International Journal of Mathematics And its Applications
Volume 5, Issue 4–F (2017), 815–820.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Existence of Solution for Functional Random Differential Equation

Research Article

S. S. Hambarde[1] and D. S. Palimkar[2]*

1 Department of Mathematics, Matoshri Polytechnic College, Khupsarwadi, Nanded, Maharashtra, India.

2 Department of Mathematics, Vasantrao Naik College, Vasarni, Nanded, Maharashtra, India.

Abstract: In this paper, we discuss the existence of solution for functional random differential equation under caratheodory, compactness and monotonic conditions. Here, we shall employees random fixed point theorem of Dhage.

MSC: 60H25, 47H40.

Keywords: Random functional differential equation, random fixed point theorem, caratheodory condition.

1. Introduction

Let R be the real line and let $I = [-r, 0]$ and $I_0 = [0, T]$ are two closed and bounded intervals in R for some real number $r > 0$ and $T > 0$. Let $J = [-r, T]$ and let $C(I_0, R)$ be the space of all continuous real valued functions on I_0. We define a supremum norm $\|.\|_c$ in $C(I_0, R)$ by $\|x\|_c = \sup\limits_{0\leq\theta\leq T} |x(\theta)|$. Clearly C is a banach space with this supremum norm.For given $t \in I$ we define a continuous real valued function $x_t : I_0 \to R$ by $x_t(t + \theta)$, for all $\theta \in I_0$. Given a measurable space (Ω, A) and for a given measurable function $f : \Omega \to C^1(J, R)$. Consider the second order Nonlinear Functional Random Differential equation (in short FRDE)

$$\begin{aligned} x''(t.\omega) &= f(t, x_t(\omega), \omega) + g(t, x_t(\omega), \omega), \quad \text{a.e. } t \in J \\ x_0(0, \omega) &= \varphi_0(\omega), \;\; x_0'(0, \omega) = \varphi_1(\omega) \end{aligned} \tag{1}$$

for all $\omega \in \Omega$ where $(f + g) : J \times R \times \Omega \to R$ and $\varphi_0, \varphi_1 : \Omega \to R$. By a random solution of (1) means a measurable function $x : \Omega \to AC^1(J, R)$ that satisfies the equation (1), where $AC^1(J, R)$ is the space of real valued functions defined and absolutely continuously differentiable on J. We discuss the existence of the solution to (1) under the caratheodory, compactness and monotonic conditions through random fixed point theorem of Dhage. The classical ordinary differential equation has been studied by several authors for different aspects of the solutions.

In this paper we discuss the second order nonlinear functional random differential equation for existence of solution under suitable condition of non-linearity of $f + g$ which generalized several existence results.

* *E-mail: dspalimkar@rediffmail.com*

2. Axillary Results

Let E be the Banach space with the norm $\|.\|$ and let $Q : E \to E$. Then Q is called compact if $Q(E)$ is relatively compact subset of $E.Q$ is called totally bounded if $Q(B)$ is totally subset of E for any bounded subset B of $E.Q$ is called completely continuous if it is continuous and totally bounded on E. We also assume that the banach space E is separable i.e. E has the countable dence subset and let β_E be the σ-algebra of Borel subsets of E. We say a mapping $x : \Omega \to E$ is measurable if for any $B \in \beta_E$, $x^{-1}(B) = \{\omega \in \Omega | x(\omega) \in B\} \in A$. Similarly, a mapping $x : \Omega \times E \to E$ is called jointly measurable if for any $B \in \beta_E$, one has

$$x^{-1}(B) = \{(\omega, x) \in \Omega \times E | x(\omega, x) \in B\} \in A \times \beta_E,$$

where, $A \times \beta_E$ is the direct product of the σ-algebra A and β_E, with those defined in Ω and E respectively.
Let $Q : \Omega \times E \to E$ be a mapping. Then Q is called a random operator if $Q(\omega, x)$ is measurable in ω for all $x \in E$ and it expressed as $Q(\omega)x = Q(\omega, x)$. In this case we also say that $Q(\omega)$ is a random operator on E. A random operator $Q(\omega)$ on E is called continuous (respectively compact, totally bounded and completely continuous) if $Q(\omega, x)$ is continuous (respectively compact, totally bounded and completely continuous) in x for all $\omega \in \Omega$. The details of completely continuous random operators in Banach space and their properties appear Itoh [8]. In this paper, we employ the following random nonlinear alternative in proving the main result.

Definition 2.1. *A random operator $Q : \Omega \times X \to X$ is called D-Lipschitz if there is a continuous and non decreasing function $\phi_\omega : \Omega \times R^+ \to R^+$ satisfying for each $\omega \in \Omega$*

$$\|Q(\omega)x - Q(\omega)y\| = \phi_\omega(\|x - y\|) \text{ for all } x, y \in X$$

Where $\phi_\omega(r) = \phi(\omega, r)$ with $\phi(\omega, 0) = 0$, for all $\omega \in \Omega$. The function ϕ is called random D-function of the random operator $\phi(\omega)$ on X. In the special case when $\phi_\omega(r) = \alpha(\omega)r$, for some measurable function $\alpha : \Omega \to R$. The random operator $Q(\omega)$ is called Lipschitz with Lipschitz constant $\alpha(\omega)$. In particular if $\alpha(\omega) < 1$ for each $\omega \in \Omega$, $Q(\omega)$ is called a contraction on X. If $\alpha(\omega) = 1$, for each $\omega \in \Omega$ then random operator $Q(\omega)$ is called uniform contraction on X.

Theorem 2.2 ([2, 3])**.** *Let X be the separable Banach space and $Q : \Omega \times X \to X$ be a completely continuous random operator then either*

(1). Random equation $Q(\omega)$ has random solution i.e. there is a measurable function $\xi : \Omega \to X$ such that $Q(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$.

(2). The set $\varepsilon = \{x : \Omega \to X$ is measurable / $\lambda(\omega)Q(\omega)x = x\}$ is unbounded for some measurable function $\lambda : \Omega \to R$ with $0 < \lambda(\omega) < 1$ on Ω.

Theorem 2.3 (Caratheodory)**.** *Let $Q : \Omega \times X \to X$ be a mapping such that $Q(., x)$ is measurable for all $x \in X$ and $Q(\omega, .)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, x) \to Q(\omega, x)$ is jointly measurable.*

3. Existence Results

If a function $t \to x(t, \omega)$ is continuous for each $\omega \in \Omega$. Then in this case we write $x : C(J, R) \times \Omega \to C(J, R)$ now the (1) is equivalent to the random integral equation

$$\begin{aligned} x(t,\omega) &= \phi_0(\omega) + \phi_1(\omega)t + \int_0^t (t-s)(f+g)(s, x_s(\omega), \omega)ds, \quad \text{for } t \in I \\ x(t,\omega) &= \phi_0(\omega), \quad x'(t,\omega) = \phi_1(\omega), \quad \text{for } t \in I_0 \end{aligned} \tag{2}$$

We need the following definitions. Let $C(J,R)$ and $BM(J,R)$ denote the spaces all continuous, bounded and measurable real valued function on J respectively. We define a norm $\|.\|_c$ in $C(J,R)$ by $\|x\|_c = \sup_{t\in J}|x(t)|$. And $\|.\|_B$ in $BM(J,R)$ by $\|x\|_B = \max_{t\in J}|x(t)|$, $C(J,R) \subset BM(J,R)$. We shall seek the random solution of (**??**) in the space $BM(J,R)$ under suitable condition.

Definition 3.1. *A mapping $\beta : J \times R \times \Omega \to R$ is called a random caratheodory if the following conditions satisfied*

(1). $t \to \beta(t,x,\omega)$ is measurable for each $x \in R$ and

(2). $x \to \beta(t,x,\omega)$ is continuous almost everywhere for $t \in J$.

Definition 3.2. *A Carathedory function $f : J \times R \times \Omega \to R$ is called a random L^1-carathe'odory if for each real number $r > 0$ there is a measurable and bounded function $h_r : \Omega \to L^1(J,R)$ such that $|f(t,x,\omega)| \le h_r(t,\omega)$ a.e. $t \in J$, for all $\omega \in \Omega$ and $x \in R$ with $|x| < r$. Similarly a Carathe'dory function $f : J \times R \times \Omega \to R$ is called a random $L^1_R - \omega-$carathe'odory if there exist a function $h : \Omega \to L^1(J,R)$ such that $|f(t,x,\omega)| \le h(t,\omega)$, a.e. for $t \in J$, for all $\omega \in \Omega$ and all $x \in R$. We consider the following hypotheses.*

(H_1) The function $\omega \to (f+g)(t,x,\omega)$ is measurable for all $t \in J$ and $x \in R$.

(H_2) The function $(f+g) : J \times R \times \Omega \to R$ is continuous and there exist a function $\alpha \in B(J,R)$. With bound $\|\alpha(\omega)\|$ satisfying for each $\omega \in \Omega$

$$|(f+g)(t,x,\omega) - (f+g)(t,y,\omega)| \le \alpha(t,\omega)\,\|x(\omega) - y(\omega)\|\,, \quad a.e. \quad t \in J, \quad \text{for all } x,y \in R.$$

(H_3) The function $(t,\omega) \to (f+g)(t,x,\omega)$ is measurable for all $t \in J$ and $x \in R$.

(H_4) The function $(f+g)$ is $L^1_R - \omega-$carathe'odory.

(H_5) There exist a function $\gamma : \Omega \to L(I,R)$ with $\gamma(t,\omega) > 0$, a.e. $t \in I$, and continuous non decreasing function $\psi : [0,\infty) \to [0,\infty)$ satisfying for each $\omega \in \Omega$

$$|(f+g)(t,x,\omega)| \le \gamma(t,\omega)\psi(\|x(\omega)\|_R), \quad a.e. \quad t \in I, \text{ for all } x \in R.$$

Moreover we assume that $\int_c^\infty \frac{dr}{\psi(r)} = \infty$, for all $c \ge 0$.

Our main existence results is

Theorem 3.3. *Suppose that the assumptions $(H_1) - (H_5)$ holds. Further if for each $\omega \in \Omega$,*

$$\int_c^\infty \frac{dr}{\psi(r)} \ge \|\gamma(\omega)\|_{L^1}, \tag{3}$$

where $c = Q_0 + Q_1 T$, then (1) has a random solution defined on J.

Proof. It is known that $BM(J,R)$ is separable Banach Space. Let $\omega \in \Omega$ be fixed. Define a mapping $Q : \Omega \times BM(J,R) \to BM(J,R)$ by $Q(\omega)x(t) = \phi_0(\omega) + \phi_1(\omega)t + \int_0^t (t-s)(f+g)(s,x_s(\omega),\omega)ds$, for $t \in J$ and $\omega \in \Omega$. Now the map $t \to \phi_0(\omega) + \phi_1(\omega)t$ is continuous for all $\omega \in \Omega$. Again as the indefinite integral is continuous on $J, Q(\omega)$ defines a mapping $Q : \Omega \times BM(J,R) \to BM(J,R)$. We shall show that $Q(\omega)$ satisfies all the condition of Theorem 2.1. First we show that Q is a random operator on $BM(J,R)$. Since $(f+g)(t,x,\omega)$ is random Carathe'odory, the map $\omega \to (f+g)(t,x,\omega)$ is measurable in the view of Theorem 2.1. Similarly the product $(t-s)(f+g)(s, x_s(\omega), \omega)$ of continuous and measurable function is again measurable. Further the integral is the limit of a finite sum of measurable functions, therefore the map

$$\omega \to \phi_0(\omega) + \phi_1(\omega) t + \int_0^t (t-s)(f+g)(s, x_s(\omega), \omega) ds = Q(\omega)x(t)$$

is measurable. As a result Q is a random operator on $\Omega \times BM(J, R)$ into $BM(J, R)$. Using the standard argument and the dominated convergence theorem it is proved that $Q(\omega)$ is continuous random operator on $BM(J, R)$. Now we show that $Q(\omega)$ is totally bounded random operator on $BM(J, R)$. We prove that $Q(\omega)(S)$ is totally bounded subset of $BM(J, R)$ for each bounded subset S of $BM(J, R)$. To finish it is to prove to prove that $Q(\omega)(S)$ is uniformly bounded and equicontinuous set in $MB(J, R)$ for each $\omega \in \Omega$. Since the map $t \to \gamma(t, \omega)$ is bounded, by hypothesis (H_1), there is a constant such that $\|\gamma(\omega)\|_{L^1} \le c$ for all $\omega \in \Omega$. Let $\omega \in \Omega$ be fixed, one has

$$\begin{aligned}
|Q(\omega)x(t)| &= \left|\phi_0(\omega) + \phi_1(\omega) t + \int_0^t (t-s)(f+g)(s, x_s(\omega), \omega) ds\right| \\
&\le |\phi_0(\omega)| + |\phi_1(\omega)| t + \int_0^t (t-s)\left|(f+g)(s, x_s(\omega), \omega)\right| ds \\
&\le \|\phi_0(\omega)\| + \|\phi_1(\omega)\| + \int_0^T (t-s)\gamma(s,\omega)\psi(\|x(s,\omega)\|) ds \\
&\le (\phi_0 + \phi_1 T) + \int_0^T T\gamma(s,\omega)\psi(r) ds \\
&\le (\phi_0 + \phi_1 T) + T\|\gamma(\omega)\|_{L^1}\psi(r) \le K
\end{aligned}$$

for all $t \in J$, where $K = (\phi_0 + \phi_1 T) + T\psi(r)$. This shows that $\{Q(\omega)(S)\}$ is uniformly bounded set in $BM(J, R)$ for each $\omega \in \Omega$. Next we show that $Q(\omega)(S)$ is equicontineous set in $BM(J, R)$. Let $t_1, t_2 \in J$, then for any $x \in S$

$$\begin{aligned}
|Q(\omega)x(t_1) - Q(\omega)x(t_2)| &\le |\phi_1(\omega) t_1 - \phi_1(\omega) t_2| \\
&+ \left|\int_0^{t_1} (t_1 - s)(f+g)(s, x_s(\omega), \omega) ds - \int_0^{t_2} (t_2 - s)(f+g)(s, x_s(\omega), \omega) ds\right| \\
&\le |\phi_1(\omega)| |t_1 - t_2| + \left|\int_0^{t_1} (t_1 - s)(f+g)(s, x_s(\omega), \omega) ds - \int_0^{t_1} (t_2 - s)(f+g)(s, x_s(\omega), \omega) ds\right| \\
&+ \left|\int_0^{t_1} (t_2 - s)(f+g)(s, x_s(\omega), \omega) ds - \int_0^{t_2} (t_2 - s)(f+g)(s, x_s(\omega), \omega) ds\right| \\
&\le |\phi_1(\omega)| |t_1 - t_2| + \left|\int_0^{T} (t_1 - t_2)(f+g)(s, x_s(\omega), \omega) ds\right| + \left|\int_{t_2}^{t_1} (t_2 - s)(f+g)(s, x_s(\omega), \omega) ds\right| \\
&\le Q_1 |t_1 - t_2| + \left|\int_0^{T} (t_1 - t_2)(f+g)(s, x_s(\omega), \omega) ds\right| + \left|\int_{t_2}^{t_1} T(f+g)(s, x_s(\omega), \omega) ds\right| \\
&\le Q_1 |t_1 - t_2| + \int_0^{T} |t_1 - t_2| \left|(f+g)(s, x_s(\omega), \omega)\right| ds + \int_{t_2}^{t_1} T\left|(f+g)(s, x_s(\omega), \omega)\right| ds \\
&\le Q_1 |t_1 - t_2| + \int_0^{T} |t_1 - t_2| \gamma(s,\omega)\psi(\|x_s(\omega)\| ds + \int_{t_2}^{t_1} T\gamma(s,\omega)\psi(\|x_s(\omega)\|) ds \\
&\le Q_1 |t_1 - t_2| + |t_1 - t_2| \|\gamma(\omega)\|_{L^1}\psi(r) + |p(t_1,\omega) - p(t_2,\omega)| \\
&\le (Q_1 + c\psi(r) |t_1 - t_2| + |p(t_1,\omega) - p(t_2,\omega)| \qquad (4)
\end{aligned}$$

For all $\omega \in \Omega$ where $p(t, \omega) = \int_{t_2}^{t_1} T\gamma(s, \omega)\psi(r) ds$. Hence for all $t_1, t_2 \in J$, $|Q(\omega)x(t_1) - Q(\omega)x(t_2)| \to 0$ as $t_1 \to t_2$ uniformly for all $x \in S$ and $\omega \in \Omega$. Therefore $Q(\omega)(S)$ is uniformly bounded and equi-continuous, it compact by Arzela-Ascolli Theorem for each $\omega \in \Omega$. Consequently $Q(\omega)$ is completely continuous random operator on S. Finally we prove that the set ε given in inclusion (2) of Theorem 2.1 does not hold. Let $x \in \varepsilon$ be arbitrary and $\omega \in \Omega$ be fixed. Then

$x(t,\omega) = \lambda Q(\omega)x(t)$ for all $t \in J$ and $\omega \in \Omega$, where $0 < \lambda < 1$. Then one has

$$\begin{aligned}|x(t,\omega)| &\leq \lambda\, |Q(\omega)x(t)| \\ &\leq |\phi_0\,(\omega)| + |\phi_1\,(\omega)|\, t + \int_0^t (t-s)\,|(f+g)\,(s, x_s\,(\omega)\,, \omega)|\, ds \\ &\leq Q_0 + Q_1 T + \int_0^t (t-s)\gamma(s,\omega)\psi((\|x_s(\omega)\|)ds \\ &\leq C + T\int_0^t \gamma(s,\omega)\psi((\|x_s(\omega)\|)ds \end{aligned} \tag{5}$$

for all $t \in J$ and $\omega \in \Omega$, where $c = Q_0 + Q_1 T$. For fixed $\omega \in \Omega$, $\mu(t,\omega) = |x(t^*,\omega)|$. Hence from (**??**) it follows that,

$$\begin{aligned}\mu(t,\omega) = |x(t^*,\omega)| &\leq c + T\int_0^t \gamma(s,\omega)\psi(\|x_s(\omega)\|)ds \\ &\leq c + T\int_0^t \gamma(s,\omega)\psi(\|\mu(s,\omega)\|)ds\end{aligned}$$

Put $w(t,\omega) = c + T\int_0^t \gamma(s,\omega)\psi(\|\mu(s,\omega)\|)ds$ for all $t \in J$. Now differentiate with respect to t. We obtain

$$\begin{aligned}&w'(t,\omega) = T\gamma(t,\omega)\psi(\mu(t,\omega)). \\ &w(0,\omega) = c\end{aligned}$$

or

$$\begin{aligned}\frac{w'(t,\omega)}{\psi(\mu(t,\omega))} &\leq T\gamma(t,\omega) \\ w(0,\omega) &= c\end{aligned}$$

Integrating from 0 to 1,

$$\int_0^1 \frac{w'(s,\omega)}{\psi(\mu(s,\omega))}ds \leq T\int_0^1 \gamma(s,\omega)ds$$

By changing of variables,

$$\int_0^{w(t,\omega)} \frac{dr}{\psi(r)} \leq T\|\gamma(\omega)\|_{L^1} < \int_0^{\infty} \frac{dr}{\psi(r)} = \infty$$

Now an application of the mean value theorem for integral calculus, there exist a constant $M > 0$ such that, $x(t,\omega) \leq \mu(t,\omega) \leq w(t,\omega) \leq M$. For all $t \in J$ and $\omega \in \Omega$. Hence the conclusion (2) of Theorem 2.1 does not hold. As a result the conclusion (1) holds and the operator equation $Q(\omega)x = x$ has a random solution. This further implies that the random differential equation (1) has a random solution defined on $\Omega \times J$. □

4. An Example

Let $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$ and let $J = [0,1]$ be a closed and bounded interval in R. Given a measurable function $x : \Omega \to C^1(J,R)$, consider the following random differential equation:

$$\begin{aligned}&x\text{"}(t,\omega) = \frac{t\omega^2 x^2(t,\omega)}{(1+\omega^2)\,[1+x^2\,(t,\omega)]} + \frac{kt\omega^2 x^2(t,\omega)}{(1+\omega^2)\,[1+x^2\,(t,\omega)]} \quad \text{a.e. } t \in J, \text{ k is any intger} \\ &x(0,\omega) = sin\omega, \quad x'(0,\omega) = cos\omega, \quad \text{for all } \omega \in \Omega\end{aligned} \tag{6}$$

Here

$$f(t,x,\omega)=\frac{t\omega^2x^2(t,\omega)}{(1+\omega^2)\left[1+x^2\left(t,\omega\right)\right]},\quad g(t,x,\omega)=\frac{kt\omega^2x^2(t,\omega)}{(1+\omega^2)\left[1+x^2\left(t,\omega\right)\right]}$$

For all $(t,x,\omega)\in J\times R\times\Omega$, and $\phi_0(0,\omega)=sin\omega$, $\phi_1(0,\omega)=cos\omega$ for all $\omega\in\Omega$. Clearly the map $(t,\omega)\to(f+g)(t,x,\omega)$ is jointly continuous for all $x\in R$ and hence jointly measurable for all $x\in R$. Also the map $x\to(f+g)(t,x,\omega)$ is continuous for all $t\in J$ and $\omega\in\Omega$. So the function f is Carathe'odory on $J\times R\times\Omega$. Moreover,

$$\left|\frac{t\omega^2x^2(t,\omega)}{(1+\omega^2)\left[1+x^2\left(t,\omega\right)\right]}+\frac{kt\omega^2x^2(t,\omega)}{(1+\omega^2)\left[1+x^2\left(t,\omega\right)\right]}\right|\le t=\gamma(t,\omega)\psi\left|x\right|$$

Where $\gamma(t,\omega)=t$ for all $t\in[0,1]$ and $\psi(r)=1$ for all real number $r\geq 0$. Clearly γ defines measurable and bounded function $\gamma:\Omega\to C^1(J,R)$. Similarly, ψ defines a continuous and non decreasing function $\psi:R_+\to R_+$ satisfying

$$\int_c^\infty\frac{dr}{\psi(r)}=\int_c^\infty dr=\infty,$$

For all $C\geq 0$. Again, the function $\phi_0,\phi_1:\Omega\to C^1(J,R)$ are measurable and bounded with $\sup\limits_{\omega\in\Omega}\phi_0(\omega)\le 1$ and $\sup\limits_{\omega\in\Omega}\phi_1(\omega)\le 1$. Now ,

$$\|\gamma(\omega)\|_{L^1}\le\int_0^1\gamma(t,\omega)dt=\frac{1}{2}<\int_2^\infty\frac{dr}{\psi(r)}.$$

Therefore, the condition (3) is satisfied. Hence by Theorem 3.1, the random differential equation (6) has a random solution defined on [0,1].

Acknowledgement

I would like to thanks the Editor Board and other authors for making their resources, enhancement and experience available.

References

[1] B.C.Dhage, *Nonlinear Functional Random Differential Equation in Banach algebra*, Tamkang Journal of Mathematics, 38(1)(2007), 57-53.

[2] B.C.Dhage, *A Nonlinear Alternative in Banach algebra with application to functional differential equation*, Nonlinear Funct. Anal. Appl., 9(2004).

[3] B.C.Dhage, *Random Fixed Point Theorem in Banach algebra with application to Random Integral Equation*, Tamkang J. Math., 34(1)(2003).

[4] B.C.Dhage, *Some algebraic and topological random fixed point theorem with application to nonlinear random integral equation*, Tamkang J. Math., 35(2004).

[5] D.S.Palimkar, *Existence Theory of second order random differential equation*, Global Journal of Mathematical Sciences Theory and Practical, 4(4)(2012).

[6] M.K.Bhosale, *Second order Nonlinear Functional Random Diffferential Equation*, Journal of Global Research in Mathematical Archives, 1(12)(2013).

[7] A.T.Bharucha-Ried, *Random Integral Equations*, Academic Press, New York, (1972).

[8] S.Itoh, *Random Fixed Point Theorems With Application to random differential equations In Banach Spaces*, J. Math. Anal. Appl., 67(1979).