International Journal of Mathematics And its Applications
Volume 5, Issue 4–E (2017), 775–791.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Common Fixed Point Theorems on Metric Space for Two and Four maps Using Generalized Altering Distance Functions in Five Variables and Applications to Integral Type Inequalities

Research Article

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Abstract: The concept of existence and uniqueness of fixed points by altering distance between points have been explored by many authors. In this paper, we obtain unique common fixed point results for two and four self mappings by altering distances sub compatible functions with generalization of contractive type condition and application to integral type inequalities.

MSC: 47H10, 54E35.

Keywords: Altering distance functions, Sub compatible, Generalized altering distance, Integral type inequalities.

1. Introduction

Fixed point theory plays an important role in functional analysis. This is a very extensive and wider field. The concept of a metric space was introduced by M. Ferchet [12]. Fixed point theory beginning from Banach contraction principle of Banach [2] (1922) with complete metric spaces as a background and went back to Brouwer fixed point theorem of Brouwer [7, 8] (1910) with $\mathbb{R}^n$ as background. It has wider applications in differential and integral equations in mathematical science and engineering. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways. The study of the existence and uniqueness of common fixed point of mappings satisfying contractive type condition has been a very active field of research. Obtaining fixed point theorems for self-maps of a metric space by altering distances between the points with the use of certain continuous control functions is an interesting aspect. The fixed point theorems related to altering distances between points in complete metric spaces have been obtained initially by D. Delbosco [11] and F. Skof [23] in 1977. M. S. Khan et al. [15] initiated the idea of obtaining fixed point of self maps of a metric space by altering distance between the points with the use of a certain continuous control function. K. P. R. Sastry and G. V. R. Babu [21] discussed and established the existence of fixed points for the orbits of single self-maps and pairs of self-maps by using a control function. K. P. R. Sastry et al. [20, 22] proved fixed point theorems in complete metric spaces by using a

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continuous control function. B. S. Choudhury et al. [9, 10], G. V. R. Babu et al. [3, 4, 5, 6], S. V. R. Naidu [16, 17], K. P. R. Rao et al. [18, 19] proved some common fixed point results by altering distances. Aliouche [1] proved common fixed point results in symmetric spaces for weakly compatible mappings under contractive condition of integral type. Hesseni [13,14] used contractive rule of integral type by altering distance and generalized common fixed point results. Recently, Mishra et al. [24] proved two common fixed point theorems in metric spaces by altering distance function. The main aim of this paper is to prove the existence and uniqueness of common fixed points of two pairs of sub compatible mappings by using a generalized altering distance function of five variables and apply them to integral type inequalities. As a result, we obtain the results of Vishnu Narayan Mishra et al. [24] as corollaries of our results.

2. Preliminaries

Definition 2.1 ([15]). *A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function, if it satisfies following conditions.*

1. *$\psi(t)$ is monotonically increasing and continuous.*
2. *$\psi(t) = 0$ iff $t = 0$.*

Definition 2.2. *A function $\psi : {R^+}^n \to R^+ = [0,\infty)$ is called a generalized altering distance function on ${R^+}^n$ if ψ is continuous, monotone increasing in each variable and*

$$\psi(x_1, x_2, \dots, x_n) = 0 \text{ if and only if } x_1 = x_2 = \cdots = x_n = 0.$$

The collection of all generalized altering distances is denoted by Ψ_n. Suppose $\psi \in \Psi_n$. Now we define a function $\phi_\psi(y)$ by $\phi_\psi(y) = \psi(y, y, \dots, y)$ for $y \in [0,\infty)$. Clearly $\phi_\psi(y) = 0$ if and only if $y = 0$.

Definition 2.3. *Two maps $p, q : X \to X$ of a metric space (X, d) are called sub compatible if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} p(x_n) = \lim_{n\to\infty} q(x_n) = t, \quad t \in X \Rightarrow \lim_{n\to\infty} d(pqx_n, qpx_n) = 0.$$

3. Main Result

Now we state and prove our first main result.

Theorem 3.1. *Let (X, d) be a complete metric space and $\psi_1, \psi_2 \in \Psi_5$. Suppose $U, V : X \to X$ are such that for all $x, y \in X$*

$$\begin{aligned}\varphi_1(d(Ux,Vy)) \le \psi_1\left(d(x,y), d(Ux,x), d(Vy,y), \frac{1}{2}\{d(Vy,x)+d(Ux,y)\}, \frac{1}{2}\{d(x,y)+\max\{d(x,Ux), d(y,Vy)\}\}\right)\\ -\psi_2\left(d(x,y), d(Ux,x), d(Vy,y), \frac{1}{2}\{d(Vy,x)+d(Ux,y)\}, \frac{1}{2}\{d(x,y)+\max\{d(x,Ux), d(y,Vy)\}\}\right) \quad (1)\end{aligned}$$

where $\varphi_1(\alpha) = \psi_1(\alpha,\alpha,\alpha,\alpha,\alpha)$, $\alpha \in [0,\infty)$. Then U and V have a unique common fixed point in X.

Proof. Suppose

$$x_0 \in X.$$

$$x_1 = Ux_0$$

$$x_2 = Vx_1 \ldots .$$

Inductively,

$$x_{2n+1} = Ux_{2n} \tag{2}$$

$$x_{2n+2} = Vx_{2n+1} \tag{3}$$

Let $x = x_{2n+2}$ and $y = x_{2n+1}$. Substituting for x and y in (1). We get,

$$\varphi_1 (d(Ux_{2n+2}, Vx_{2n+1})) \leq \psi_1 \begin{pmatrix} d(x_{2n+2}, x_{2n+1}), d(Ux_{2n+2}, x_{2n+2}), d(Vx_{2n+1}, x_{2n+1}), \\ \frac{1}{2}\{d(Vx_{2n+1}, x_{2n+2}) + d(Ux_{2n+2}, x_{2n+1})\}, \\ \frac{1}{2}\{d(x_{2n+2}, x_{2n+1}) + \max\{d(x_{2n+2}, Ux_{2n+2}), d(x_{2n+1}, Vx_{2n+1})\}\} \end{pmatrix}$$
$$- \psi_2 \begin{pmatrix} d(x_{2n+2}, x_{2n+1}), d(Ux_{2n+2}, x_{2n+2}), d(Vx_{2n+1}, x_{2n+1}), \\ \frac{1}{2}\{d(Vx_{2n+1}, x_{2n+2}) + d(Ux_{2n+2}, x_{2n+1})\}, \\ \frac{1}{2}\{d(x_{2n+2}, x_{2n+1}) + \max\{d(x_{2n+2}, Ux_{2n+2}), d(x_{2n+1}, Vx_{2n+1})\}\} \end{pmatrix}$$

From equations (2) and (3)

$$\varphi_1 (d(x_{2n+3}, x_{2n+2})) \leq \psi_1 \begin{pmatrix} d(x_{2n+2}, x_{2n+1}), d(x_{2n+3}, x_{2n+2}), d(x_{2n+2}, x_{2n+1}), \\ \frac{1}{2}\{d(x_{2n+2}, x_{2n+2}) + d(x_{2n+3}, x_{2n+1})\}, \\ \frac{1}{2}[d(x_{2n+2}, x_{2n+1}) + \max\{d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2})\}] \end{pmatrix}$$
$$- \psi_2 \begin{pmatrix} d(x_{2n+2}, x_{2n+1}), d(x_{2n+3}, x_{2n+2}), d(x_{2n+2}, x_{2n+1}), \\ \frac{1}{2}\{d(x_{2n+2}, x_{2n+2}) + d(x_{2n+3}, x_{2n+1})\}, \\ \frac{1}{2}[d(x_{2n+2}, x_{2n+1}) + \max\{d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2})\}] \end{pmatrix} \tag{4}$$

Write

$$d(x_n, x_{n+1}) = a_n \tag{5}$$

From (4) and (5) we get,

$$\varphi_1 (a_{2n+2}) \leq \psi_1 \begin{pmatrix} a_{2n+1}, a_{2n+2}, a_{2n+1}, \frac{1}{2}\{d(x_{2n+3}, x_{2n+1})\}, \\ \frac{1}{2}\{a_{2n+1} + \max\{a_{2n+2}, a_{2n+1}\}\} \end{pmatrix} - \psi_2 \begin{pmatrix} a_{2n+1}, a_{2n+2}, a_{2n+1}, \frac{1}{2}\{d(x_{2n+3}, x_{2n+1})\}, \\ \frac{1}{2}\{a_{2n+1} + \max\{a_{2n+2}, a_{2n+1}\}\} \end{pmatrix}$$

$$\varphi_1 (a_{2n+2}) \leq \psi_1 \begin{pmatrix} a_{2n+1}, a_{2n+2}, a_{2n+1}, \frac{1}{2}[a_{2n+1} + a_{2n+2}], \\ \frac{1}{2}[a_{2n+1} + \max\{a_{2n+2}, a_{2n+1}\}] \end{pmatrix} - \psi_2 \begin{pmatrix} a_{2n+1}, a_{2n+2}, a_{2n+1}, \frac{1}{2}[a_{2n+1} + a_{2n+2}], \\ \frac{1}{2}[a_{2n+1} + \max\{a_{2n+2}, a_{2n+1}\}] \end{pmatrix} \tag{6}$$

$$\leq \psi_1 (\alpha, \alpha, \alpha, \alpha, \alpha) - \psi_2 \begin{pmatrix} a_{2n+1}, a_{2n+2}, a_{2n+1}, \frac{1}{2}[a_{2n+1} + a_{2n+2}], \\ \frac{1}{2}[a_{2n+1} + \max\{a_{2n+2}, a_{2n+1}\}] \end{pmatrix} \quad where \quad \alpha = \max\{a_{2n+2}, a_{2n+1}\} \tag{7}$$

$$\leq \varphi_1 (\alpha) - \psi_2 \begin{pmatrix} a_{2n+1}, a_{2n+2}, a_{2n+1}, \frac{1}{2}[a_{2n+1} + a_{2n+2}], \\ \frac{1}{2}[a_{2n+1} + \max\{a_{2n+2}, a_{2n+1}\}] \end{pmatrix}$$

Therefore, $\varphi_1 (a_{2n+2}) < \varphi_1 (\alpha)$ (since $\psi_2 \begin{pmatrix} a_{2n+1}, a_{2n+2}, a_{2n+1}, \frac{1}{2}[a_{2n+1} + a_{2n+2}], \\ \frac{1}{2}[a_{2n+1} + \max\{a_{2n+2}, a_{2n+1}\}] \end{pmatrix} \neq 0$)

$$a_{2n+2} < \alpha \Rightarrow a_{2n+2} < a_{2n+1} \tag{8}$$

Now let $x = x_{2n}$ and $y = x_{2n+1}$. Then from (1), we get

$$\varphi_1(a_{2n+1}) \leq \psi_1\begin{pmatrix} a_{2n}, a_{2n}, a_{2n+1}, \frac{1}{2}\{d(x_{2n}, x_{2n+2})\}, \\ \frac{1}{2}\{a_{2n} + \max\{a_{2n+1}, a_{2n}\}\} \end{pmatrix} - \psi_2\begin{pmatrix} a_{2n}, a_{2n},\ a_{2n+1}, \frac{1}{2}\{d(x_{2n}, x_{2n+2})\}, \\ \frac{1}{2}\{a_{2n} + \max\{a_{2n+1}, a_{2n}\}\} \end{pmatrix}$$

$$\begin{aligned} \varphi_1(a_{2n+1}) &\leq \psi_1\begin{pmatrix} a_{2n}, a_{2n}, a_{2n+1}, \frac{1}{2}\{a_{2n} + a_{2n+1}\}, \\ \frac{1}{2}\{a_{2n} + \max\{a_{2n+1}, a_{2n}\}\} \end{pmatrix} - \psi_2\begin{pmatrix} a_{2n}, a_{2n}, a_{2n+1}, \frac{1}{2}\{a_{2n} + a_{2n+1}\}, \\ \frac{1}{2}\{a_{2n} + \max\{a_{2n+1}, a_{2n}\}\} \end{pmatrix} \\ &\leq \psi_1(\beta, \beta, \beta, \beta, \beta) - \psi_2\begin{pmatrix} a_{2n}, a_{2n}, a_{2n+1}, \frac{1}{2}\{a_{2n} + a_{2n+1}\}, \\ \frac{1}{2}\{a_{2n} + \max\{a_{2n+1}, a_{2n}\}\} \end{pmatrix} \quad \textit{where} \quad \beta = \max\{a_{2n}, a_{2n+1}\} \qquad (9) \\ &\leq \varphi_1(\beta) - \psi_2(\beta, \beta, \beta, \beta, \beta) \end{aligned}$$

Therefore, $\varphi_1(a_{2n+1}) < \varphi_1(\beta)$ (since $\psi_2\begin{pmatrix} a_{2n}, a_{2n+1}, a_{2n}\frac{1}{2}\{a_{2n} + a_{2n+1}\}, \\ \frac{1}{2}\{a_{2n} + \max\{a_{2n+1}, a_{2n}\}\} \end{pmatrix} \neq 0$). Therefore,

$$a_{2n+1} < \beta \Rightarrow a_{2n+1} < a_{2n} \qquad (10)$$

From (8) and (10), we get, $a_{2n+2} < a_{2n+1} < a_{2n}$. Hence, $a_{n+1} < a_n$ so that $\{a_n\} \downarrow$ strictly, say, to d. Therefore $\{\varphi_1(a_n)\} \downarrow$ strictly, say, to δ. On letting $n \to \infty$ in (6), we get

$$\varphi_1(\delta) \leq \psi_1(\delta, \delta, \delta, \delta, \delta) - \psi_2(\delta, \delta, \delta, \delta, \delta) = \varphi_1(\delta) - \varphi_2(\delta)$$

Therefore, $\varphi_1(\delta) \leq \varphi_1(\delta) - \varphi_2(\delta) < \varphi_1(\delta)$ if $\delta > 0$ a contradiction. Therefore, $\delta = 0$. But $\gamma = \lim_{n\to\infty} \varphi_1(a_n) = \varphi_1(\delta) \Rightarrow \gamma = 0$. Therefore, $a_n = d(x_n, x_{n+1}) \to 0$. Now, we show that the sequence $\{x_n\}$ is a Cauchy sequence in X. For this we first show that $\{x_{2n}\}$ is a Cauchy sequence. If $\{x_{2n}\}$ is not a Cauchy sequence then there exist $\epsilon > 0$ and natural numbers $\{2m(k), 2n(k)\}$ such that $k < m(k) < n(k)$, $d\left(x_{2m(k)}, x_{2n(k)}\right) \geq \epsilon$ and $d\left(x_{2m(k)}, x_{2n(k)-2}\right) < \epsilon$.

$$\begin{aligned} \epsilon &\leq d\left(x_{2m(k)}, x_{2n(k)}\right) \\ &\leq d\left(x_{2m(k)}, x_{2n(k)-2}\right) + d\left(x_{2n(k)-2}, x_{2n(k)}\right) \\ &\leq \epsilon + d\left(x_{2n(k)}, x_{2n(k)-1}\right) + d\left(x_{2n(k)-1}, x_{2n(k)-2}\right) \end{aligned}$$

Taking $k \to \infty$ in the inequality, it follows that

$$\lim_{k\to\infty} d\left(x_{2m(k)}, x_{2n(k)}\right) = \epsilon \qquad (11)$$

Consider,

$$\begin{aligned} d\left(x_{2n(k)+1}, x_{2m(k)}\right) &\leq d\left(x_{2n(k)+1}, x_{2n(k)}\right) + d\left(x_{2n(k)}, x_{2m(k)}\right) \\ d\left(x_{2n(k)}, x_{2m(k)}\right) &\leq d\left(x_{2n(k)}, x_{2n(k)+1}\right) + d\left(x_{2n(k)+1}, x_{2m(k)}\right) \end{aligned}$$

On letting $k \to \infty$ we get,

$$\limsup_{k\to\infty} d\left(x_{2n(k)+1}, x_{2m(k)}\right) \leq \epsilon \;\; \textit{and} \;\; \liminf_{k\to\infty} d\left(x_{2n(k)+1}, x_{2m(k)}\right) \geq \epsilon \Rightarrow \lim_{k\to\infty} d\left(x_{2n(k)+1}, x_{2m(k)}\right) = \epsilon \qquad (12)$$

Consider,

$$d\left(x_{2n(k)}, x_{2m(k)-1}\right) \leq d\left(x_{2n(k)}, x_{2m(k)}\right) + d\left(x_{2m(k)}, x_{2m(k)-1}\right)$$
$$d\left(x_{2n(k)}, x_{2m(k)}\right) \leq d\left(x_{2n(k)}, x_{2m(k)-1}\right) + d\left(x_{2m(k)-1}, x_{2m(k)}\right)$$

On letting $k \to \infty$ we get,

$$\limsup_{k\to\infty} d\left(x_{2n(k)}, x_{2m(k)-1}\right) \leq \epsilon \;\; and \;\; \liminf_{k\to\infty} d\left(x_{2n(k)}, x_{2m(k)-1}\right) \geq \epsilon \Rightarrow \lim_{k\to\infty} d\left(x_{2n(k)}, x_{2m(k)-1}\right) = \epsilon \tag{13}$$

Consider,

$$d\left(x_{2n(k)+1}, x_{2m(k)-1}\right) \leq d\left(x_{2n(k)+1}, x_{2m(k)}\right) + d\left(x_{2m(k)}, x_{2m(k)-1}\right)$$
$$d\left(x_{2n(k)+1}, x_{2m(k)}\right) \leq d\left(x_{2n(k)+1}, x_{2m(k)-1}\right) + d\left(x_{2m(k)-1}, x_{2m(k)}\right)$$

On letting $k \to \infty$ we get,

$$\limsup_{k\to\infty} d\left(x_{2n(k)+1}, x_{2m(k)-1}\right) \leq \epsilon \;\; and \;\; \liminf_{k\to\infty} d\left(x_{2n(k)+1}, x_{2m(k)-1}\right) \geq \epsilon \Rightarrow \lim_{k\to\infty} d\left(x_{2n(k)+1}, x_{2m(k)-1}\right) = \epsilon \tag{14}$$

Substituting $x = x_{2n(k)}$ and $y = x_{2m(k)-1}$ in (1), we get

$$\begin{aligned}\varphi_1\left(d\left(x_{2n(k)+1}, x_{2m(k)}\right)\right) \leq{}& \psi_1\begin{pmatrix} d\left(x_{2n(k)}, x_{2m(k)-1}\right), d\left(x_{2n(k)+1}, x_{2n(k)}\right), d\left(x_{2m(k)}, x_{2m(k)-1}\right), \\ \frac{1}{2}\left\{d\left(x_{2m(k)}, x_{2n(k)}\right) + d\left(x_{2n(k)+1}, x_{2m(k)-1}\right)\right\}, \\ \frac{1}{2}[d\left(x_{2n(k)}, x_{2m(k)-1}\right) + \max\left\{d\left(x_{2n(k)}, x_{2n(k)+1}\right), d\left(x_{2m(k)-1}, x_{2m(k)}\right)\right\}] \end{pmatrix} \\ &- \psi_2\begin{pmatrix} d\left(x_{2n(k)}, x_{2m(k)-1}\right), d\left(x_{2n(k)+1}, x_{2n(k)}\right), d\left(x_{2m(k)}, x_{2m(k)-1}\right), \\ \frac{1}{2}\left\{d\left(x_{2m(k)}, x_{2n(k)}\right) + d\left(x_{2n(k)+1}, x_{2m(k)-1}\right)\right\}, \\ \frac{1}{2}[d\left(x_{2n(k)}, \; x_{2m(k)-1}\right) + \max\left\{d\left(x_{2n(k)}, \; x_{2n(k)+1}\right), d\left(x_{2m(k)-1}, x_{2m(k)}\right)\right\}\,] \end{pmatrix}\end{aligned}$$

On letting $k \to \infty$ we get, $\varphi_1(\epsilon) \leq \psi_1\left(\epsilon, 0, 0, \epsilon, \frac{\epsilon}{2}\right) - \psi_2\left(\epsilon, 0, 0, \epsilon, \frac{\epsilon}{2}\right)$ (from (11), (12), (13) and (14)). Therefore, $\varphi_1(\epsilon) \leq \varphi_1(\epsilon) - \psi_2\left(\epsilon, 0, 0, \epsilon, \frac{\epsilon}{2}\right)$. Therefore, $\varphi_1(\epsilon) < \varphi_1(\epsilon)$ (since $\epsilon > 0$ and hence $\psi_2\left(\epsilon, 0, 0, \epsilon, \frac{\epsilon}{2}\right) \neq 0$) a contradiction. Therefore, $\{x_{2n}\}$ is a Cauchy sequence. Similarly, we can show that $\{x_{2n+1}\}$ is a Cauchy sequence. Since $a_n = d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, now follows that $\{x_n\}$ is a Cauchy sequence. Suppose $\{x_n\} \to l$. Substituting, $x = x_{2n}$ and $y = l$ in (1), we get

$$\begin{aligned}\varphi_1\left(d\left(Ux_{2n}, Vl\right)\right) \leq{}& \psi_1\begin{pmatrix} d\left(x_{2n}, l\right), d\left(Ux_{2n}, x_{2n}\right), d\left(Vl, l\right), \frac{1}{2}\left\{d\left(Vl, x_{2n}\right) + d\left(Ux_{2n}, l\right)\right\}, \\ \frac{1}{2}\left\{d\left(x_{2n}, l\right) + \max\left\{d\left(x_{2n}, Ux_{2n}\right), d\left(l, \; Vl\right)\right\}\right\} \end{pmatrix} \\ &- \psi_2\begin{pmatrix} d\left(x_{2n}, l\right), d\left(Ux_{2n}, x_{2n}\right), d\left(Vl, l\right), \frac{1}{2}\left\{d\left(Vl, x_{2n}\right) + d\left(Ux_{2n}, l\right)\right\}, \\ \frac{1}{2}\left\{d\left(x_{2n}, l\right) + \max\left\{d\left(x_{2n}, Ux_{2n}\right), d\left(l, Vl\right)\right\}\right\} \end{pmatrix} \\ \varphi_1\left(d\left(x_{2n+1}, Vl\right)\right) \leq{}& \psi_1\begin{pmatrix} d\left(x_{2n}, l\right), d\left(x_{2n+1}, x_{2n}\right), d\left(Vl, l\right), \frac{1}{2}\left\{d\left(Vl, x_{2n}\right) + d\left(x_{2n+1}, l\right)\right\}, \\ \frac{1}{2}\left\{d\left(x_{2n}, l\right) + \max\left\{d\left(x_{2n}, x_{2n+1}\right), d\left(l, Vl\right)\right\}\right\} \end{pmatrix} \\ &- \psi_2\begin{pmatrix} d\left(x_{2n}, l\right), d\left(x_{2n+1}, x_{2n}\right), d\left(Vl, l\right), \frac{1}{2}\left\{d\left(Vl, x_{2n}\right) + d\left(x_{2n+1}, l\right)\right\}, \\ \frac{1}{2}\left\{d\left(x_{2n}, l\right) + \max\left\{d\left(x_{2n}, x_{2n+1}\right), d\left(l, Vl\right)\right\}\right\} \end{pmatrix}\end{aligned}$$

On letting $n \to \infty$ we get,

$$\begin{aligned}\varphi_1 (d(l, Vl)) &\le \psi_1 \begin{pmatrix} d(l,l), d(l,l), d(Vl,l), \frac{1}{2}\{d(Vl,l) + d(l,l)\}, \\ \frac{1}{2}\{d(l,l) + \max\{d(l,l), d(l,Vl)\}\} \end{pmatrix} - \psi_2 \begin{pmatrix} d(l,l), d(l,l), d(Vl,l), \frac{1}{2}\{d(Vl,l) + d(l,l)\}, \\ \frac{1}{2}\{d(l,l) + \max\{d(l,l), d(l,Vl)\}\} \end{pmatrix} \\ &= \psi_1 \left(0, 0, d(Vl,l), \frac{1}{2}\{d(Vl,l)\}, \frac{1}{2}\{d(l,Vl)\}\right) - \psi_2 \left(0, 0, d(Vl,l), \frac{1}{2}\{d(Vl,l)\}, \frac{1}{2}\{d(l,Vl)\}\right) \\ &\le \psi_1 (d(Vl,l), d(Vl,l), d(Vl,l), d(Vl,l), d(Vl,l)) - \psi_2 \left(0, 0, d(Vl,l), \frac{1}{2}\{d(Vl,l)\}, \frac{1}{2}\{d(l,Vl)\}\right) \\ &= \varphi_1 (d(Vl,l)) - \psi_2 \left(0, 0, d(Vl,l), \frac{1}{2}\{d(Vl,l)\}, \frac{1}{2}\{d(l,Vl)\}\right)\end{aligned}$$

Therefore, $\varphi_1 (d(Vl,l)) < \varphi_1 (d(Vl,l))$ (if $Vl \neq l$) a contradiction. Therefore, $Vl = l$. Therefore, l is a fixed point of V. Similarly, we prove $Ul = l$. Therefore, $Ul = Vl = l$. Therefore, l is a common fixed point of U and V. Suppose, h and l are common fixed point of U and V. From (1) we get,

$$\begin{aligned}\varphi_1 (d(Uh, Vl)) &\le \psi_1 \left(d(h,l), d(Uh,h), d(Vl,l), \frac{1}{2}\{d(Vl,h) + d(Uh,l)\}, \frac{1}{2}\{d(h,l) + \max\{d(h,Uh), d(l,Vl)\}\}\right) \\ &- \psi_2 \left(d(h,l), d(Uh,h), d(Vl,l), \frac{1}{2}\{d(Vl,h) + d(Uh,l)\}, \frac{1}{2}\{d(h,l) + \max\{d(h,Uh), d(l,Vl)\}\}\right) \\ &= \psi_1 \left(d(h,l), 0, 0, d(h,l), \frac{1}{2}d(h,l)\right) - \psi_2 \left(d(h,l), 0, 0, d(h,l), \frac{1}{2}d(h,l)\right) \\ &= \varphi_1 (d(h,l)) - \psi_2 \left(d(h,l), 0, 0, d(h,l), \frac{1}{2}d(h,l)\right)\end{aligned}$$

Therefore, $\varphi_1 (d(h,l)) < \varphi_1 (d(h,l))$ (if $h \neq l$) a contradiction. Therefore, $h = l$. Therefore, U and V have a unique common fixed point in X. □

Corollary 3.2. *Let (X, d) be a complete metric space and $\psi_1, \psi_2 \in \Psi_5$. Suppose $U, V : X \to X$ are such that for all $x, y \in X$*

$$\begin{aligned}\varphi_1 (d(Ux, Vy)) &\le \psi_1 \left(d(x,y), d(Ux,x), d(Vy,y), \frac{1}{2}\{d(Vy,x) + d(Ux,y)\}, \frac{1}{2}\{d(x, y) + d(x,Ux)\}\right) \\ &- \psi_2 \left(d(x,y), d(Ux,x), d(Vy,y), \frac{1}{2}\{d(Vy,x) + d(Ux,y)\}, \frac{1}{2}\{d(x,y) + d(x,Ux)\}\right) \qquad (15)\end{aligned}$$

where $\varphi_1 (\alpha) = \psi_1 (\alpha, \alpha, \alpha, \alpha, \alpha)$, $\alpha \in [0, \infty)$. Then U and V have a unique common fixed point in X.

Proof. Since (15) $\to$ (1), the result follows from Theorem 3.1. □

The following theorem is an application of Theorem 3.1 to integral type inequalities.

Theorem 3.3. *Suppose (X, d) is a complete metric space and $U, V : X \to X$ be such that for all $x, y \in X$*

$$\begin{aligned}\int_0^{\varphi_1(d(Ux,Vy))} \eta(t)\, dt &\le \int_0^{\psi_1\left(d(x,y), d(Ux,x), d(Vy,y), \frac{1}{2}\{d(Vy,x)+d(Ux,y)\}, \frac{1}{2}\{d(x,y)+\max\{d(x,Ux), d(y,Vy)\}\}\right)} \eta(t)\, dt \\ &- \int_0^{\psi_2\left(d(x,y), d(Ux,x), d(Vy,y), \frac{1}{2}\{d(Vy,x)+d(Ux,y)\}, \frac{1}{2}\{d(x,y)+\max\{d(x,Ux), d(y,Vy)\}\}\right)} \eta(t)\, dt \qquad (16)\end{aligned}$$

where $\psi_1, \psi_2 \in \Psi_5$ with $\varphi_1 (\alpha) = \psi_1 (\alpha, \alpha, \alpha, \alpha, \alpha)$, $\alpha \in [0, \infty)$ and $\eta : R^+ \longrightarrow R^+$ is a Lebesgue-integrable function, which is non negative, summable, and $\int_0^{\epsilon} \eta(t)\, dt > 0$ for each $\epsilon > 0$. Then U and V have unique common fixed point in X.

Proof. We first show that (1) holds for U and V. Suppose for some $x, y \in X$, (1) does not hold. Then

$$\varphi_1 (d(Ux, Vy)) > \psi_1 \left(d(x,y), d(Ux,x), d(Vy,y), \frac{1}{2}\{d(Vy,x) + d(Ux,y)\}, \frac{1}{2}\{d(x,y) + \max\{d(x, Ux), d(y,Vy)\}\}\right)$$

$$- \psi_2 \left(d(x,y), d(Ux,x), d(Vy,y), \frac{1}{2}\{d(Vy,x)+d(Ux,y)\}, \frac{1}{2}\{d(x,y)+\max\{d(x,\ Ux), d(y,Vy)\}\} \right)$$

Write,

$$\epsilon = \varphi_1(d(Ux,Vy)) - \begin{pmatrix} \psi_1\left(d(x,y), d(Ux,x), d(Vy,y), \frac{1}{2}\{d(Vy,\ x)+d(Ux,y)\}, \frac{1}{2}\{d(x,y)+\max\{d(x,Ux), d(y,Vy)\}\ \}\right) \\ -\psi_2\left(d(x,y), d(Ux,x), d(Vy,y), \frac{1}{2}\{d(Vy,x)+d(Ux,\ y)\}, \frac{1}{2}\{d(x,y)+\max\{d(x,Ux), d(y,Vy)\}\}\right) \end{pmatrix}$$

Then $\epsilon > 0$. By hypothesis, $\int_0^{\epsilon} \eta(t)\,dt > 0$. Therefore,

$$\int_0^{\varphi_1(d(Ux,Vy))} \eta(t)\,dt > \int_0^{\psi_1\left(d(x,y),d(Ux,x),d(Vy,y),\frac{1}{2}\{d(Vy,x)+d(Ux,y)\},\frac{1}{2}\{d(x,y)+\max\{d(x,\ Ux),d(y,Vy)\}\}\right)} \eta(t)\,dt - \int_0^{\psi_2\left(d(x,y),d(Ux,x),d(Vy,y),\frac{1}{2}\{d(Vy,x)+d(Ux,y)\},\frac{1}{2}\{d(x,y)+\max\{d(x,\ Ux),d(y,Vy)\}\}\right)} \eta(t)\,dt$$

a contradiction. Therefore,

$$\varphi_1(d(Ux,Vy)) \le \psi_1\left(d(x,y), d(Ux,\ x), d(Vy,y), \frac{1}{2}\{d(Vy,x)+d(Ux,y)\}, \frac{1}{2}\{d(x,y)+\max\{d(x,Ux), d(y,Vy)\}\right) - \psi_2\left(d(x,y), d(Ux,x), d(Vy,y), \frac{1}{2}\{d(Vy,x)+d(Ux,y)\}, \frac{1}{2}\{d(x,y)+\max\{d(x,Ux), d(y,Vy)\}\right)$$

Thus (1) holds for U and V. Therefore, by Theorem 3.1, U and V have a unique fixed point. □

Now we have the following Corollary which is due to Mishra et al. [24].

Corollary 3.4 ([24])**.** *Suppose (X,d) is a complete metric space and $U,V : X \to X$ be such that for all $x,y \in X$*

$$\int_0^{\varphi_1(d(Ux,Vy))} \eta(t)\,dt \le \int_0^{\psi_1\left(d(x,y),d(Ux,x),d(Vy,y),\frac{1}{2}\{d(Vy,x)+d(Ux,y)\},\frac{1}{2}\{d(x,y)+d(x,Ux)\}\right)} \eta(t)\,dt - \int_0^{\psi_2\left(d(x,y),d(Ux,x),d(Vy,y),\frac{1}{2}\{d(Vy,x)+d(Ux,y)\},\frac{1}{2}\{d(x,y)+d(x,Ux)\}\right)} \eta(t)\,dt$$

where $\psi_1, \psi_2 \in \Psi_5$ with $\varphi_1(\alpha) = \psi_1(\alpha,\alpha,\alpha,\alpha,\alpha)$, $\alpha \in [0,\infty)$ and $\eta : R^+ \to R^+$ is a Lebesgue-integrable function, which is non negative, summable, and $\int_0^{\epsilon} \eta(t)\,dt > 0$ for each $\epsilon > 0$. Then U and V have unique common fixed point in X.

Proof. This follows from Corollary 3.2. □

Now we extend Corollary 3.2 to four maps.

Theorem 3.5. *Let (X,d) be complete metric space and f,g,U and V be four mappings from X to itself such that*

$$\varphi_1(d(fx,gy)) \le \psi_1\left(d(Ux,Vy), d(Ux,fx), d(Vy,gy), \frac{1}{2}[d(gy,Ux)+d(fx,Vy)], \frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right) - \psi_2\left(d(Ux,Vy), d(Ux,fx), d(Vy,gy), \frac{1}{2}[d(gy,Ux)+d(fx,Vy)], \frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right) \quad (17)$$

for all $x,y \in X$, where $\psi_1, \psi_2 \in \Psi_5$ with $\varphi_1(\alpha) = \psi_1(\alpha,\alpha,\alpha,\alpha,\alpha)$, for $\alpha \in [0,\infty)$.

(1). One of the four mappings f,g,U and V is continuous.

(2). (f,U) and (g,V) are sub compatible.

(3). $f(X) \subseteq V(X)$ *and* $g(X) \subseteq U(X)$.

Then f, g, U *and* V *have unique common fixed point in* X.

Proof. Let $s \in X$. Then there exist t and r such that

$$fs = Vt \tag{18}$$

$$gt = Ur \tag{19}$$

From (17), (18) and (19), we get

$$\begin{aligned}\varphi_1(d(fs,gt)) &\le \psi_1\left(d(Us,fs), d(Us,fs), d(fs,gt), \frac{1}{2}[d(gt,Us)+d(fs,fs)], \frac{1}{2}[d(Us,fs)+d(Us,fs)]\right)\\ &- \psi_2\left(d(Us,fs), d(Us,fs), d(fs,gt), \frac{1}{2}[d(gt,Us)+d(fs,fs)], \frac{1}{2}[d(Us,fs)+d(Us,fs)]\right)\\ &= \psi_1\left(d(Us,fs), d(Us,fs), d(fs,gt), \frac{1}{2}d(gt,Us), d(Us,fs)\right)\\ &- \psi_2\left(d(Us,fs), d(Us,fs), d(fs,gt), \frac{1}{2}d(gt,Us), d(Us,fs)\right)\\ &\le \psi_1\left(d(Us,fs), d(Us,fs), d(fs,gt), \frac{1}{2}[d(Us,fs)+d(fs,gt)], d(Us,fs)\right)\\ &- \psi_2\left(d(Us,fs), d(Us,fs), d(fs,gt), \frac{1}{2}[d(Us,fs)+d(fs,gt)], d(Us,fs)\right)\end{aligned}$$

write $\alpha = \max\{d(Us,fs), d(fs,gt)\}$. Then, since ψ is increasing in each variable, we get

$$\varphi_1(d(fs,gt)) \le \psi_1(\alpha,\alpha,\alpha,\alpha,\alpha) - \psi_2\left(d(Us,fs), d(Us,fs), d(fs,gt), \frac{1}{2}[d(Us,fs)+d(fs,gt)], d(Us,fs)\right)$$
$$\varphi_1(d(fs,gt)) \le \varphi_1(\alpha) - \psi_2\left(d(Us,fs), d(Us,fs), d(fs,gt), \frac{1}{2}[d(Us,fs)+d(fs,gt)], d(Us,fs)\right)$$

Suppose $d(Us,fs) \le d(fs,gt)$. Then

$$\begin{aligned}\varphi_1(d(fs,gt)) &\le \varphi_1(d(fs,gt)) - \psi_2\left(d(Us,fs), d(Us,fs), d(fs,gt), \frac{1}{2}[d(Us,fs)+d(fs,gt)], d(Us,fs)\right)\\ &< \varphi_1(d(fs,gt)) \quad if \quad d(fs,gt) > 0\end{aligned}$$

a contradiction. Therefore, $d(fs,gt) = 0$. Consequently, $d(Us,fs) = d(fs,gt) = 0$ so that,

$$\varphi_1(d(fs,gt)) \le \varphi_1(d(Us,fs)) \tag{20}$$

Suppose, $d(Us,fs) > d(fs,gt)$. Then

$$\begin{aligned}\varphi_1(d(fs,gt)) &\le \varphi_1(d(Us,fs)) - \psi_2\left(d(Us,fs), d(Us,fs), d(fs,gt), \frac{1}{2}[d(Us,fs)+d(fs,gt)], d(Us,fs)\right)\\ &< \varphi_1(d(Us,fs))\,(since \quad 0 \le d(fs,gt) \le d(Us,fs))\end{aligned} \tag{21}$$

From (20) and (21) follows that, therefore, $\varphi_1(d(fs,gt)) \le \varphi_1(d(Us,fs))$

$$\varphi_1(d(fs,gt)) \le \varphi_1(d(Us,fs)) = \varphi_1(d(Us,Vt)) \quad (from \quad (18))$$

Hence,

$$d\left(fs,gt\right)\le d\left(Us,Vt\right) \tag{22}$$

Let $x_0 \in X$.

$$fx_0 = Vx_1$$
$$gx_1 = Ux_2 \ldots$$

In general,

$$fx_{2n} = Vx_{2n+1} \quad \text{and} \quad gx_{2n+1} = Ux_{2n+2} \tag{23}$$

Substituting $s = x_{2n}$ and $t = x_{2n+1}$ in (22)

$$d\left(fx_{2n},gx_{2n+1}\right)\le d\left(Ux_{2n},Vx_{2n+1}\right)\Rightarrow d\left(fx_{2n},gx_{2n+1}\right)\le d\left(gx_{2n-1},fx_{2n}\right) \tag{24}$$

Substituting $s = x_{2n+2}$ and $t = x_{2n+1}$ in (22)

$$d\left(fx_{2n+2},gx_{2n+1}\right)\le d\left(Ux_{2n+2},Vx_{2n+1}\right)\Rightarrow d\left(fx_{2n+2},gx_{2n+1}\right)\le d\left(gx_{2n+1},fx_{2n}\right) \tag{25}$$

From (24) and (25)

$$d\left(fx_{2n+2},gx_{2n+1}\right)\le d\left(fx_{2n},gx_{2n+1}\right)\le d\left(fx_{2n},gx_{2n-1}\right)$$

In general,

$$d\left(fx_{2n+2},gx_{2n+1}\right)\le d\left(fx_{2n},gx_{2n+1}\right)\le d\left(fx_{2n},gx_{2n-1}\right)\le\cdots\le d\left(fx_2,gx_1\right)\le d\left(fx_0,gx_1\right)$$

From this it follows that the sequences $\{d\left(fx_{2n},gx_{2n-1}\right)\}$ and $\{d\left(fx_{2n+2},gx_{2n+1}\right)\}$ are the both decreasing sequences, decreasing to the same limit. Suppose,

$$\beta = \lim_{n\to\infty} d\left(fx_{2n+2},gx_{2n+1}\right) = \lim_{n\to\infty} d\left(fx_{2n},gx_{2n-1}\right) \tag{26}$$

From, (17)

$$\begin{aligned}\varphi_1\left(d\left(fx_{2n+2},gx_{2n+1}\right)\right)\le{}&\psi_1\begin{pmatrix} d\left(Ux_{2n+2},Vx_{2n+1}\right),d\left(Ux_{2n+2},fx_{2n+2}\right),d\left(Vx_{2n+1},gx_{2n+1}\right),\\ \frac{1}{2}\left[d\left(gx_{2n+1},Ux_{2n+2}\right)+d\left(fx_{2n+2},Vx_{2n+1}\right)\right],\\ \frac{1}{2}\left[d\left(Ux_{2n+2},Vx_{2n+1}\right)+d\left(Ux_{2n+2},fx_{2n+2}\right)\right]\end{pmatrix}\\ &-\psi_2\begin{pmatrix} d\left(Ux_{2n+2},Vx_{2n+1}\right),d\left(Ux_{2n+2},fx_{2n+2}\right),d\left(Vx_{2n+1},gx_{2n+1}\right),\\ \frac{1}{2}\left[d\left(gx_{2n+1},Ux_{2n+2}\right)+d\left(fx_{2n+2},Vx_{2n+1}\right)\right],\\ \frac{1}{2}\left[d\left(Ux_{2n+2},Vx_{2n+1}\right)+d\left(Ux_{2n+2},fx_{2n+2}\right)\right]\end{pmatrix}\end{aligned}$$

From, (23)

$$\begin{aligned}\psi_1\left(d\left(fx_{2n+2},gx_{2n+1}\right)\right)\le{}&\psi_1\begin{pmatrix} d\left(gx_{2n+1},fx_{2n}\right),d\left(gx_{2n+1},fx_{2n+2}\right),d\left(fx_{2n},gx_{2n+1}\right),\frac{1}{2}\left[d\left(fx_{2n+2},fx_{2n}\right)\right],\\ \frac{1}{2}\left[d\left(gx_{2n+1},fx_{2n}\right)+d\left(gx_{2n+1},fx_{2n+2}\right)\right]\end{pmatrix}\\ &-\psi_2\begin{pmatrix} d\left(gx_{2n+1},fx_{2n}\right),d\left(gx_{2n+1},fx_{2n+2}\right),d\left(fx_{2n},gx_{2n+1}\right),\frac{1}{2}\left[d\left(fx_{2n+2},fx_{2n}\right)\right],\\ \frac{1}{2}\left[d\left(gx_{2n+1},fx_{2n}\right)+d\left(gx_{2n+1},fx_{2n+2}\right)\right]\end{pmatrix}\end{aligned}$$

$$\varphi_1\left(d\left(fx_{2n+2},gx_{2n+1}\right)\right)\leq\psi_1\begin{pmatrix}d\left(gx_{2n+1},fx_{2n}\right),d\left(gx_{2n+1},fx_{2n+2}\right),d\left(fx_{2n},gx_{2n+1}\right),\\ \frac{1}{2}\left[d\left(fx_{2n+2},gx_{2n+1}\right)+d\left(gx_{2n+1},fx_{2n}\right)\right],\\ \frac{1}{2}\left[d\left(gx_{2n+1},fx_{2n}\right)+d\left(gx_{2n+1},fx_{2n+2}\right)\right]\end{pmatrix}$$
$$-\psi_2\begin{pmatrix}d\left(gx_{2n+1},fx_{2n}\right),d\left(gx_{2n+1},fx_{2n+2}\right),d\left(fx_{2n},gx_{2n+1}\right),0,\\ \frac{1}{2}\left[d\left(gx_{2n+1},fx_{2n}\right)+d\left(gx_{2n+1},fx_{2n+2}\right)\right]\end{pmatrix}$$

On letting $n\to\infty$, we get

$$\varphi_1(\beta)\leq\psi_1(\beta,\beta,\beta,\beta,\beta)-\psi_2(\beta,\beta,\beta,0,\beta)\quad(from\ (26))$$
$$\varphi_1(\beta)\leq\varphi_1(\beta)-\psi_2(\beta,\beta,\beta,0,\beta)$$

Therefore, $\varphi_1(\beta)<\varphi_1(\beta)$ if $\beta>0$ a contradiction. Therefore, $\beta=0$. Therefore,

$$\beta=\lim_{n\to\infty}d\left(fx_{2n+2},gx_{2n+1}\right)=\lim_{n\to\infty}d\left(fx_{2n},gx_{2n-1}\right)=0\quad(from\ \ (26))$$

Write

$$y_n=\begin{cases}fx_n & \text{if n is even}\\ gx_n & \text{if n is odd}\end{cases}$$

Therefore, $d(y_{2n+2},y_{2n+1})\leq d(y_{2n},y_{2n-1})$. Therefore,

$$d(y_{n+1},y_n)\to 0\quad as\quad n\to\infty \tag{27}$$

Now, to prove the sequence $\{y_n\}$ is a Cauchy sequence in X, it is sufficient to prove that $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there is an $\epsilon>0$, sequence $\{2m(k),2n(k)\}$ with $k<m(k)<n(k)$

$$d\left(y_{2m(k)},y_{2n(k)}\right)\geq\epsilon\quad and\quad d\left(y_{2m(k)},y_{2n(k)-2}\right)<\epsilon$$
$$\begin{aligned}\epsilon&\leq d\left(y_{2m(k)},y_{2n(k)}\right)\\ &\leq d\left(y_{2m(k)},y_{2n(k)-2}\right)+d\left(y_{2n(k)-2},y_{2n(k)}\right)\\ &\leq\epsilon+d\left(y_{2n(k)-2},y_{2n(k)}\right)\\ &\leq\epsilon+d\left(y_{2n(k)},y_{2n(k)-1}\right)+d\left(y_{2n(k)-1},y_{2n(k)-2}\right)\end{aligned}$$

Taking $n\to\infty$ in the inequality

$$\lim_{k\to\infty}d\left(y_{2m(k)},y_{2n(k)}\right)=\epsilon \tag{28}$$

Consider,

$$d\left(y_{2m(k)},y_{2n(k)-1}\right)\leq d\left(y_{2m(k)},x_{2n(k)}\right)+d\left(y_{2n(k)},y_{2n(k)+1}\right)$$
$$d\left(y_{2m(k)},y_{2n(k)}\right)\leq d\left(y_{2m(k)},y_{2n(k)-1}\right)+d\left(y_{2n(k)-1},y_{2n(k)}\right)$$

On letting $k\to\infty$ we get,

$$\limsup_{k\to\infty}d\left(y_{2m(k)},y_{2n(k)-1}\right)\leq\epsilon\ \ and\ \ \liminf_{k\to\infty}d\left(y_{2m(k)},y_{2n(k)-1}\right)\geq\epsilon\Rightarrow\lim_{k\to\infty}d\left(y_{2m(k)},y_{2n(k)-1}\right)=\epsilon \tag{29}$$

Consider,

$$d\left(y_{2m(k)-1},y_{2n(k)-1}\right)\leq d\left(y_{2m(k)-1},y_{2m(k)}\right)+d\left(y_{2m(k)},y_{2n(k)-1}\right)$$
$$d\left(y_{2m(k)},y_{2n(k)-1}\right)\leq d\left(y_{2m(k)-1},y_{2n(k)-1}\right)+d\left(y_{2m(k)-1},y_{2m(k)}\right)$$

On letting $k\to\infty$ we get,

$$\limsup_{k\to\infty} d\left(y_{2m(k)-1},y_{2n(k)-1}\right)\leq\epsilon \;\; and \;\; \liminf_{k\to\infty} d\left(y_{2m(k)-1},y_{2n(k)-1}\right)\geq\epsilon\Rightarrow\lim_{k\to\infty} d\left(y_{2m(k)-1},y_{2n(k)-1}\right)=\epsilon \tag{30}$$

Consider,

$$d\left(y_{2m(k)-1},y_{2n(k)-2}\right)\leq d\left(y_{2m(k)-1},y_{2n(k)-1}\right)+d\left(y_{2n(k)-1},y_{2n(k)-2}\right)$$
$$d\left(y_{2m(k)-1},y_{2n(k)-1}\right)\leq d\left(y_{2m(k)-1},\ y_{2n(k)-2}\right)+d\left(y_{2n(k)-2},\ y_{2n(k)-1}\right)$$

On letting $k\to\infty$ we get,

$$\limsup_{k\to\infty} d\left(y_{2m(k)-1},y_{2n(k)-2}\right)\leq\epsilon \;\; and \;\; \liminf_{k\to\infty} d\left(y_{2m(k)-1},y_{2n(k)-2}\right)\geq\epsilon\Rightarrow\lim_{k\to\infty} d\left(y_{2m(k)-1},y_{2n(k)-2}\right)=\epsilon \tag{31}$$

Consider,

$$d\left(y_{2m(k)},y_{2n(k)-2}\right)\leq d\left(y_{2m(k)},y_{2n(k)-1}\right)+d\left(y_{2n(k)-1},y_{2n(k)-2}\right)$$
$$d\left(y_{2m(k)},y_{2n(k)-1}\right)\leq d\left(y_{2m(k)},y_{2n(k)-2}\right)+d\left(y_{2n(k)-2},y_{2n(k)-1}\right)$$

On letting $k\to\infty$ we get,

$$\limsup_{k\to\infty} d\left(y_{2m(k)},y_{2n(k)-2}\right)\leq\epsilon \;\; and \;\; \liminf_{k\to\infty} d\left(y_{2m(k)},y_{2n(k)-2}\right)\geq\epsilon\Rightarrow\lim_{k\to\infty} d\left(y_{2m(k)},y_{2n(k)-2}\right)=\epsilon \tag{32}$$

Substituting $s=x_{2m(k)}$ and $t=x_{2n(k)-1}$ in (17), we get

$$\begin{aligned}\varphi_1\left(d\left(fx_{2m(k)},gx_{2n(k)-1}\right)\right)\leq\psi_1&\begin{pmatrix} d\left(Ux_{2m(k)},\ Vx_{2n(k)-1}\right),d\left(Ux_{2m(k)},fx_{2m(k)}\right),d\left(Vx_{2n(k)-1},gx_{2n(k)-1}\right),\\ \frac{1}{2}\left[d\left(gx_{2n(k)-1},Ux_{2m(k)}\right)+d\left(fx_{2m(k)},Vx_{2n(k)-1}\right)\right],\\ \frac{1}{2}[d\left(Ux_{2m(k)},Vx_{2n(k)-1}\right)+d\left(Ux_{2m(k)},fx_{2m(k)}\right)]\end{pmatrix}\\ -\psi_2&\begin{pmatrix} d\left(Ux_{2m(k)},Vx_{2n(k)-1}\right),d\left(Ux_{2m(k)},fx_{2m(k)}\right),d\left(Vx_{2n(k)-1},gx_{2n(k)-1}\right),\\ \frac{1}{2}\left[d\left(gx_{2n(k)-1},Ux_{2m(k)}\right)+d\left(fx_{2m(k)},Vx_{2n(k)-1}\right)\right],\\ \frac{1}{2}[d\left(Ux_{2m(k)},Vx_{2n(k)-1}\right)+d\left(Ux_{2m(k)},fx_{2m(k)}\right)]\end{pmatrix}\end{aligned}$$

From, (23)

$$\begin{aligned}\varphi_1\left(d\left(fx_{2m(k)},gx_{2n(k)-1}\right)\right)\leq\psi_1&\begin{pmatrix} d\left(gx_{2m(k)-1},fx_{2n(k)-2}\right),d\left(gx_{2m(k)-1},fx_{2m(k)}\right),d\left(fx_{2n(k)-2},gx_{2n(k)-1}\right),\\ \frac{1}{2}\left[d\left(gx_{2n(k)-1},gx_{2m(k)-1}\right)+d\left(fx_{2m(k)},fx_{2n(k)-2}\right)\right],\\ \frac{1}{2}[d\left(gx_{2m(k)-1},fx_{2n(k)-2}\right)+d\left(gx_{2m(k)-1},fx_{2m(k)}\right)]\end{pmatrix}\\ -\psi_2&\begin{pmatrix} d\left(gx_{2m(k)-1},fx_{2n(k)-2}\right),d\left(gx_{2m(k)-1},fx_{2m(k)}\right),d\left(fx_{2n(k)-2},gx_{2n(k)-1}\right),\\ \frac{1}{2}\left[d\left(gx_{2n(k)-1},gx_{2m(k)-1}\right)+d\left(fx_{2m(k)},fx_{2n(k)-2}\right)\right],\\ \frac{1}{2}[d\left(gx_{2m(k)-1},fx_{2n(k)-2}\right)+d\left(gx_{2m(k)-1},fx_{2m(k)}\right)]\end{pmatrix}\end{aligned}$$

$$\varphi_1\left(d\left(y_{2m(k)},y_{2n(k)-1}\right)\right)\le\psi_1\begin{pmatrix}d\left(y_{2m(k)-1},y_{2n(k)-2}\right),d\left(y_{2m(k)-1},y_{2m(k)}\right),d\left(y_{2n(k)-2},y_{2n(k)-1}\right),\\ \frac{1}{2}\left[d\left(y_{2n(k)-1},y_{2m(k)-1}\right)+d\left(y_{2m(k)},y_{2n(k)-2}\right)\right],\\ \frac{1}{2}[d\left(y_{2m(k)-1},y_{2n(k)-2}\right)+d\left(y_{2m(k)-1},y_{2m(k)}\right)]\end{pmatrix}$$
$$-\psi_2\begin{pmatrix}d\left(y_{2m(k)-1},y_{2n(k)-2}\right),d\left(y_{2m(k)-1},y_{2m(k)}\right),d\left(y_{2n(k)-2},y_{2n(k)-1}\right),\\ \frac{1}{2}\left[d\left(y_{2n(k)-1},y_{2m(k)-1}\right)+d\left(y_{2m(k)},y_{2n(k)-2}\right)\right],\\ \frac{1}{2}[d\left(y_{2m(k)-1},y_{2n(k)-2}\right)+d\left(y_{2m(k)-1},y_{2m(k)}\right)]\end{pmatrix}$$

On letting $k\to\infty$. From (30), (31) and (32)), we get,

$$\varphi_1(\epsilon)\le\psi_1\left(\epsilon,0,0,\epsilon,\frac{\epsilon}{2}\right)-\psi_2\left(\epsilon,0,0,\epsilon,\frac{\epsilon}{2}\right)$$

Therefore, $\varphi_1(\epsilon)\le\psi_1(\epsilon,\epsilon,\epsilon,\epsilon,\epsilon)-\psi_2\left(\epsilon,0,0,\epsilon,\frac{\epsilon}{2}\right)$. Therefore, $\varphi_1(\epsilon)\le\varphi_1(\epsilon)-\psi_2\left(\epsilon,0,0,\epsilon,\frac{\epsilon}{2}\right)$. Therefore, $\varphi_1(\epsilon)<\varphi_1(\epsilon)$ (since $\epsilon>0$ and hence $\psi_2\left(\epsilon,0,0,\epsilon,\frac{\epsilon}{2}\right)>0$) a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence.

Similarly we can show that, $\{y_{2n+1}\}$ is a Cauchy's sequence. Since $d(y_n,y_{n+1})\to0$ as $n\to\infty$; $\{y_n\}$ is a Cauchy sequence. There exists l such that $\{y_n\}\to l$ as $n\to\infty$. Without loss of generality we suppose that U is continuous. Suppose U is continuous function then $Ux_{2n}\to l\Rightarrow gx_{2n-1}\to l$ and $Ufx_{2n}\to Ul, UUx_{2n}\to Ul$. Since (f,U) is sub compatible, we have $fUx_{2n}\to Ul$. Substituting $s=Ux_{2n}$ and $t=x_{2n-1}$ in (17), we get

$$\varphi_1(d(fUx_{2n},gx_{2n-1}))\le\psi_1\begin{pmatrix}d(UUx_{2n},Vx_{2n-1}),d(UUx_{2n},fUx_{2n}),d(Vx_{2n-1},gx_{2n-1}),\\ \frac{1}{2}[d(gx_{2n-1},UUx_{2n})+d(fUx_{2n},Vx_{2n-1})],\\ \frac{1}{2}[d(UUx_{2n},Vx_{2n-1})+d(UUx_{2n},fUx_{2n})]\end{pmatrix}$$
$$-\psi_2\begin{pmatrix}d(UUx_{2n},Vx_{2n-1}),d(UUx_{2n},fUx_{2n}),d(Vx_{2n-1},gx_{2n-1}),\\ \frac{1}{2}[d(gx_{2n-1},UUx_{2n})+d(fUx_{2n},Vx_{2n-1})],\\ \frac{1}{2}[d(UUx_{2n},Vx_{2n-1})+d(UUx_{2n},fUx_{2n})]\end{pmatrix}$$
$$\varphi_1(d(Ul,l))\le\psi_1\begin{pmatrix}d(Ul,l),d(Ul,Ul),d(l,l),\frac{1}{2}[d(l,Ul)+d(Ul,l)],\\ \frac{1}{2}[d(Ul,l)+d(Ul,Ul)]\end{pmatrix}$$
$$-\psi_2\begin{pmatrix}d(Ul,l),d(Ul,Ul),d(l,l),\frac{1}{2}[d(l,Ul)+d(Ul,l)],\\ \frac{1}{2}[d(Ul,l)+d(Ul,Ul)]\end{pmatrix}$$
$$=\psi_1\left(d(Ul,l),0,0,d(l,Ul),\frac{1}{2}d(Ul,l)\right)-\psi_2\left(d(Ul,l),0,0,d(l,Ul),\frac{1}{2}d(Ul,l)\right)$$
$$\le\psi_1(d(Ul,l),d(Ul,l),d(Ul,l),d(Ul,l),d(Ul,l))-\psi_2\left(d(Ul,l),0,0,d(l,Ul),\frac{1}{2}d(Ul,l)\right)$$
$$=\varphi_1(d(Ul,l))-\psi_2\left(d(fl,l),0,0,d(fl,l),\frac{1}{2}[d(fl,\ l)]\right)$$

$\varphi_1(d(Ul,l))<\varphi_1(d(Ul,l))$ if $Ul\ne l$ a contradiction. Therefore,

$$Ul=l \tag{33}$$

Substituting $s=l$ and $t=x_{2n-1}$ in (17)

$$\varphi_1(d(fl,gx_{2n-1}))\le\psi_1\begin{pmatrix}d(Ul,Vx_{2n-1}),d(Ul,fl),d(Vx_{2n-1},gx_{2n-1}),\frac{1}{2}[d(gx_{2n-1},Ul)+d(fl,Vx_{2n-1})],\\ \frac{1}{2}[d(Ul,Vx_{2n-1})+d(Ul,fl)]\end{pmatrix}$$

$$- \psi_2 \begin{pmatrix} d(Ul, Vx_{2n-1}), d(Ul, fl), d(Vx_{2n-1}, gx_{2n-1}), \frac{1}{2}[d(gx_{2n-1}, Ul) + d(fl, Vx_{2n-1})], \\ \frac{1}{2}[d(Ul, Vx_{2n-1}) + d(Ul, fl)] \end{pmatrix}$$

$$\varphi_1(d(fl, l)) \le \psi_1 \begin{pmatrix} d(l, l), d(l, fl), d(l, l), \frac{1}{2}[d(l, l) + d(fl, l)], \\ \frac{1}{2}[d(l, l) + d(l, fl)] \end{pmatrix}$$

$$- \psi_2 \begin{pmatrix} d(l, l), d(l, fl), d(l, l), \frac{1}{2}[d(l, l) + d(fl, l)], \\ \frac{1}{2}[d(l, l) + d(l, fl)] \end{pmatrix}$$

$$= \psi_1\left(0, d(l, fl), 0, \frac{1}{2}d(fl, l), \frac{1}{2}d(l, fl)\right) - \psi_2\left(0, d(l, fl), 0, \frac{1}{2}d(fl, l), \frac{1}{2}d(l, fl)\right)$$

$$\le \psi_1(d(fl, l), d(fl, l), d(fl, l), d(fl, l), d(fl, l)) - \psi_2\left(0, \ d(l, fl), 0, \frac{1}{2}d(fl, l), \frac{1}{2}d(l, fl)\right)$$

$$= \varphi_1(d(fl, l)) - \psi_2\left(d(fl, l), 0, 0, d(fl, l), \frac{1}{2}[d(fl, l)]\right)$$

$\varphi_1(d(fl, l)) < \varphi_1(d(fl, l))$ if $fl \neq l$ a contradiction. Therefore,

$$fl = l \tag{34}$$

From (27), $l = fl \in f(X) \subseteq V(X)$ then there exists $h \in X$ such that $l = Vh$. Substituting $s = x_{2n}$ and $t = h$ in (17), we get

$$\varphi_1(d(fx_{2n}, gh)) \le \psi_1 \begin{pmatrix} d(Ux_{2n}, Vh), d(Ux_{2n}, fx_{2n}), d(Vh, gh), \frac{1}{2}[d(gh, Ux_{2n}) + d(fx_{2n}, Vh)], \\ \frac{1}{2}[d(Ux_{2n}, Vh) + d(Ux_{2n}, fx_{2n})] \end{pmatrix}$$

$$- \psi_2 \begin{pmatrix} d(Ux_{2n}, Vh), d(Ux_{2n}, fx_{2n}), d(Vh, gh), \frac{1}{2}[d(gh, Ux_{2n}) + d(fx_{2n}, Vh)], \\ \frac{1}{2}[d(Ux_{2n}, Vh) + d(Ux_{2n}, fx_{2n})] \end{pmatrix}$$

$$\varphi_1(d(l, gh)) \le \psi_1 \begin{pmatrix} d(l, \ l), d(l, l), d(l, gh), \frac{1}{2}[d(gh, l) + d(l, l)], \\ \frac{1}{2}[d(l, l) + d(l, l)] \end{pmatrix}$$

$$- \psi_2 \begin{pmatrix} d(l, l), d(l, l), d(l, gh), \frac{1}{2}[d(gh, l) + d(l, l)], \\ \frac{1}{2}[d(l, l) + d(l, l)] \end{pmatrix}$$

$$= \psi_1\left(0, 0, d(l, gh), \frac{1}{2}d(gh, l), 0\right) - \psi_2\left(0, 0, d(l, gh), \frac{1}{2}d(gh, l), 0\right)$$

$$\le \psi_1(d(l, gh), d(l, gh), d(l, gh), d(l, gh), d(l, gh)) - \psi_2\left(0, 0, d(l, gh), \frac{1}{2}d(gh, l), 0\right)$$

$$= \varphi_1(d(l, gh)) - \psi_2\left(0, 0, d(l, gh), \frac{1}{2}d(gh, l), 0\right)$$

$\varphi_1(d(l, gh)) < \varphi_1(d(l, gh))$ if $gh \neq l$ a contradiction. Therefore,

$$gh = l = Vh \tag{35}$$

Therefore, $gVh = Vgh$ (since (V, g) is sub compatible)

$$gl = gVh = Vgh = Vl \tag{36}$$

Substituting $s = x_{2n}$ and $t = l$ in (17), we get

$$\varphi_1(d(fx_{2n}, gl)) \le \psi_1 \begin{pmatrix} d(Ux_{2n}, Vl), d(Ux_{2n}, fx_{2n}), d(Vl, gl), \frac{1}{2}[d(gl, Ux_{2n}) + d(fx_{2n}, Vl)], \\ \frac{1}{2}[d(Ux_{2n}, Vl) + d(Ux_{2n}, fx_{2n})] \end{pmatrix}$$

$$- \psi_2 \begin{pmatrix} d(Ux_{2n}, Vl), d(Ux_{2n}, fx_{2n}), d(Vl, gl), \frac{1}{2}[d(gl, Ux_{2n}) + d(fx_{2n}, Vl)], \\ \frac{1}{2}[d(Ux_{2n}, Vl) + d(Ux_{2n}, fx_{2n})] \end{pmatrix}$$

$$\begin{aligned} \varphi_1(d(l, gl)) \leq \psi_1 &\begin{pmatrix} d(l, gl), d(l, l), d(gl, gl), \frac{1}{2}[d(gl, l) + d(l, gl)], \\ \frac{1}{2}[d(l, gl) + d(l, l)] \end{pmatrix} \\ &- \psi_2 \begin{pmatrix} d(l, gl), d(l, l), d(gl, gl), \frac{1}{2}[d(gl, l) + d(l, gl)], \\ \frac{1}{2}[d(l, gl) + d(l, l)] \end{pmatrix} \\ &= \psi_1\left(d(l, gl), 0, 0, d(l, gl), \frac{1}{2}d(l, gl)\right) - \psi_2\left(d(l, gl), 0, 0, d(l, gl), \frac{1}{2}d(l, gl)\right) \\ &\leq \psi_1(d(l, gl), d(l, gl), d(l, gl), d(l, gl), d(l, gl)) - \psi_2\left(d(l, gl), 0, 0, d(l, gl), \frac{1}{2}d(l, gl)\right) \\ &= \varphi_1(d(l, gl)) - \psi_2\left(d(l, gl), 0, 0, d(l, gl), \frac{1}{2}d(l, gl)\right) \end{aligned}$$

$\varphi_1(d(l, gl)) < \varphi_1(d(l, gl))$ if $gl \neq l$ a contradiction. Therefore,

$$gl = l. \tag{37}$$

From (36) and (37), we get, therefore,

$$gl = Vl = l \tag{38}$$

From (33), (34) and (38), f, g, U and V have a common fixed point in X. If f, g or V is continuous, similarly we can prove f, g, U and V have a common fixed point in X. Suppose, l and h are common fixed point of f, g, U and V. From (17),

$$\begin{aligned} \varphi_1(d(fl, gh)) \leq \psi_1 &\begin{pmatrix} d(Ul, Vh), d(Ul, fl), d(Vh, gh), \\ \frac{1}{2}[d(gh, Ul) + d(fl, Vh)], \frac{1}{2}[d(Ul, Vh) + d(Ul, fl)] \end{pmatrix} \\ &- \psi_2 \begin{pmatrix} d(Ul, Vh), d(Ul, fl), d(Vh, gh), \\ \frac{1}{2}[d(gh, Ul) + d(fl, Vh)], \frac{1}{2}[d(Ul, Vh) + d(Ul, fl)] \end{pmatrix} \\ \varphi_1(d(l, h)) \leq \psi_1 &\begin{pmatrix} d(l, h), d(l, l), d(h, h), \\ \frac{1}{2}[d(h, l) + d(l, h)], \frac{1}{2}[d(l, h) + d(l, l)] \end{pmatrix} \\ &- \psi_2 \begin{pmatrix} d(l, h), d(l, l), d(h, h), \\ \frac{1}{2}[d(h, l) + d(l, h)], \frac{1}{2}[d(l, h) + d(l, l)] \end{pmatrix} \\ &= \psi_1\left(d(l, h), 0, 0, d(h, l), \frac{1}{2}d(l, h)\right) - \psi_2\left(d(l, h), 0, 0, d(h, l), \frac{1}{2}d(l, h)\right) \\ \varphi_1(d(l, h)) &\leq \psi_1(d(l, h), d(l, h), d(l, h), d(h, l), d(l, h)) - \psi_2\left(d(l, h), 0, 0, d(h, l), \frac{1}{2}d(l, h)\right) \\ &= \varphi_1(d(l, h)) - \psi_2\left(d(l, h), 0, 0, d(h, l), \frac{1}{2}d(l, h)\right) \end{aligned}$$

$\varphi_1(d(l, h)) < \varphi_1(d(l, h))$ if $h \neq l$ a contradiction. Therefore, $h = l$. Hence, f, g, U and V have a unique common fixed point in X. □

The following is an application of Theorem 3.5 to integral type inequalities.

Theorem 3.6. *Suppose (X, d) is a complete metric space and f, g, U and $V : X \to X$ be such that for all $x, y \in X$*

$$\int_0^{\varphi_1(d(fx,gy))} \eta(t)\,dt \leq \int_0^{\psi_1\left(d(Ux,Vy), d(Ux,fx), d(Vy,gy), \frac{1}{2}[d(gy,Ux)+d(fx,Vy)], \frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right)} \eta(t)\,dt$$

$$- \int_0^{\psi_2\left(d(Ux,Vy),d(Ux,fx),d(Vy,gy),\frac{1}{2}[d(gy,Ux)+d(fx,Vy)],\frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right)} \eta(t)\,dt \qquad (39)$$

where $\psi_1, \psi_2 \in \Psi_5$ *with* $\varphi_1(a) = \psi_1(\alpha,\alpha,\alpha,\alpha,\alpha)$, $\alpha \in [0,\infty)$ *and* $\eta : R^+ \longrightarrow R^+$ *is a Lebesgue-integrable function, which is non negative, summable, and* $\int_0^\epsilon \eta(t)\,dt > 0$ *for each* $\epsilon > 0$.

(1). One of the four mappings f, g, U *and* V *is continuous.*

(2). (f,U) *and* (g,V) *are sub compatible.*

(3). $f(X) \subseteq V(X)$ *and* $g(X) \subseteq U(X)$.

Then f, g, U *and* V *have a unique common fixed point in* X.

Proof. We first show that (17) holds for f, g, U and V. Suppose for some $x, y \in X$, (17) does not hold. Then

$$\begin{aligned}\varphi_1(d(fx,gy)) > \psi_1&\left(d(Ux,Vy), d(Ux,fx), d(Vy,gy), \frac{1}{2}[d(gy,Ux)+d(fx,Vy)], \frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right)\\ - \psi_2&\left(d(Ux,Vy), d(Ux,fx), d(Vy,gy), \frac{1}{2}[d(gy,Ux)+d(fx,Vy)], \frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right)\end{aligned}$$

Write

$$\epsilon = \varphi_1(d(fx,gy)) - \begin{pmatrix} \psi_1\left(d(Ux,Vy), d(Ux,fx), d(Vy,gy), \frac{1}{2}[d(gy,Ux)+d(fx,Vy)], \frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right) \\ -\psi_2\left(d(Ux,Vy), d(Ux,fx), d(Vy,gy), \frac{1}{2}[d(gy,Ux)+d(fx,Vy)], \frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right) \end{pmatrix}$$

Then $\epsilon > 0$. By hypothesis, $\int_0^\epsilon \eta(t)\,dt > 0$. Therefore,

$$\begin{aligned}\int_0^{\varphi_1(d(fx,gy))} \eta(t)\,dt > &\int_0^{\psi_1\left(d(Ux,Vy),d(Ux,fx),d(Vy,gy),\frac{1}{2}[d(gy,Ux)+d(fx,Vy)],\frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right)} \eta(t)\,dt\\ &- \int_0^{\psi_2\left(d(Ux,Vy),d(Ux,fx),d(Vy,gy),\frac{1}{2}[d(gy,Ux)+d(fx,Vy)],\frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right)} \eta(t)\,dt\end{aligned}$$

a contradiction. Therefore,

$$\begin{aligned}\varphi_1(d(fx,gy)) \le \psi_1&\left(d(Ux,Vy), d(Ux,fx), d(Vy,gy), \frac{1}{2}[d(gy,Ux)+d(fx,Vy)], \frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right)\\ - \psi_2&\left(d(Ux,Vy), d(Ux,fx), d(Vy,gy), \frac{1}{2}[d(gy,Ux)+d(fx,Vy)], \frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right)\end{aligned}$$

Thus, (17) holds for f, g, U and V. Therefore, by Theorem 3.5, f, g, U and V have a unique common fixed point in X. □

The proof of the following theorem similar to that of Theorem 3.5.

Theorem 3.7. *Let* (X,d) *be complete metric space and* f, g, U *and* V *be four mappings from* X *to itself such that*

$$\begin{aligned}\varphi_1(d(fx,gy)) \le \psi_1&\begin{pmatrix} d(Ux,Vy), d(Ux,fx), d(Vy,gy), \frac{1}{2}[d(gy,Ux)+d(fx,Vy)], \\ \frac{1}{2}[d(Ux,Vy)+\max\{d(Ux,fx),(gy,Vy)\}] \end{pmatrix}\\ - \psi_2&\begin{pmatrix} d(Ux,Vy), d(Ux,fx), d(Vy,gy), \frac{1}{2}[d(gy,Ux)+d(fx,Vy)], \\ \frac{1}{2}[d(Ux,Vy)+\max\{d(Ux,fx),(gy,Vy)\}] \end{pmatrix}\end{aligned}$$

for all $x, y \in X$, *where* $\psi_1, \psi_2 \in \Psi_5$ *with* $\varphi_1(\alpha) = \psi_1(\alpha,\alpha,\alpha,\alpha,\alpha)$, *for* $\alpha \in [0,\infty)$.

(1). One of the four mappings f, g, U and V is continuous.

(2). (f,U) and (g,V) are sub compatible.

(3). $f(X) \subseteq V(X)$ and $g(X) \subseteq U(X)$.

Then f, g, U and V have unique common fixed point in X.

The proof of the following theorem similar to that of Theorem 3.6.

Theorem 3.8. *Suppose (X,d) is a complete metric space and f,g,U and $V : X \to X$ be such that for all $x,y \in X$*

$$\int_0^{\varphi_1(d(fx,gy))} \eta(t)\,dt \leq \int_0^{\psi_1\left(d(Ux,Vy),\ d(Ux,fx),d(Vy,\ gy),\frac{1}{2}[d(gy,Ux)+d(fx,Vy)],\frac{1}{2}[d(Ux,Vy)+\max\{d(Ux,fx),(gy,Vy)\}]\right)} \eta(t)\,dt$$
$$- \int_0^{\psi_2\left(d(Ux,Vy),d(Ux,fx),d(Vy,gy),\frac{1}{2}[d(gy,Ux)+d(fx,Vy)],\frac{1}{2}[d(Ux,Vy)+\max\{d(Ux,fx),(gy,Vy)\}\right)} \eta(t)\,dt$$

where $\psi_1, \psi_2 \in \Psi_5$ with $\varphi_1(\alpha) = \psi_1(\alpha,\alpha,\alpha,\alpha,\alpha)$, $\alpha \in [0,\infty)$ and $\eta : R^+ \longrightarrow R^+$ is a Lebesgue-integrable function, which is non negative, summable, and $\int_0^{\epsilon} \eta(t)\,dt > 0$ for each $\epsilon > 0$.

(1). One of the four mappings f, g, U and V is continuous.

(2). (f,U) and (g,V) are sub compatible.

(3). $f(X) \subseteq V(X)$ and $g(X) \subseteq U(X)$.

Then f, g, U and V have a unique common fixed point in X.

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