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Three Point Boundary Value Problem for Random Differential Inclusions

Research Article

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Abstract: In this paper, we prove the existence of solution on a compact interval to a-three-point boundary value problem for a class of second order random differential inclusions. We shall employees random fixed point theorem for condensing maps due to Martelli.

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Keywords: Three point boundary value problems, random differential inclusion, random fixed point.

1. Description of the Problem

Consider the three-point boundary value problem of the second order random differential inclusions,

$$y''(t,\omega) \in F(t, y(t,\omega)),\ t \in J = [0,1] \tag{1}$$

$$y(0,\omega) = 0,\ y(\eta,\omega) = y(1,\omega) \tag{2}$$

where $F : J \times R \times \Omega \to 2^R$ is a multi-valued map with compact convex values and $t \in J$, $\omega \in \Omega$, $\eta \in (0,1)$. The study of multi-point boundary value problems for second order ordinary differential equations was initiated by Il' In, Moiseev [12, 13] and work of Bitsadze on nonlocal elliptic boundary value problems [14]. The methods used are usually the topological transversality of Granas .The method we are use is to reduce the existence of solutions to problem (1)-(2) to the search for random fixed points of a suitable multi-valued map on the Banach space $C(J,R)$. In order to prove the existence of random fixed points, we shall employees a random fixed point theorem for condensing maps due to Martelli [15].

2. Preliminaries

Let $(X, \|\|)$ be a Banach space. A multi-valued map $G : X \to 2^X$ is convex valued if $G(x)$ is convex for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x\in B} G(x)$ is bounded in X for any bounded set B of X. G is called upper semi-continuous (u.s.c.) on X if for each $x_* \in X$ the set, $G(x_*)$ is a nonempty, closed subset of X, and if for each open set B of X containing

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$G(x_*)$, there exists an open neighborhood V of x_* such that $G(V) \subseteq B$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multi-valued map G is completely continuous with nonempty compact values, then G is upper semicontinuous (u.s.c.) if and only if G has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in Gx_n$ imply $y_* \in Gx_*$). G has a fixed point if there is $x \in X$ such that $x \in Gx$. In the following $CC(X)$ denotes the set of all nonempty compact and convex subsets of X. A multi-valued map $G : J \to CC(E)$ is said to be measurable if for each $x \in X$ the function $Y : J \to R$ defined by $Y(t) = d(x, G(t) = \inf\{|x - z| : z \in G(t)\}$ is measurable.

Definition 2.1. *A multi-valued map $F : J \times R \times \Omega \to 2^R$ is said to be an random L^1-Caratheodory if*

(1). $t \mapsto F(t, y, \omega)$ is measurable for each $y \in R$;

(2). $y \mapsto F(t, y, \omega)$ is upper semi-continuous for almost all $t \in J$, $\omega \in \Omega$.

(3). For each $k > 0$, there exists $h_k \in L^1(J, \Omega, R+)$ such that

$\|F(t, y, \omega)\| = \sup\{\|v\| : v \in F(t, y, \omega)\} \leq h_k(t, \omega)$ for all $|y| \leq k$, $\omega \in \Omega$ and for almost all $t \in J$.
An upper semi-continuous map $G : X \to 2^X$ is said to be condensing if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of non-compactness. We need the following hypotheses:

(A1). $F : J \times R \times \Omega \to CC(R)$ is random L^1-Caratheodory multi-valued map.

(A2). There exists a function $H \in L^1(\Omega, J, R+)$ such that

$\|F(t, y, \omega)\| = \sup\{\|v\| : v \in F(t, y, \omega)\} \leq H_k(t, \omega)$ for all $|y| \leq k$, $\omega \in \Omega$. $\|F(t, y, \omega)\| = \sup\{\|v\| : v \in F(t, y, \omega)\} \leq H(t, \omega)$ for all for almost all $t \in J$ and all $\omega \in \Omega$, $y \in R$.

Definition 2.2. *A function $y : \Omega \to R$ is called a solution for the BVP (1)-(2) if y and its first derivative are absolutely continuous and y'' which exists almost everywhere satisfies the differential inclusion (1) a.e. on J and the condition (2). For the multi-valued map F and for each $y \in C(J, R)$ we define $S_F^1(y, \omega)$ by*

$$S_F^1(y, \omega) = \{v \in L^1(J, \Omega, R) : v(t, \omega) \in F(t, y(t, \omega), \omega)) \text{ for a.e. } t \in J, \omega \in \Omega.\}$$

Our considerations are based on the following lemmas.

Lemma 2.3. *Let I be a compact real interval and X be a Banach space. Let F be a multi-valued map satisfying (A1) and $S_F^1(y, \omega) \neq \phi$ for any $y \in C(J, X)$. Let Γ be a linear continuous mapping from $L^1(I, X)$ to $C(I, X)$ then the operator*

$$\Gamma o S_F(y, \omega) : C(I, X) \to CC(C(I, X)), y \mapsto (\Gamma o S_F)(y, \omega) := \Gamma(S_F(y, \omega))$$

is a closed graph operator in $C(I, X) \times C(I, X)$.

Lemma 2.4. *Let (Ω, X) be a Banach space and $N : \Omega \times X \to CC(X)$ be a condensing map. If the set $P := \{y \in X : \lambda(\omega)y \in N(\omega)(y)$ for some $\lambda(\omega) > 1\}$, where $\lambda : \Omega \times X$, is bounded, then $N(\omega)$ has a random fixed point.*

3. Main Result

Our main result is

Theorem 3.1. *Assume that hypotheses (A1)-(A2) hold. Then the BVP (1)-(2) has at least one random solution on J.*

Proof. Let $C(J,R)$ be the Banach space provided with the norm $\|y\|_\infty := \sup\{\|y(t,\omega)\| : t\in J, \omega\in\Omega\}, \quad for\ y\in C(J,R)$. Transform the problem into a random fixed point problem. Consider the multi-valued map, $N : C(J,\Omega,R)\to 2^{C(J,R)}$ defined by

$$N(\omega)(y)=\left\{h\in C(J,\Omega,R) : h(t,\omega)\int_0^t (t-s)g(s,\omega)ds+\frac{t}{1-\eta}\int_0^\eta(\eta-s)g(s,\omega)ds-\frac{t}{1-\eta}\int_0^t(1-s)g(s,\omega)ds\right\}$$

where $g\in S_F(y,\omega)=\{g\in L^1(J,\Omega,R) : g(t,\omega)\in F(t,y(t,\omega),\omega))\ for\ a.e.\ t\in J,\omega\in\Omega\}$. We shall show that $N(\omega)$ satisfies the assumptions of Lemma 2.2. The proof will be given in following several steps.

Step 1: $N(\omega)$ is convex for each $y\in C(J,R)$. Indeed, if h_1,h_2 belong to $N(\omega)(y)$ then there exist $g_1,g_2\in S_F(y,\omega)$ such that for each $t\in J$, $\omega\in\Omega$, we have

$$h_i(t,\omega)=\int_0^t(t-s)g_1(s,\omega)ds+\frac{t}{1-\eta}\int_0^\eta(\eta-s)g_1(s,\omega)ds-\frac{t}{1-\eta}\int_0^1(1-s)g_i(s,\omega)ds,\quad i=1,2.$$

Let $0\le\alpha\le1$. Then for each $t\in J$, $\omega\in\Omega$, we have

$$\begin{aligned}(\alpha h_1+(1-\alpha)h_2)(t,\omega)&=\int_0^t(t-s)[\alpha g_1(s,\omega)+(1-\alpha)g_2(s,\omega)]ds+\frac{t}{1-\eta}\int_0^\eta(\eta-s)[\alpha g_1(s,\omega)+(1-\alpha)g_2(s,\omega)]ds\\&\quad-\frac{t}{1-\eta}\int_0^1(1-s)[\alpha g_1(s,\omega)+(1-\alpha)g_2(s,\omega)]ds.\end{aligned}$$

Since $S_F(y,\omega)$ is convex since F has convex values, then $\alpha h_1+(1-\alpha)h_2\in N(y,\omega)$.

Step 2: $N(\omega)(y)$ is bounded on bounded sets of $C(J,R)$. Indeed, it is enough to show that there exists a positive constant c such that for each $h\in N(y), y\in B_r=\{y\in C(J,R) : \|y\|_\infty\le r\}$ one has $\|h\|_\infty\le c$. If $h\in N(y)$, then there exists $g\in S_F(y,\omega)$ such that for each $t\in J$, $\omega\in\Omega$ we have

$$h(t,\omega)=\int_0^t(t-s)g(s,\omega)ds+\frac{t}{1-\eta}\int_0^\eta(\eta-s)g(s,\omega)ds-\frac{t}{1-\eta}\int_0^1(1-s)g(s,\omega)ds.$$

By (A1) we have for each $\omega\in\Omega$, $t\in J$ that

$$\|h(t,\omega)\|\le\int_0^t h_r(s,\omega)ds+\frac{1}{1-\eta}\int_0^\eta(\eta-s)h_r(s,\omega)ds+\frac{1}{1-\eta}\int_0^1(1-s)h_r(s,\omega)ds$$

Then

$$|h|_\infty\le\int_0^1 h_r(s,\omega)ds+\frac{1}{1-\eta}\int_0^\eta(\eta-s)h_r(s,\omega)ds+\frac{1}{1-\eta}\int_0^1(1-s)h_r(s,\omega)ds=c.$$

Step 3: N sends bounded sets of $C(J,R)$ into equi-continuous sets. Let, $t_1,t_2\in J$, $t_1<t_2$ and B_r be a bounded set of $C(J,R)$. For each $y\in B_r$ and $h\in N(\omega)(y)$ there exists $g\in S_F(y,\omega)$ such that

$$h(t,\omega)=\int_0^t(t-s)g(s,\omega)ds+\frac{t}{1-\eta}\int_0^\eta(\eta-s)g(s,\omega)ds-\frac{t}{1-\eta}\int_0^1(1-s)g(s,\omega)ds,\quad t\in J,\omega\in\Omega.$$

Thus we obtain

$$\begin{aligned}\|h(t_2,\omega)-h(t_1\omega)\|&\le\int_0^{t_2}(t_2-s)\,|g(s,\omega)|\,ds+\int_{t_1}^{t_2}(t_1-s)\,\|g(s,\omega)\|\,ds\\&\quad+\frac{t_2-t_1}{1-\eta}\int_0^\eta(t-s)\,\|g(s,\omega)\|\,ds+\frac{t_2-t_1}{1-\eta}\int_0^1(1-s)\,\|g(s,\omega)\|\,ds\\&\le\int_0^{t_2}(t_2-s)h_r(s,\omega)ds+\int_{t_1}^{t_2}(t_1-s)h_r(s,\omega)ds\end{aligned}$$

$$+\frac{t_2-t_1}{1-\eta}\int_0^{\eta}(t-s)h_r(s,\omega)ds+\frac{t_2-t_1}{1-\eta}\int_0^1(1-s)h_r(s,\omega)ds.$$

As $t_1 \to t_2$ the right-hand side of the above inequality tends to zero. As a consequence of Step 2, Step 3 together with the Ascoli-Arzela theorem we can conclude that $N(\omega)(y)$ is completely continuous.

Step 4: $N(\omega)$ has a closed graph. Let $y_n \to y_*$, $h_n \in N(\omega)(y_n)$, and $h_n \to h_*$. We shall prove that $h_* \in N(\omega)(y_*)$. $h_n \in N(\omega)(y_n)$ means that there exists $g_n \in S_F(y_n,\omega)$ such that

$$h_n(t,\omega)=\int_0^t(t-s)g_n(s,\omega)ds+\frac{t}{1-\eta}\int_0^{\eta}(\eta-s)g_n(s,\omega)ds-\frac{t}{1-\eta}\int_0^1(1-s)g_n(s,\omega)ds,\ \ t\in J,\omega\in\Omega.$$

We must prove that there exists $g_* \in S_F(y_*,\omega)$ such that

$$h_*(t,\omega)=\int_0^t(t-s)g_*(s,\omega)ds+\frac{t}{1-\eta}\int_0^{t}(\eta-s)g_*(s,\omega)ds-\frac{t}{1-\eta}\int_0^t(t-s)g_*(s,\omega)ds,\ \ t\in J,\omega\in\Omega.$$

Now, we consider the linear continuous operator $\Gamma : L^1(J,\Omega,R) \to C(J,\Omega,R)$

$$g\mapsto\Gamma(g,\omega)(t)=\int_0^t(t-s)g(s,\omega)ds+\frac{t}{1-\eta}\int_0^{\eta}(\eta-s)g(s,\omega)ds-\frac{t}{1-\eta}\int_0^1(1-s)g(s,\omega)ds,\ \ t\in J,\omega\in\Omega.$$

From Lemma, it follows that $\Gamma o S_F(y,\omega)$ is a closed graph operator. Moreover, from the definition of Γ we have $h_n(t,\omega) \in \Gamma(S_F(y_n,\omega))$. Since $y_n \to y_0$, it follows from Lemma that

$$h_*(t,\omega)=\int_0^t(t-s)g_*(s,\omega)ds+\frac{t}{1-\eta}\int_0^{\eta}(\eta-s)g_*(s,\omega)ds-\frac{t}{1-\eta}\int_0^1(1-s)g_*(s,\omega)ds,\ \ t\in J,\omega\in\Omega.$$

For some $g_* \in S_F(y_*,\omega)$.

Step 5: The set $Q := \{y \in C(J,R) : \lambda(\omega)y \in N(\omega)(y) \textit{ for some } \lambda(\omega) > 1\}$ is bounded. Let $y \in Q$. Then $\lambda(\omega)y \in N(\omega)(y)$ for some $\lambda(\omega) > 1$. Thus there exists $g \in S_F(y,\omega)$ such that

$$y(t,\omega)=\lambda^{-1}\int_0^t(t-s)g(s,\omega)ds+\lambda^{-1}\frac{t}{1-\eta}\int_0^{\eta}(\eta-s)g(s,\omega)ds-\lambda^{-1}\frac{t}{1-\eta}\int_0^1(1-s)g(s,\omega)ds,\ \ t\in J,\omega\in\Omega.$$

This implies by (A2) that for each $t \in J$ we have

$$\|y(t,\omega)\|\leq\int_0^t(t-s)H(s,\omega)ds+\frac{1}{1-\eta}\int_0^{\eta}(\eta-s)H(s,\omega)ds+\frac{1}{1-\eta}\int_0^1(1-s)H(s,\omega)ds.$$

Thus

$$\|y(t,\omega)\|_{\infty}\leq\int_0^1(1-s)H(s,\omega)ds+\frac{1}{1-\eta}\int_0^{\eta}(\eta-s)H(s,\omega)ds+\frac{1}{1-\eta}\int_0^1(1-s)H(s,\omega)ds:=K.$$

This shows that Q is bounded. Set $X := C(J,R)$. As a consequence of Lemma 2.2, we deduce that $N(\omega)$ has a fixed point which is a solution of (1)-(2) on J. □

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