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Edge Cover in a Hypergraph

Research Article

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Abstract: In this paper we introduce a new concept called an edge cover of a hypergraph. We characterize a minimal edge cover in hypergraph. We proved that if G is a hypergraph with minimum edge degree ≥ 2 then the complement of a minimal edge cover is an edge h-dominating set of G. We also proved a necessary & sufficient condition under which the edge covering number of a hypergraph decreases when a vertex v is removed from the hypergraph. We also prove corresponding results for the partial sub hypergraph obtained by removing a vertex from the hypergraph.

MSC: 05C15, 05C69, 05C65.

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1. Introduction

The concept of edge cover is well known for graphs. An edge cover of a graph is a set of edges such that every vertex of the graph is an end vertex of some member of this set. We introduce the concept of an edge cover for hypergraphs. An edge cover of a hypergraph is a set of edges which covers all the vertices of the hypergraph. Every edge cover is an edge dominating set of hypergraph but an edge dominating set may not be an edge cover. In this paper we begin with the characterization of a minimal edge cover in hypergraph. We also consider the operation of removing a vertex from the hypergraph on the edge covering number of a hypergraph.

2. Preliminaries

Definition 2.1 (Hypergraph [4])**.** *A hypergraph G is an ordered pair $(V(G),\ E(G))$ where $V(G)$ is a non-empty finite set and $E(G)$ is a family of non-empty subsets of $V(G) \ni$ their union $= V(G)$. The elements of $V(G)$ are called vertices and the members of $E(G)$ are called edges of the hypergraph G.*

We make the following assumption about the hypergraph.

(1). Any two distinct edges intersect in at most one vertex.

(2). If e_1 and e_2 are distinct edges with $|e_1|, |e_2| > 1$ then $e_1 \nsubseteq e_2$ and $e_2 \nsubseteq e_1$.

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Definition 2.2 (Edge Degree [4]). *Let G be a hypergraph and $v \in V(G)$ then the edge degree of $v = d_E(v) =$ the number of edges containing the vertex v. The minimum edge degree among all the vertices of G is denoted as $\delta_E(G)$ and the maximum edge degree is denoted as $\Delta_E(G)$.*

Definition 2.3 (Dominating Set in Hypergraph [4]). *Let G be a hypergraph and $S \subseteq V(G)$ then S is said to be a dominating set of G if for every $v \in V(G)-S$ there is $u \in S \ni u$ and v are adjacent vertices. A dominating set with minimum cardinality is called minimum dominating set and cardinality of such a set is called domination number of G and it is denoted as $\gamma(G)$.*

Definition 2.4 (Edge Dominating Set [7]). *Let G be a hypergraph and $S \subseteq E(G)$ then S is said to be an edge dominating set of G if for every $e \in E(G) - S$ there is some f in $S \ni e$ and f are adjacent edges. An edge dominating set with minimum cardinality is called a minimum edge dominating set and cardinality of such a set is called edge domination number of G and it is denoted as $\gamma_E(G)$.*

Definition 2.5 (Sub hypergraph and Partial sub hypergraph [3]). *Let G be a hypergraph and $v \in V(G)$. Consider the subset $V(G) - \{v\}$ of $V(G)$. This set will induce two types of hypergraphs from G.*

(1). First type of hypergraph: Here the vertex set $= V(G) - \{v\}$ and the edge set $= \{e'/e' = e - \{v\}$ for some $e \in E(G)\}$. This hypergraph is called the sub hypergraph of G and it is denoted as $G - \{v\}$.

(2). Second type of hypergraph: Here also the vertex set $= V(G) - \{v\}$ and edges in this hypergraph are those edges of G which do not contain the vertex v.This hypergraph is called the partial sub hypergraph of G.

Definition 2.6 (Edge h-Dominating Set [10]). *Let G be a hypergraph. A collection F of edges of G is called an edge h-dominating set of G if*

(1). All isolated edges of G are in F.

(2). If f is not an isolated edge and $f \notin F$ then there is a vertex x in $f \ni$ edge degree of $x \geq 2$ and all the edges containing x except f are in F.

An edge h-dominating set with minimum cardinality is called a minimum edge h-dominating set of G and its cardinality is called edge h-domination number of G and it is denoted as $\gamma'_h(G)$.

We assume that all the hypergraphs considered here are linear. This means that for any two distinct vertices u and v there is at most one edge which contains both u and v.

3. Edge Cover in a Hypergraph

An edge cover of a graph is a set of edges $\ni$ every vertex of the graph is an end vertex of some member of this set. We may similarly define the concept of an edge cover in the hypergraph.

Definition 3.1 (Edge Cover in Hypergraph). *Let G be a hypergraph and F be a set of edges of G then F is said to be an edge cover of G if for every vertex x there is an edge e in $F \ni x \in e$.*

Example 3.2. *Consider the hypergraph G whose vertex set $V(G) = \{1, 2, 3, 4, 5\}$ and $E(G) = \{e_1, e_2, e_3\}$.*

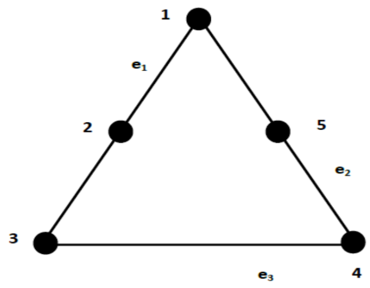

Figure 1:

Here, $F = \{e_1, e_2\}$ is an edge cover of this hypergraph.

Definition 3.3 (Minimal Edge Cover in Hypergraph)**.** *Let G be a hypergraph and F be an edge cover of G then F is said to be a minimal edge cover of G if no proper subset of F is an edge cover of G. Equivalently for every e in F, $F - \{e\}$ is not an edge cover of G.*

Definition 3.4 (Minimum Edge Cover in Hypergraph)**.** *An edge cover with minimum cardinality is called a minimum edge cover of G.*

Definition 3.5 (Edge Covering Number)**.** *Let G be a hypergraph. The cardinality of a minimum edge cover is called the edge covering number of the hypergraph G and it is denoted as $\alpha_1(G)$.*

Obviously every minimum edge cover is a minimal edge cover of a hypergraph but converse is not true.

Example 3.6. *Consider the finite projective plane with $r^2 - r + 1$ vertices and $r^2 - r + 1$ edges, $r \geq 3$ Here edge degree of each vertex is r and cardinality of each edges is also r. Let v be any fixed vertex of r. In each edge containing v there are $r - 1$ vertices different from v. For different edges containing v these $r - 1$ vertices are disjoint. Thus, the total number of vertices contained in these edges $= r(r-1) + 1 = r^2 - r + 1$. Thus, the edge covering number of this finite projective plane $\leq r$. If we consider any collection of $r - 1$ edges, their union will contain at most $r(r-1)$ vertices and $r(r-1) < r^2 - r + 1$. Thus, any collection of $r - 1$ edges cannot cover all the vertices of a finite projective plane. Thus, the edge covering number of the finite projective plane $= r$.*

Example 3.7. *Consider the hypergraph G whose vertex set $V(G) = \{1, 2, 3, 4\}$ and $E(G) = \{e_1, e_2, e_3, e_4\}$ where $e_1 = \{1, 2, 3\}$, $e_2 = \{1, 4\}$, $e_3 = \{2, 4\}$, $e_4 = \{3, 4\}$.*

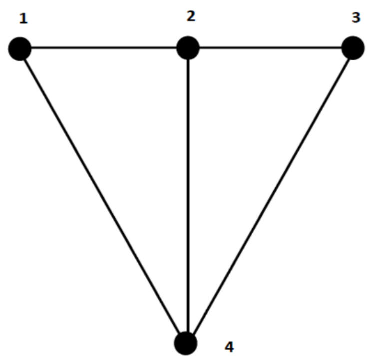

Figure 2:

Here, $\{e_2, e_3, e_4\}$ is a minimal edge cover but not minimum edge cover of this hypergraph. Here, $\{e_1, e_2\}$ is a minimum edge cover of G. Therefore $\alpha_1(G) = 2$.

Remark 3.8. *Let F be an edge cover of a hypergraph G and e be an edge of G and suppose there is a vertex x in $e \ni$ edge degree of $x = 1$ then $e \in F$.*

Proposition 3.9. *Let G be a hypergraph and F be an edge cover of G then F is an edge dominating set of G.*

Proof. Let e be any edge of G. If there is a vertex x in $e \ni$ edge degree of $x = 1$ then $e \in F$. Suppose for every x in e, edge degree of $x \geq 2$ and suppose $e \notin F$. Since F is an edge cover of G there is an edge f in $F \ni x \in f$. Thus, e is adjacent to f and $f \in F$. Thus, F is an edge dominating set of G. □

Example 3.10.

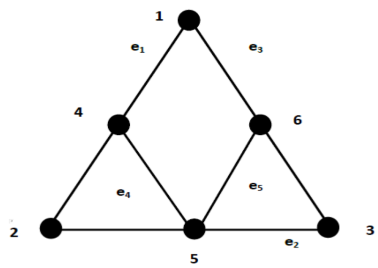

Figure 3:

Here, $F = \{e_1, e_5\}$ is an edge dominating set but it is not an edge cover as $3 \notin F$.

Now, we give a characterization of a minimal edge cover of a hypergraph.

Theorem 3.11. *Let G be a hypergraph and F be an edge cover of G. Then F is a minimal edge cover of G iff for every $e \in F$ one of the following conditions is satisfied.*

(1). There is a vertex x in $e \ni$ edge degree of $x = 1$

(2). The edge degree of every vertex x in e is at least 2 and there is a vertex x_0 in $e \ni$ all the edges containing x_0 except e are in $E(G) - F$.

Proof. Suppose F is a minimal edge cover of G. Let $e \in F$. If condition (1) is satisfied then the requirement is fulfilled. Suppose condition (1) is not satisfied. Then edge degree of $x \geq 2, \forall\, x \in e$. Now, $F - \{e\}$ is not an edge cover of G. Therefore, there is a vertex $x_0 \ni x_0 \notin f, \forall\, f \in F - \{e\}$. However, there is some g in $F \ni x_0 \in g$. It implies that $g = e$. Thus, $x_0 \in e$ and $x_0 \notin f, \forall\, f \in F - \{e\}$. It follows that all the edges containing x_0 except e are in $E(G) - F$. Thus, condition (1) is satisfied.

Conversely suppose F is an edge cover of G and let $e \in F$, then condition (1) or (2) is satisfied. Suppose condition (1) is satisfied. Then $F - \{e\}$ is obviously not an edge cover of G. Suppose condition (2) is satisfied. Then e is the only edge of F which contains x_0. Therefore, there is no edge of $F - \{e\}$ which contains x_0. Thus, it follows that $F - \{e\}$ is not an edge cover of G for each edge $e \in F$. Thus, F is a minimal edge cover of G. □

Theorem 3.12. *Let G be a hypergraph with minimum edge degree ≥ 2. Let F be a set of edges of G if F is a minimal edge cover of G then $E(G) - F$ is an edge h-dominating set of G.*

Proof. Let F be a minimal edge cover of G. Let $e \in F$. By the above theorem there is a vertex x_0 in $e \ni$ all the edges containing x_0 except e are in $E(G) - F$. Thus, $E(G) - F$ is an edge h-dominating set of G. □

Corollary 3.13. *Let G be a hypergraph with minimum edge degree ≥ 2 then $\alpha_1(G) + \gamma'_h(G) \leq m$. Where $m =$ the number of edges in the hypergraph G.*

Proof. Let F be a minimum edge cover of G then F is also a minimal edge cover of G. By the above theorem $E(G) - F$ is an edge h-dominating set of G. Therefore

$$\gamma'_h(G) \leq |E(G) - F|$$
$$= |E(G)| - |F| = m - \alpha_1(G)$$
$$\alpha_1(G) + \gamma'_h(G) \leq m$$

□

Example 3.14. *Consider the hypergraph G whose vertex set $V(G) = \{1, 2, 3, 4, 5, 6\}$ and $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$*

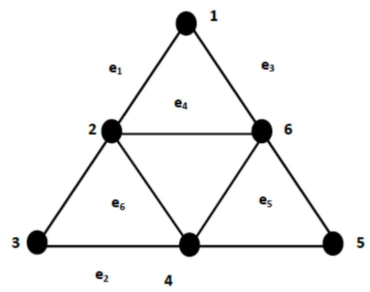

Figure 4:

Here, $\alpha_1(G) = 3$, $\gamma'_h(G) = 3$. Therefore $\alpha_1(G) + \gamma'_h(G) = 6 = m$.

Example 3.15. *Consider the hypergraph G whose vertex set $V(G) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.*

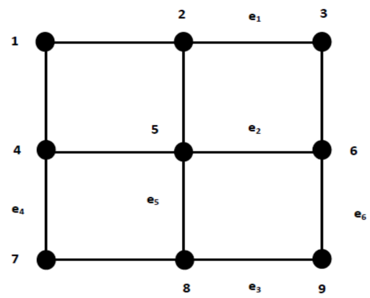

Figure 5:

Here, $\alpha_1(G) = 3$, $\gamma'_h(G) = 2$. Therefore $\alpha_1(G) + \gamma'_h(G) = 5 < m$.

Let G be a hypergraph and $v \in V(G) \ni \{v\}$ is not an edge of G. Consider the subhypergraph $G - v$. Let F be an edge cover of G. Let $F' = \{e\} - v \ni e \in F$. It is obvious that F' is an edge cover of $G - v$. In particular if F is a minimum edge cover of G then F' is an edge cover of $G - v$. Thus, we have following proposition.

Proposition 3.16. *Let G be a hypergraph and $v \in V(G) \ni \{v\}$ is not an edge of G. Consider the subhypergraph $G - v$ then $\alpha_1(G - v) \leq \alpha_1(G)$.*

Example 3.17. *Consider the hypergraph G whose vertex set $V(G) = \{1, 2, 3, 4, 5, 6\}$ and $E(G) = \{e_1, e_2, e_3\}$.*

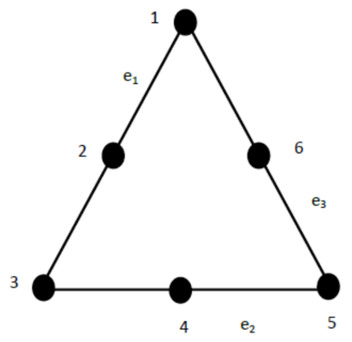

Figure 6:

Here, $\alpha_1(G) = 3$.

(1). Consider $v = 2$. The edges of this subhypergraph $e_1' = \{1, 3\}$, $e_2' = \{3, 4, 5\}$, $e_3' = \{5, 6, 1\}$ obviously, $\{e_2', e_3'\}$ is a minimum edge cover of $G - v$. Therefore $\alpha_1(G - 2) = 2$. Therefore $\alpha_1(G - 2) < \alpha_1(G)$.

(2). In above hypergraph let $v = 1$. The edges of the subhypergraph $G - \{1\}$ are $e_1' = \{2, 3\}$, $e_2' = \{3, 4, 5\}$, $e_3' = \{5, 6\}.\{e_1', e_2', e_3'\}$ is a minimum edge cover of $G - \{1\}$. Therefore $\alpha_1(G - 1) = 3 = \alpha_1(G)$.

Now, we prove a necessary and sufficient condition under which the edge covering number of a hypergraph decreases when a vertex v is removed from the hypergraph.

Theorem 3.18. *Let G be a hypergraph and $v \in V(G) \ni \{v\}$ is not an edge of G. Then $\alpha_1(G - v) < \alpha_1(G)$ iff there is a minimum edge cover F of G and an edge e in $F \ni \forall\, x \in e$ with $x \neq v$ there is another edge f in $F \ni x \in f$.*

Proof. Suppose $\alpha_1(G - v) < \alpha_1(G)$. Let F_1' be a minimum edge cover of $G - v$ and $F_1 = \{e \in E(G) \ni e - \{v\} \in F_1'\}$ then $|F_1| = |F_1'|$ and F_1 is a set of edges of G. Since $|F_1| < \alpha_1(G)$, F_1 cannot be an edge cover of G. Let e be any edge containing v. Let $F = F_1 \cup \{e\}$ then obviously F is an edge cover of G. Since $\alpha_1(G) > \alpha_1(G - v)$, F is a minimum edge cover of G also $e \in F$. Let $x \in e \ni x \neq v$. Since x is a vertex of $G - v$, there is an edge f' in $F_1' \ni x \in F_1'$. Let f be the edge of $G \ni f - \{v\} = f'$ then $f \in F_1$ and $x \in f$ also $f \neq e$. Thus, F satisfies the required condition.
Conversely suppose the condition is satisfied. Let F be a minimum edge cover of G and let $e \in F$ be $\ni \forall\, x \in e$ with $x \neq v$ there is an edge f in $F \ni x \in f$. Let $F_1 = F - \{e\}$ and $F_1' = \{f - v \ni f \in F_1\}$ then F_1' is an edge cover of $G - v$ with $|F_1'| < |F|$. Therefore $\alpha_1(G - v) < \alpha_1(G)$. □

Corollary 3.19. *Let G be a hypergraph and $v \in V(G)$ if $\alpha_1(G - v) < \alpha_1(G)$ then $\alpha_1(G - v) = \alpha_1(G) - 1$.*

Proof. It follows from the first part of the proof of the above theorem. □

Corollary 3.20. *Let G be a hypergraph and $v \in V(G)$ if $\alpha_1(G - v) < \alpha_1(G)$ then for every edge e containing v there is a minimum edge cover F of $G \ni e \in F$.*

Proof. It follows from the first part of the proof of the above theorem. □

Remark 3.21. *Let G be a hypergraph and $v \in V(G) \ni \{v\}$ is not an edge of G and suppose $\alpha_1(G - v) < \alpha_1(G)$. Suppose edge degree of $v = g$ and suppose there are k minimum edge covers of $G - v$. Every minimum edge cover of $G - v$ will give rise to j minimum edge covers of G by the above corollary. Therefore, these k minimum edge covers of $G - v$ will give rise to $j.k$ edge covers of G, each containing some edge containing v.*

3.1. Partial Subhypergraph

Example 3.22. *Consider the hypergraph in Example 3.15. Here, $\alpha_1(G) = 3$ and $\alpha_1(G-5) = 4$. Therefore $\alpha_1(G-v) > \alpha_1(G)$.*

Example 3.23. *Consider the hypergraph in Example 3.17. Here, $\alpha_1(G) = 3$ and $\alpha_1(G-4) = 2$. Therefore $\alpha_1(G-v) < \alpha_1(G)$.*

Example 3.24. *Consider the hypergraph in Example 3.2. Here, $\alpha_1(G) = 2$ and $\alpha_1(G-2) = 2$. Therefore $\alpha_1(G-v) = \alpha_1(G)$.*

Now, we stat and prove a necessary and sufficient condition under which the edge covering number of a hypergraph decreases when a vertex is removed from the hypergraph and the partial subhypergraph is considered.

Theorem 3.25. *Let G be a hypergraph and $v \in V(G)$. Then $\alpha_1(G-v) < \alpha_1(G)$ iff there is a minimum edge covering set F of G and there is exactly one edge e containing $v \ni e \in F$ and $\forall\, x \in e$ with $x \neq v$ there is an edge e_x containing $x \ni e_x \neq e$ and $e_x \in F$.*

Proof. First suppose that the condition is satisfied. Let $F_1 = F - \{e\}$ then F_1 is a set of edges of $G-v$. Let x be any vertex of $G-v$. If $x \in e$ then by our assumption there is an edge e_x containing $x \ni e_x \in F_1$. If $x \notin e$ then there is an edge h in $F \ni x \in h$. Obviously $v \notin h$ because $h \neq e$. Therefore, h is an edge of F_1 containing x. Thus, F_1 is an edge covering set of $G-v$. Thus, $\alpha_1(G-v) \leq |F_1| < |F| = \alpha_1(G)$. Therefore $\alpha_1(G-v) < \alpha_1(G)$.
Conversely suppose $\alpha_1(G-v) < \alpha_1(G)$. Let F_1 be a minimum edge covering set of $G-v$. Then no edge of F_1 contains the vertex v. Let e be any edge of G containing v. Let $F = F_1 \cup \{e\}$. Then obviously F is a minimum edge covering set of G and e is the only edge containing $v \ni e \in F$. Let $x \in e \ni x \neq v$. Since x is a vertex of $G-v$ there is an edge e_x of F_1 containing x. Obviously $e_x \neq e$ and $e_x \in F$. □

Corollary 3.26. *Let G be a hypergraph and $v \in V(G)$ if $\alpha_1(G-v) < \alpha_1(G)$ then $\alpha_1(G-v) = \alpha_1(G) - 1$.*

Proof. It follows from the proof of the above theorem. □

Now, we consider the possibility that the edge covering number of a hypergraph increases when a vertex is removed from the hypergraph.

Notation 3.27. *If v is a vertex of hypergraph G then $N_e(v) = \{e \in E(G) \ni v \in e\}$.*

Theorem 3.28. *Let G be a hypergraph and $v \in V(G)$ then $\alpha_1(G-v) > \alpha_1(G)$ iff the following two conditions are satisfied.*

(1). If F is a minimum edge cover of G then there is an edge e containing $v \ni e \in F$.

(2). There is no set F of edges of $G-v \ni F \subseteq E(G) - N_e(v)$, $|F| \leq \alpha_1(G)$ and F is an edge covering set of $G-v$.

Proof. First suppose that $\alpha_1(G-v) > \alpha_1(G)$.

(1). Suppose there is a minimum edge covering set F of $G \ni F$ does not contain any edge containing v. Then F is a set of edges of $G-v$ and it is obviously an edge cover of $G-v$. Therefore $\alpha_1(G-v) \leq |F| = \alpha_1(G)$. This is a contradiction. Thus, it follows that some minimum set F contains an edge which contains v.

(2). Suppose there is a set F of edges of $G-v \ni F \subseteq E(G) - N_e(v)$, $|F| \leq \alpha_1(G)$ and F is an edge covering set of $G-v$ then $\alpha_1(G-v) \leq |F| \leq \alpha_1(G)$.

This is a contradiction. Hence, condition (2) holds.

Conversely suppose condition (1) and (2) holds. First suppose that $\alpha_1(G-v) = \alpha_1(G)$. Let F be a minimum edge covering set of $G-v$ then F can not contain any edge containing v. Therefore F is a set of edges of $G-v \ni F \subseteq E(G) - N_e(v)$ and F is an edge covering set of $G-v$. This contradicts condition (1) and (2). Now, suppose that $\alpha_1(G-v) < \alpha_1(G)$. Let F be a minimum edge covering set of $G-v$. Then $|F| < \alpha_1(G)$. Here also F can not contain any edge containing v. Thus, F is a set of edges of $G-v \ni F \subseteq E(G) - N_e(v)$, $|F| \leq \alpha_1(G)$ and F is an edge covering set of $G-v$. This contradicts condition (2). Thus, $\alpha_1(G-v) \leq \alpha_1(G)$ is not possible and hence $\alpha_1(G-v) > \alpha_1(G)$. □

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