mm] -(\bar{D}_1[s + \lambda - \lambda C(z)]r_1 + \bar{D}_2[s + \lambda - \lambda C(z)]r_2)m \\[2mm] \bar{M}_1[s + \lambda - \lambda C(z)]\bar{M}_2[s + \lambda - \lambda C(z)]\bar{M}_i[s + \lambda - \lambda C(z)]\end{array}} \tag{73}$$

$$V^{(2)}(z,s) = \cfrac{r_2 m\bar{M}_1[s + \lambda - \lambda C(z)]\bar{M}_2[s + \lambda - \lambda C(z)][-1 + \bar{D}_2[s + \lambda - \lambda C(z)]]Q}{\begin{array}{c}Z - r_3\bar{M}_1[s + \lambda - \lambda C(z)]\bar{M}_2[s + \lambda - \lambda C(z)]\bar{M}_i[s + \lambda - \lambda C(z)] \\[2mm] -(\bar{D}_1[s + \lambda - \lambda C(z)]r_1 + \bar{D}_2[s + \lambda - \lambda C(z)]r_2)m \\[2mm] \bar{M}_1[s + \lambda - \lambda C(z)]\bar{M}_2[s + \lambda - \lambda C(z)]\bar{M}_i[s + \lambda - \lambda C(z)]\end{array}} \tag{74}$$

Adding the above equations we have,

$$P_q(z) = P^{(1)}(z) + P^{(2)}(z) + P^{(i)}(z) + V^{(1)}(z) + V^{(2)}(z)$$

To determine Q, the normalizing condition is used $P_q(z) + Q = 1$. For $Z = 1$ the above equations becomes indeterminate. Using L's Hopital's rule, we get

$$P^{(1)}(1) = \frac{\lambda E(I)E(M_1)Q}{D'(1)} \tag{75}$$

$$P^{(2)}(1) = \frac{\lambda E(I)E(M_2)Q}{D'(1)} \tag{76}$$

$$P^{(i)}(1) = \frac{\lambda E(I)E(M_i)Q}{D'(1)} \tag{77}$$

$$V^{(1)}(1) = \frac{r_1 m\lambda E(I)E(D_1)Q}{D'(1)} \tag{78}$$

$$V^{(2)}(1) = \frac{r_2 m\lambda E(I)E(D_2)Q}{D'(1)} \tag{79}$$

Therefore $P_q(1) = P^{(1)}(1) + P^{(2)}(1) + P^{(i)}(1) + V^{(e)}(1) + V^{(f)}(1)$. The Steady state Probability where the system is empty is given by

$$Q = \frac{1 - m\lambda E(I)[E(M_1) + E(M_2) + E(M_i) + r_1 E(D_1) + r_2 E(D_2)]}{1 + \lambda E(I)[1 - m][E(M_1) + E(M_2)]} \tag{80}$$

Also the utilization factor ρ can be found. Let L_q denote the mean queue size. Then

$$L_q = \frac{d}{dz}(P_q(z)|_{z=1} = \frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \tag{81}$$

$$D'(1) = 1 - m\lambda E(I)[E(M_1) + E(M_2) + E(M_i) + r_1 E(D_1) + r_2 E(D_2)] \tag{82}$$

$$D''(1) = 1 - m\{-\lambda E(I(I-1))(E(M_1) + E(M_2) + E(M_i))$$

$$+ (\lambda E(I))^2 \begin{bmatrix} E(M_1^2) + E(M_2^2) + E(M_3^2) \\ +2(E(M_1)E(M_2)) + E(M_2) + E(M_i) + E(M_1) + E(M_i) \end{bmatrix}$$

$$+ (\lambda E(I))^2[E(M_1) + E(M_2) + E(M_i)][r_1 E(D_1) + r_2 E(D_2)] - \lambda E(I(I-1))[r_1 E(D_1) + r_2 E(D_2)]$$

$$+ (\lambda E(I))^2[E(M_1) + E(M_2) + E(M_3)][r_1 E(D_1) + r_2 E(D_2)] + (\lambda E(I))^2[r_1 E(D_1^2) + r_2 E(D_2^2)] \tag{83}$$

$$N'(1) = \lambda E(I)\{E(M_1) + E(M_2) + mE(M_i) + mr_1 E(D_1) + r_2 E(D_2)\} \tag{84}$$

$$N''(1) = (1 - m)[E(I(I-1)[E(M_1) + E(M_2)]] + (\lambda E(I))^2[E(M_1^2) + 2E(M_1)E(M_2) + E(M_2^2)]$$

$$+ m(-\lambda E(I(I-1)))(E(M_1) + E(M_2) + E(M_i)) + m(\lambda E(I))^2[E(M_1^2) + E(M_2^2) + E(M_i^2)$$

$$+ 2(E(M_1)E(M_2) + E(M_2)E(M_i) + E(M_i) + E(M_1))] + m(-\lambda E(I(I-1)))[r_1 E(D_1) + r_2 E(D_2)]$$

$$+ m(\lambda E(I))^2[r_1 E(D_1^2) + r_2 E(D_2^2)] \tag{85}$$

Substituting $D'(1)$, $D''(1)$, $N'(1)$, $N''(1)$ in (81), we get the mean queue size. The other performance measures L, W_q, W can be derived using Little's formula.

3.1. Special Cases

Case (i): Service time & Vacation time follows exponential distribution. Here we have $E(I) = 1$, $E(I(I-1) = 0$, $E(M_1) = \frac{1}{\mu_1}$, $E(M_2) = \frac{1}{\mu_2}$, $E(M_i) = \frac{1}{\mu_i}$,

$$E(D_1) = \frac{1}{\gamma_1}, E(D_2) = \frac{1}{\gamma_2}, E(D_1^2) = \frac{1}{\gamma_1^2}, E(D_2^2) = \frac{1}{\gamma_2^2}$$

$$E(M_1^2) = \frac{2}{\mu_1^2}, E(M_2^2) = \frac{2}{\mu_2^2}, E(M_3^2) = \frac{2}{\mu_3^2}$$

$$P^{(1)}(1) = \frac{\lambda \frac{1}{\mu_1} Q}{D'(1)} \tag{86}$$

$$P^{(2)}(1) = \frac{\lambda \frac{1}{\mu_2} Q}{D'(1)} \tag{87}$$

$$P^{(i)}(1) = \frac{m\lambda \frac{1}{\mu_i} Q}{D'(1)} \tag{88}$$

$$V^{(1)}(1) = \frac{r_1 m \lambda \frac{1}{\gamma_1} Q}{D'(1)} \tag{89}$$

$$V^{(2)}(1) = \frac{r_2 m \lambda \frac{1}{\gamma_2} Q}{D'(1)} \tag{90}$$

$$Q = \frac{1 - m\lambda \left[\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_i} + r_1 \frac{1}{\gamma_1} + r_2 \frac{1}{\gamma_2} \right]}{1 + \lambda \left[1 - m \right] \left[\frac{1}{\mu_1} + \frac{1}{\mu_2} \right]} \tag{91}$$

Case (ii): No Optional Service. In this case put $m = 0$ in the results of the above case.

4.   Conclusion

In this model, we examined a queuing system with batch arrival. Service is provided in three stages. After the completion of first two essential stages of services, the third stage of service is left to the customer?s choice. The third stage of service is provided with n number of voluntary services which leads to a intact approval of the customer. After close of the service, the server takes a vacation. To one side from the vacation policies like single, multiple server and Bernoulli schedule server vacations, a new concept of short and long vacation is discussed here. We investigated this system by means of this new vacation conjecture and extended it in many directions. By adding a new assumption in this model, a diverse and more highly developed queuing system is urbanized. The steady state solution and the performance measures of the queuing system are derived. This model also plays a prominent role in mechanical system like factories, machineries, textile mills, cotton mills etc. As future work, developing the above queuing model with service interruption, delay time in repair process and close down time is suggested.

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