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Rainbow Connection in Some Brick Product Graphs

Research Article

Kulkarni Sunita Jagannatharao[1]* and R. Murali[1]

1 Department of Mathematics, Dr.Ambedkar Institute of Technology, Bengaluru, Karnataka, India.

Abstract: Let G be a non-trivial connected graph on which is defined a colouring $c : E(G) \to \{1, 2, 3, \ldots, k\}$, $k \in N$ of edges of G, where adjacent edges may be coloured the same. A path P in G is a rainbow path if no two edges of P are coloured the same. G is rainbow-connected if it contains a rainbow $u - v$ path for every two vertices u and v of G. The minimum k for which there exists such a k-edge colouring is called rainbow connection number of G, denoted by $r_c(G)$. In this paper we determine $r_c(G)$ for some brick product graphs $C(2n, m, r)$ associated with even cycles for $m = 2$.

Keywords: Edge colouring, rainbow colouring, brick product graph.

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1. Introduction

Let G be a nontrivial connected graph with an edge coloring $c : E(G) \to \{1, 2, 3, \ldots, k\}$, $k \in N$ where adjacent edges may be colored the same. A path in G is called a rainbow path if no two edges of it are colored the same. An edge colored graph G is said to be rainbow connected if for any two vertices in G, there is a rainbow path in G connecting them. Clearly, if a graph is rainbow connected, then it must be connected. Conversely, any connected graph has a trivial edge coloring that makes it rainbow connected, i.e., a coloring such that each edge has a distinct color. The minimum k for which there exists a rainbow k-coloring of G is called the rainbow connection number of G, denoted by $rc(G)$. If u and v are any two vertices in G, a rainbow $u - v$ geodesic in G is a rainbow $u - v$ path of length $d(u, v)$. G is termed strongly rainbow connected if G contains a rainbow $u - v$ geodesic for every pair of vertices u and v in it. The concept of rainbow connection number was introduced by Chartrand et.al. [2] in 2008. The rainbow connection number of line graphs and some upper bounds for the same were studied by Li and Sun in [3] and [4]. The rainbow connection number of the fan graph, sun graph, gear graph, book graph and cycle-chain graph was obtained by Syafrizal et.al. in [5] and [6]. Various results on the rainbow connection number can be found in [7–10]. An overview about the rainbow connection number can be found in a book in Li and Sun in [11]. In [1], Alspach et.al. have proved that brick product graphs associated with even cycles C_{2n} are Hamiltonian laceable, in the sense that any two vertices at an odd distance apart have a Hamiltonian path. Brick product graphs of even cycles, introduced by Alspach et.al., are a class of three regular graphs that exhibit interesting graph properties. Some results on the rainbow connection number of brick product graphs and modified brick product graphs have been determined by Srinivasa Rao and Murali in [12–14]. We begin with the formal definition of the brick product graph [1] associated with an even cycle.

Definition 1.1. *Let m, n and r be a positive integers. Let $C_{2n} = v_0, v_1, \ldots, v_{2n-1}, v_{2n} = v_0$ denote a cycle of order $2n$. The (m, r) brick product of C_{2n}, denoted by $C(2n, m, r)$ is defined in two cases as follows:*

* *E-mail: sunitambb73@gmail.com*

1. *For $m = 1$, we require that r be odd and greater than 1. Then $C(2n, m, r)$ is obtained from C_{2n} by adding chords $v_{2k}\,(v_{2k+r})$, $k = 1, 2, \ldots, n$, where the computation is performed modulo $2n$.*

2. *For $m > 1$, we require that $m + r$ be even. Then $C(2n, m, r)$ is obtained by first taking the disjoint union of m copies of C_{2n}, namely $C_{2n}(1), C_{2n}(2), \ldots, C_{2n}(m)$ where for each $i = 1, 2, \ldots, m$, $C_{2n}\,(i) = v_{i1}, v_{i2}, \ldots, v_{i2n}, v_{i0}$. Next for each odd $i = 1, 2, \ldots, m-1$ and each even $k = 0, 1, 2, \ldots, 2n-2$, an edge (called a brick edge) is drawn to join (v_i, v_k) to (v_{i+1}, v_k) whereas for each even $i = 1, 2, \ldots, m-1$ and each odd $k = 1, 2, \ldots, 2n-1$ an edge (also called a brick edge) is drawn to join (v_i, v_k) to (v_{i+1}, v_k). Finally, for each odd $k = 1, 2, \ldots, 2n-1$, an edge (called a hooking edge) is drawn to join (v_1, v_k) to (v_m, v_{k+r}). An edge in $C(2n, m, r)$ which is neither a brick edge nor a hooking edge is called a flat edge.*

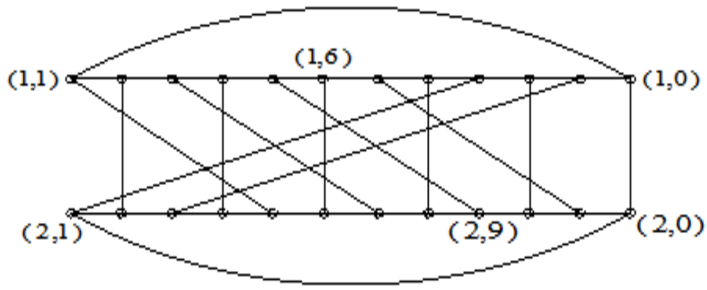

Figure 1. **The brick product graph** $C(12, 2, 4)$

Definition 1.2. *A graph G is termed k-strongly (k^*-strongly) rainbow connected if for every pair of vertices (at least one pair of vertices) u, v such that $d(u, v) = k$, there exists a rainbow path where $1 \leq k \leq diam\,G$. By definition, every strongly rainbow connected graph is 1-strongly rainbow connected.*

2. Results

Theorem 2.1. *Let $G = C(2n, 2, r)$. Then, for $r = 4$ and $n \geq 5$, $r_c(G) = n + 1$.*

Proof. Consider two copies of C_{2n} namely $C_{2n}(1)$ and $C_{2n}(2)$. Let $(V_0)_1, (V_1)_1, (V_2)_1, \ldots, (V_{2n})_1 = (V_0)_1$ be the vertices of the cycle $C_{2n}(1)$ and let $(V_0)_2, (V_1)_2, (V_2)_2, \ldots, (V_{2n})_2 = (V_0)_2$ be the vertices of the cycle $(C_{2n})_2$. Let the edges of G be $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ where

$$E_1 = \{(e_i)_1/(e_i)_1 = ((v_i)_1, (v_{i+1})_1); 0 \leq i < 2n\} \text{ under modulo 2n.}$$

$$E_2 = \{(e_i)_2/(e_i)_2 = ((v_i)_2, (v_{i+1})_2); 0 \leq i < 2n\} \text{ under modulo 2n}$$

$$E_3 = \{(e_{bi})/(e_{bi}) = ((v_i)_1, (v_i)_2); 0 \leq i \leq 2n-2, \text{ where i is even}\} \text{ (Brick edges)}$$

$$E_4 = \{(e_{hi})/(e_{hi}) = ((v_{2i+1})_1, (v_{2i+5})_2) for 0 \leq i < n-2\} \text{ (Hooking edges)}$$

$$E_5 = \{(e_{hi'})/(e_{hi'}) = ((v_{2i+1})_2, (v_{2n+2i-3})_1); i = 0, 1\} \text{ (Brick edges)}$$

Now, let us define a colouring C to the edges of G as follows: $C : e(G) \to \{1, 2, 3, 4, \ldots, n, n+1\}$ such that

$$C : (e_i)_1 = \begin{cases} i, & \text{for } 1 \leq i \leq n \\ i - n, & \text{for } n+1 \leq i \leq 2n \end{cases}$$

$$C:(e_i)_2 = \begin{cases} (n+1)-i, & \text{for } 1 \le i \le n \\ (2n+1)-i, & \text{for } n+1 \le i \le 2n \end{cases}$$

$$C:(e_{bi}) = (n+1), 0 \le i \le 2n-2, \text{ where i is even}$$

$$C:(e_{hi}) = (n+1), 0 \le i \le n-2 \text{ and}$$

$$C:(e_{hi'}) = (n+1), 0 \le i \le 1.$$

Using this assignment of colours it is clear that $r_c(G) = n+1$. (An illustration for the assignment of colors in brick product $C(10,2,4)$ is provided in figure 2)

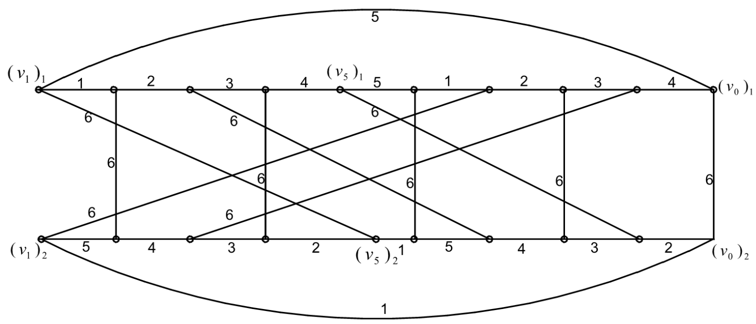

Figure 2. **Assignment of colors in** $C(10,2,4)$

□

Theorem 2.2. *Let $G = (2n, 2, r)$. Then, for $r = 6$ and $n \ge 5$, $r_c(G) = n+1$.*

Proof. We consider the vertices of G as in Theorem 2.1. Let the edges of G be E be $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$, where

$$E_1 = \{(e_i)_1/(e_i)_1 = ((v_i)_1, (v_{i+1})_1); 0 \le i < 2n\} \text{ under modulo 2n.}$$

$$E_2 = \{(e_i)_2/(e_i)_2 = ((v_i)_2, (v_{i+1})_2); 0 \le i < 2n\} \text{ under modulo 2n.}$$

$$E_3 = \{(e_{bi})/(e_{bi}) = ((v_i)_1, (v_i)_2), 0 \le i \le 2n-2 \text{ where i is even }\}$$

$$E_4 = \{(e_{hi})/(e_{hi}) = ((v_{2i+1})_1, (v_{2i+7})_2) for 0 \le i < n-3\}$$

$$E_5 = \{(e_{hi'})/(e_{hi'}) = ((v_{2i+1})_2, (v_{2n+2i-5})_1); 0 \le i \le 2\}$$

Now let us define colouring C to the edges of G as follows: $C: e(G) \to \{1,2,3,4,\ldots,n,n+1\}$ such that

$$C:(e_i)_1 = \begin{cases} i, & \text{for } 1 \le i \le n \\ i-n, & \text{for } n+1 \le i \le 2n \end{cases}$$

$$C:(e_i)_2 = \begin{cases} (n+1)-i, & \text{for } 1 \le i \le n \\ (2n+1)-i, & \text{for } n+1 \le i \le 2n \end{cases}$$

$$C:(e_{bi}) = (n+1), 0 \le i \le 2n-2, \text{ where i is even}$$

$$C:(e_{hi}) = (n+1), 0 \le i \le n-4$$

$$C:(e_{hi'}) = (n+1), 0 \le i \le 2.$$

Using this assignment of colours it is clear that $r_c(G) = n+1$. (An illustration for the assignment of colors in brick product $C(26,2,6)$ is provided in figure 3)

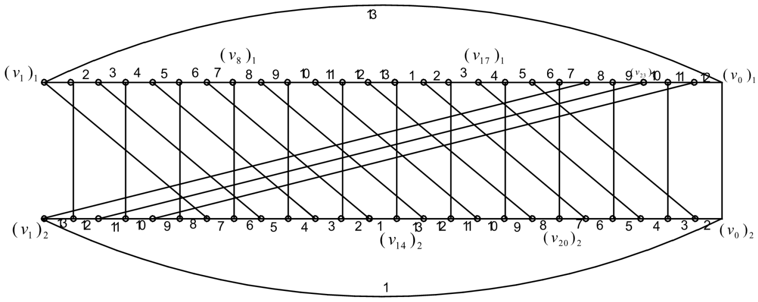

Figure 3. **Assignment of colors in** $C(26, 2, 6)$

□

Theorem 2.3. *Let* $G = (2n, 2, r)$. *Then, for* $r = 8$ *and* $n \geq 5$, $r_c(G) = n + 1$.

Proof. We consider the vertices of G as in Theorem 2.1. Let the edges of G be$E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ where

$$E_1 = \{(e_i)_1/(e_i)_1 = ((v_i)_1, (v_{i+1})_1); 0 \leq i < 2n\} \text{ under modulo 2n.}$$

$$E_2 = \{(e_i)_2/(e_i)_2 = ((v_i)_2, (v_{i+1})_2); 0 \leq i < 2n\} \text{ under modulo 2n.}$$

$$E_3 = \{(e_{bi})/(e_{bi}) = ((v_i)_1, (v_i)_2), 0 \leq i \leq 2n - 2 \text{ where i is even } \}$$

$$E_4 = \{(e_{hi})/(e_{hi}) = ((v_{2i+1})_1, (v_{2i+9})_2) \text{ for } 0 \leq i < n - 4\}$$

$$E_5 = \{(e_{hi'})/(e_{hi'}) = ((v_{2i+1})_2, (v_{2n+2i-7})_1); 0 \leq i \leq 3\}$$

Now let us define colouring C to the edges of G as follows: $C : e(G) \to \{1, 2, 3, 4, \ldots, n, n + 1\}$ such that

$$C : (e_i)_1 = \begin{cases} i, & \text{for } 1 \leq i \leq n \\ i - n, & \text{for } n + 1 \leq i \leq 2n \end{cases}$$

$$C : (e_i)_2 = \begin{cases} (n+1) - i, & \text{for } 1 \leq i \leq n \\ (2n+1) - i, & \text{for } n + 1 \leq i \leq 2n \end{cases}$$

$$C : (e_{bi}) = (n + 1), 0 \leq i \leq 2n - 2, \text{ where 'i' is even}$$

$$C : (e_{hi}) = (n + 1), 0 \leq i \leq n - 5$$

$$C : (e_{hi'}) = (n + 1), 0 \leq i \leq 3.$$

Using this assignment of colours it is clear that $r_c(G) = n+1$. (An illustration for the assignment of colors in brick product $C(16, 2, 8)$ is provided in figure 4)

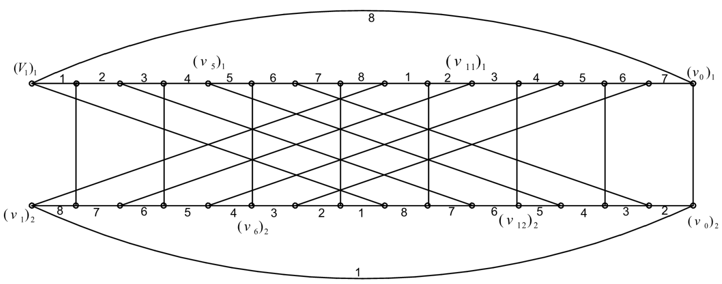

Figure 4. **Assignment of colors in** $C(16, 2, 8)$

□

Theorem 2.4. *Let $G = C(2n, 2, r)$. Then, for $r \geq 4$ and $n \geq 5$, G is 2-strongly rainbow connected.*

Proof. We have the following cases:

Case 1: For every $u, v, w \in C_{2n}(1)$ or $C_{2n}(2)$ such that $u - v - w$ is a path of length two, the edges (u, v) and (v, w) are assigned with colours as in Theorem 2.1, i.e.,

$$C : (e_i)_1 or (e_i)_2 = \begin{cases} i, & 1 \leq i \leq n \\ i - n, & n + 1 \leq i \leq 2n \end{cases} \tag{1}$$

From this assignment of colors, it is clear that $u - v - w$ is a rainbow path.

Case 2: For every $u \in C_{2n}(1)$ and $v, w \in C_{2n}(2)$ such that $u - v - w$ is a path of length two, the edge (u, v) is assigned with the color $(n + 1)$ and, for the edge (v, w) we consider the coloring in (1) above. From this assignment of colors, it is clear that $u - v - w$ is a rainbow path.

Case 3: For every $u \in C_{2n}(2)$ and $v, w \in C_{2n}(1)$ such that $u - v - w$ is a path of length two, the edge (u, v) is assigned with the color $(n + 1)$ and, for the edge (v, w) we consider the coloring in (1) above. From this assignment of colors, it is clear that $u - v - w$ is a rainbow path. □

Consider the graph $G = C(2n, 2, r)$. For $r \geq 4$ and $n \geq 5$, G is only 3*-strongly rainbow connected since for $(v_1)_1, (v_1)_2 \in G$, we have $d((v_1)_1, (v_1)_2) = 3$ and $(v_1)_1 - (v_2)_1 - (v_2)_2 - (v_1)_2$ is a rainbow path. But, for $(v_7)_1, (v_5)_2 \in G$ again we have $d((v_7)_1, (v_5)_2) = 3$ but $(v_7)_1 - (v_6)_1 - (v_6)_2 - (v_5)_2$ is not a rainbow path (figure 2). This observation leads to the following result:

Theorem 2.5. *Let $G = C(2n, 2, r)$. Then, for $r \geq 4$ and $n \geq 5$, G is 3*-strongly rainbow connected.*

3. Conclusion

In this paper, we obtain the rainbow connection number of some brick product graphs $C(2n, m, r)$ for $m = 2$ and $r = 4$, 6 and 8. We also show that for $m = 2$ and $r = 4$, the brick product graphs are 2-strongly rainbow connected and 3*-strongly rainbow connected respectively.

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