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The Determinant and Adjoint of Fuzzy Neutrosophic Soft Matrices

Research Article

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Abstract: In this paper, we have introduced the determinant and adjoint of a square Fuzzy Neutrosophic Soft Matrices (FNSMs). Also we define the circular FNSM and study some relations on square FNSM such as reflexivity, transitivity and circularity.

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1. Introduction

The theory of fuzzy set was introduced by Zadeh [16] as an appropriate mathematical instrument for description of uncertainty observed in nature. Since the inception it has got intensive acceptability in various fields. The traditional fuzzy sets is characterised by the membership value or the grade of membership value. Some times it may be very difficult to assign the membership value for fuzzy sets. Consequently the concept of interval valued fuzzy sets was proposed [17] to capture the uncertainty of grade of membership value. In some real life problems in expert system, belief system, information fusion and so on, we must consider the truth membership as well as the falsity-membership for proper description of an object in uncertain, ambiguous environment. Neither the fuzzy sets nor the interval valued fuzzy sets is appropriate for such a situation. Intuitionistic fuzzy sets introduced by Atanassov [1] is appropriate for such a situation. The intuitionistic fuzzy sets can only handle the incomplete information considering both the truth membership (or simply membership) and falsity-membership (or non membership) values. It does not handle the indeterminate and inconsistent information which exists in belief system.

Smarandache [9] introduced the concept of neutrosophic set which is a Mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. The concept of soft set theory was introduced by Molodtsov [6] in 1999, it is a new approach for modeling vagueness and uncertainty. Soft set theory has a rich potential for applications in several directions, few of which had been shown by Molodtsov in his pioneer work. It is well known that the matrix formulation of a Mathematical formula gives extra advantages to handle the problem. The classical matrix theory cannot solve the problems involving various types of uncertainities. That type of problems are solved by using fuzzy matrix [11]. Fuzzy matrix has

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been proposed to represent fuzzy relation in a system based on fuzzy set theory, Ovehinnikov [7].

Fuzzy matrices play an important role in Science and Technology. Kim [2–4] has explored some important result on the determinant of a square matrix. In Yong Yang and Chenli Ji [15], introduced a matrix representation of soft set and applied it in decision making problems. Rajarajeswari and Dhanalakshmi [8] introduced fuzzy soft matrix and its application in Medical diagnosis. Sumathi and Arockiarani [10] introduced new operations on fuzzy neutrosophic soft matrices. Mamouni Dhar [5] et al., have also defined Neutrosophic fuzzy matrices and studied about square neutrosophic fuzzy matrices.Uma et al., [12–14] introduced two types of fuzzy neutrosophic soft matrices, determinant theory for fuzzy neutrosophic matrices and generalized inverse of fuzzy neutrosophic soft matrix.

In this article our main intention is to define determinant and adjoint of FNSMs. Furthermore, efforts have been made to establish some properties with the help of the new introduced definition of determinant of square FNSMs. In section 1 we have introduced determinant of two types FNSM and its properties. In section 2, the definition of adjoint of FNSM is given and some related Theorems are asserted.

2. Preliminaries

Definition 2.1 ([6]). *Let U be an initial universe set and E be a set of parameters. Let P(U) denotes the power set of U. Consider a nonempty set A, $A \subset E$. A pair (F,A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.*

Definition 2.2 ([6]). *Let U be an initial universe set and E be a set of parameters. Consider a non empty set A, $A \subset E$. Let $P(U)$ denotes the set of all fuzzy neutrosophic sets of U. The collection (F, A) is termed to be the Fuzzy Neutrosophic Soft Set (FNSS) over U , Where F is a mapping given by $F : A \to P(U)$. Hereafter we simply consider A as FNSM over U instead of (F, A).*

Definition 2.3 ([9]). *A neutrosophic set A on the universe of discourse X is defined as $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle, x \in X\}$, where $T, I, F : X \to]^-0, 1^+[$ and*

$$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+ \tag{1}$$

From philosophical point of view the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application especially in scientific and Engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0, 1]$. Therefore we can rewrite the equation (1) as $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$. In short an element $\tilde{a}$ in the neutrosophic set A, can be written as $\tilde{a} = \langle a^T, a^I, a^F\rangle$, where a^T denotes degree of truth, a^I denotes degree of indeterminacy, a^F denotes degree of falsity such that $0 \leq a^T + a^I + a^F \leq 3$.

Example 2.4. *Assume that the universe of discourse $X = \{x_1, x_2, x_3\}$, where x_1, x_2, and x_3 characterises the quality, relaibility, and the price of the objects. It may be further assumed that the values of $\{x_1, x_2, x_3\}$ are in $[0, 1]$ and they are obtained from some investigations of some experts. The experts may impose their opinion in three components viz; the degree of goodness, the degree of indeterminacy and the degree of poorness to explain the characteristics of the objects. Suppose A is a Neutrosophic Set (NS) of X, such that $A = \{\langle x_1, 0.4, 0.5, 0.3\rangle, \langle x_2, 0.7, 0.2, 0.4\rangle, \langle x_3, 0.8, 0.3, 0.4\rangle\}$, where for x_1 the degree of goodness of quality is 0.4 , degree of indeterminacy of quality is 0.5 and degree of falsity of quality is 0.3 etc,.*

Let $\mathcal{F}_{m\times n}$ denotes FNSM of order $m \times n$ and $\mathcal{F}_n$ denotes FNSM of order $n \times n$. Operations on FNSM of type-I are defined as follows.

Definition 2.5 ([12]). *Let* $A = (\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle), B = (\langle b^T_{ij}, b^I_{ij}, b^F_{ij}\rangle) \in \mathcal{F}_{m\times n}$. *The componentwise addition and componentwise multiplication is defined as*

$$A \oplus B = \left(\sup\left\{a^T_{ij}, b^T_{ij}\right\}, \sup\left\{a^I_{ij}, b^I_{ij}\right\}, \inf\left\{a^F_{ij}, b^F_{ij}\right\}\right).$$
$$A \odot B = \left(\inf\{a^T_{ij}, b^T_{ij}\}, \inf\{a^I_{ij}, b^I_{ij}\}, \sup\{a^F_{ij}, b^F_{ij}\}\right).$$

Definition 2.6 ([12]). *Let* $A \in \mathcal{F}_{m\times n}, B \in \mathcal{F}_{n\times p}$, *the composition of* A *and* B *is defined as*

$$A \circ B = \left(\sum_{k=1}^{n}(a^T_{ik} \wedge\ b^T_{kj}),\quad \sum_{k=1}^{n}(a^I_{ik} \wedge\ b^I_{kj}),\quad \prod_{k=1}^{n}(a^F_{ik} \vee b^F_{kj})\right)$$

equivalently we can write the same as

$$= \left(\bigvee_{k=1}^{n}(a^T_{ik} \wedge b^T_{kj}),\quad \bigvee_{k=1}^{n}(a^I_{ik} \wedge b^I_{kj}),\quad \bigwedge_{k=1}^{n}(a^F_{ik} \vee b^F_{kj})\right).$$

The product $A \circ B$ *is defined if and only if the number of columns of* A *is same as the number of rows of* B. A *and* B *are said to be conformable for multiplication. We shall use* AB *instead of* $A \circ B$.

Definition 2.7 ([12]). *Let* $A = (\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle)$ *and* $c \in \mathcal{F} = [0, 1]$. *Define the fuzzy neutrosophic scalar multiplication as* $cA = (\langle inf\{c, a^T_{ij}\}, inf\{c, a^I_{ij}\}, \sup\{c, a^F_{ij}\}\rangle) \in \mathcal{F}_{m\times n}$. *For the universal matrix* J_1, *by the Definition 2.5,* $cJ_1 = inf(c \odot \langle 1, 1, 0\rangle) = (\langle \inf\{c, 1\}, inf\{c, 1\}, \sup\{c, 0\}\rangle) = (\langle c, c, c\rangle)$ *is the constant matrix all of whose entries are* c. *Further under componentwise multiplication*

$$\begin{aligned} cJ_1 \odot A &= (\langle c, c, c\rangle) \odot (\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle) \\ &= (\langle \min\{c, a^T_{ij}\}, \min\{c, a^I_{ij}\}, \max\{c, a^F_{ij}\}\rangle) \\ &= cA \end{aligned} \tag{2}$$

Definition 2.8. *If* $A = (a_{ij}) \in \mathcal{F}_{m\times n}$, *where* $(a_{ij}) = (\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle)$, *then* $A^c = (b_{ij})_{m\times n}$ *where* $(b_{ij}) = (\langle a^F_{ij}, 1 - a^I_{ij}, a^T_{ij}\rangle)$, *is the complement of* A.

Definition 2.9 ([12]). *The* $n \times m$ *Zero matrix* O_1 *is the matrix all of whose entries are of the form* $\langle 0, 0, 1\rangle$. *The* $n \times n$ *identity matrix* $\mathcal{I}_1$ *is the matrix* $\mathcal{I}_1 = \begin{cases} \langle 1, 1, 0\rangle & \text{if } i = j \\ \langle 0, 0, 1\rangle & \text{if } i \neq j \end{cases}$. *The* $n \times m$ *universal matrix* $\mathcal{J}_1$ *is the matrix all of whose entries are of the form* $\langle 1, 1, 0\rangle$.

Operations on FNSM of type-II are defined as follows.

Definition 2.10 ([12]). *Let* $A = (a^T_{ij}, a^I_{ij}, a^F_{ij}), B = (b^T_{ij}, b^I_{ij}, b^F_{ij}) \in \mathcal{F}_{m\times n}$, *the componentwise addition and componentwise multiplication is defined as*

$$A \oplus B = \left(\left\langle \sup\left\{a^T_{ij}, b^T_{ij}\right\}, \inf\left\{a^I_{ij}, b^I_{ij}\right\}, \inf\left\{a^F_{ij}, b^F_{ij}\right\}\right\rangle\right).$$
$$A \odot B = \left(\left\langle \inf\{a^T_{ij}, b^T_{ij}\}, \sup\{a^I_{ij}, b^I_{ij}\}, \sup\{a^F_{ij}, b^F_{ij}\}\right\rangle\right).$$

Analogous to FNSM of type-I we can define FNSM of type -II in the following way

Definition 2.11 ([12]). *Let $A = (\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle) = (a_{ij}) \in \mathcal{F}_{m\times n}$ and $B = (\langle b_{ij}^T, b_{ij}^I, b_{ij}^F \rangle) = (b_{ij}) \in \mathcal{F}_{n\times p}$ the product of A and B is defined as*

$$A * B = \left(\sum_{k=1}^{n} \left\langle a_{ik}^T \wedge \; b_{kj}^T \right\rangle, \quad \prod_{k=1}^{n} \left\langle a_{ik}^I \vee \; b_{kj}^I \right\rangle, \quad \prod_{k=1}^{n} \left\langle a_{ik}^F \vee b_{kj}^F \right\rangle \right)$$

equivalently we can write the same as

$$= \left(\bigvee_{k=1}^{n} \left\langle a_{ik}^T \wedge b_{kj}^T \right\rangle, \quad \bigwedge_{k=1}^{n} \left\langle a_{ik}^I \vee b_{kj}^I \right\rangle, \quad \bigwedge_{k=1}^{n} \left\langle a_{ik}^F \vee b_{kj}^F \right\rangle \right).$$

the product $A * B$ is defined if and only if the number of columns of A is same as the number of rows of B. A and B are said to be conformable for multiplication.

Definition 2.12 ([12]). *The $n \times m$ Zero matrix O_2 is the matrix all of whose entries are of the form $\langle 0, 1, 1 \rangle$. The $n \times n$ identity matrix $\mathcal{I}_2$ is the matrix $= \begin{cases} \langle 1,0,0 \rangle & \text{if } \; i = j \\ \langle 0,1,1 \rangle & \text{if } \; i \neq j \end{cases}$. The $n \times m$ universal matrix $\mathcal{J}_2$ is the matrix all of whose entries are of the form $\langle 1, 0, 0 \rangle$.*

Definition 2.13 ([12]). *Let $A = (\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle)$ and $c \in \mathcal{F}$, then the fuzzy neutrosophic scalar multiplication is defined by $cA = (\inf\{c, a_{ij}^T\}, \sup\{c, a_{ij}^I\}, \sup\{c, a_{ij}^F\})$.*

Proposition 2.14 ([11]). *If $A \leq B$, then $AC \leq BC$.*

3. The Determinant and Adjoint of FNSM of Type-I

Definition 3.1. *The determinant $|A|$ of $n \times n$ FNSM $A = (\langle a_{ij}^T, a_{ij}^I . a_{ij}^F \rangle)$ is defined as follows*

$$|A| = \langle \bigvee_{\sigma\in S_n} a_{1\sigma(1)}^T \wedge ... \wedge a_{n\sigma(n)}^T, \bigvee_{\sigma\in S_n} a_{1\sigma(1)}^I \wedge ... \wedge a_{n\sigma(n)}^I, \bigwedge_{\sigma\in S_n} a_{1\sigma(1)}^F \vee ... \vee a_{n\sigma(n)}^F \rangle$$

where S_n denotes the symmetric group of all permutations of the indices $(1, 2, ...n)$.

Example 3.2. *Let $A = (\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle)$ be a FNSM such that*

$$A = \begin{bmatrix} (0.5, 0.3, 0.4) & (0.6, 0.7, 0.8) \\ (0.9, 0.6, 0.7) & (0.5, 0.6, 0.7) \end{bmatrix}$$

$$|A| = (\langle 0.5, 0.3, 0.4 \rangle \wedge \langle 0.5, 0.6, 0.7 \rangle) \vee (\langle 0.6, 0.7, 0.8 \rangle \wedge \langle 0.9, 0.6, 0.7 \rangle)$$

$$= \langle 0.5, 0.3, 0.7 \rangle \vee \langle 0.6, 0.6, 0.8 \rangle$$

$$= \langle 0.6, 0.6, 0.7 \rangle$$

Theorem 3.3. *If a FNSM B is obtained from an $n \times n$ FNSM $A = (\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle)$ by multiplying the $i-th$ row of A ($i-th$ column) by $k \in [0, 1]$, then $|B| = k\,|A|$.*

Proof. Suppose that $B = (\langle b_{ij}^T, b_{ij}^I, b_{ij}^F \rangle)$, then

$$|B| = \langle \bigvee_{\sigma\in S_n} b_{1\sigma(1)}^T \wedge ... \wedge b_{n\sigma(n)}^T, \bigvee_{\sigma\in S_n} b_{1\sigma(1)}^I \wedge ... \wedge b_{n\sigma(n)}^I, \bigwedge_{\sigma\in S_n} b_{1\sigma(1)}^F \vee ... \vee b_{n\sigma(n)}^F \rangle$$

$$= \langle \bigvee_{\sigma\in S_n} a_{1\sigma(1)}^T \wedge ... \wedge k\; a_{i\sigma(i)} \wedge ... \wedge a_{n\sigma(n)}^T, \bigvee_{\sigma\in S_n} a_{1\sigma(1)}^I \wedge ... \wedge \wedge k\; a_{i\sigma(i)}^I ... \wedge ... a_{n\sigma(n)}^I, \bigwedge_{\sigma\in S_n} a_{1\sigma(1)}^F \vee ... \vee k\; a_{i\sigma(i)}^F ... a_{n\sigma(n)}^F \rangle$$

$$= \langle k \bigvee_{\sigma\in S_n} a^T_{1\sigma(1)} \wedge ... \wedge a^T_{n\sigma(n)}, k \bigvee_{\sigma\in s_n} a^I_{1\sigma(1)} \wedge ... \wedge a^I_{n\sigma(n)}, k \wedge_{\sigma\in S_n} a^F_{1\sigma(1)} \vee ... \vee a^F_{n\sigma(n)}\rangle$$

$$= k\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle$$

$$= k\,|A|\,.$$

□

Theorem 3.4. *Let $A = (\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle)$ be an $n\times n$ FNSM then $det(PA) = det(A) = det(AP)$, where P is a permutation FNSM which is obtained from the identity FNSM by interchanging row i and row j.*

Proof. Let $A = (\langle c^T_{ij}, c^I_{ij}, c^F_{ij}\rangle)$. Then for any i, j, the $i-th(j-th)$ row of PA is the $j-th(i-th$ respectively) row of A. Infact, P is a permutation FNSM which is generated by a permutation $\begin{bmatrix} i & j \\ j & i \end{bmatrix}$. Since, for any permutation $\sigma \in S_n$,

$$\begin{bmatrix} i & j \\ j & i \end{bmatrix}_\sigma = \zeta \in S_n,$$

$$|PA| = \langle \bigvee_{\sigma\in S_n} c^T_{1\sigma(1)} \wedge ... \wedge c^T_{n\sigma(n)}, \bigvee_{\sigma\in S_n} c^I_{1\sigma(1)} \wedge ... \wedge c^I_{n\sigma(n)}, \bigwedge_{\sigma\in S_n} c^F_{1\sigma(1)} \vee ... \vee c^F_{n\sigma(n)}\rangle$$

$$= \langle \bigvee_{\zeta\in S_n} a^T_{1\zeta(1)} \wedge ... \wedge a^T_{n\zeta(n)}, \bigvee_{\zeta\in S_n} a^I_{1\zeta(1)} \wedge ... \wedge a^I_{n\zeta(n)}, \bigwedge_{\zeta\in S_n} a^F_{1\zeta(1)} \vee ... \vee a^F_{n\zeta(n)}\rangle$$

$$= |A|\,.$$

The case of AP is similar to the above proof. □

Definition 3.5. *Let $A = (\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle)$, be a $m\times n$ FNSM then the transpose of A is defined by, $A^T = (\langle a^T_{ji}, a^I_{ji}, a^F_{ji}\rangle)$.*

Theorem 3.6. *Let $A = (\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle)$ be a FNSM then $det(A) = det(A^T)$, where A^T denotes the transpose of A..*

Proof. Let $A^T = (\langle b^T_{ij}, b^I_{ij}, b^F_{ij}\rangle)$. Since each permutation σ is one -to-one function, we have

$$\left|A^T\right| = \langle \bigvee_{\sigma\in S_n} b^T_{1\sigma(1)} \wedge ... \wedge b^T_{n\sigma(n)}, \bigvee_{\sigma\in S_n} b^I_{1\sigma(1)} \wedge ... \wedge b^I_{n\sigma(n)}, \bigwedge_{\sigma\in S_n} b^F_{1\sigma(1)} \vee ... \vee b^F_{n\sigma(n)}\rangle$$

$$= \langle \bigvee_{\sigma\in S_n} a^T_{\sigma(1)1} \wedge ... \wedge a^T_{\sigma(n)n}, \bigvee_{\sigma\in S_n} a^I_{\sigma(1)1} \wedge ... \wedge a^I_{\sigma(n)n}, \bigwedge_{\sigma\in S_n} a^F_{\sigma(1)1} \vee ... \vee a^F_{\sigma(n)n}\rangle$$

$$= \langle \bigvee_{\zeta\in S_n} a^T_{1\zeta(1)} \wedge ... \wedge a^T_{n\zeta(n)}, \bigvee_{\zeta\in S_n} a^I_{1\zeta(1)} \wedge ... \wedge a^I_{n\zeta(n)}, \bigwedge_{\zeta\in S_n} a^F_{1\zeta(1)} \vee ... \vee a^F_{n\zeta(n)}\rangle,$$

where the permutations ζ is induced by the rearrangement of each σ in $S_n = |A|$. □

Theorem 3.7. *Let $A = (\langle a^T_{ij}, a^I_{ij}, {}^F_{ij}\rangle)$ be an $n\times n$ FNSM. If A contains a zero row (column) then $|A| = \langle 0, 0, 1\rangle$.*

Proof. Each term in $|A|$ contains a factor of each row(column) and hence a factor of zero row (column). Thus each term of $|A|$ is equal to zero, and consequently $|A| = \langle 0, 0, 1\rangle$. Hence zero means element of the form $\langle 0, 0, 1\rangle$. □

Theorem 3.8. *Let $A = \langle(a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle)$ be an $n\times n$ FNSM, If A is triangular, then the determinant of A,*

$$|A| = \left\langle \bigwedge_{1\leq i\leq n} a^T_{ii}, \bigwedge_{1\leq i\leq n} a^I_{ii}, \bigvee_{1\leq i\leq n} a^F_{ii} \right\rangle.$$

Proof. Suppose that $A = \langle (a_{ij}^T, a_{ij}^I, a_{ij}^F) \rangle$ is lower triangular. We consider the term of $|A|$ that $t_{a^T} = \bigwedge_{1 \leq i \leq n} a_{i\sigma(i)}$, $t_{a^I} = \bigwedge_{1 \leq i \leq n} a_{i\sigma(i)}$, $t_{a^F} = \bigvee_{1 \leq i \leq n} a_{i\sigma(i)}$. Let $\sigma(1) \neq 1$. Then $1 < \sigma(1)$ and so $a_{1\sigma(1)}^T = 0$, $a_{1\sigma(1)}^I = 0$, $a_{1\sigma(1)}^F = 1$. This means that $t_{a^T} = 0$, $t_{a^I} = 0$, $t_{a^F} = 1$. If $\sigma(1) \neq 1$. Now, let $\sigma(1) = 1$ and $\sigma(2) \neq 2$ then $2 < \sigma(2)$ and $a_{2\sigma(2)}^T = 0$, $a_{2\sigma(2)}^I = 0$, $a_{2\sigma(2)}^F = 1$, and $t_{a^T} = 0$, $t_{a^I} = 0$, $t_{a^F} = 1$. This means that $t_{a^T} = 0$, $t_{a^I} = 0$, $t_{a^F} = 1$, if $\sigma(1) \neq 1$ or $\sigma(2) \neq 2$. Therefore, in this method, we know that each of terms t_{a^T}, t_{a^I}, t_{a^F}, for $\sigma(1) \neq 1, \sigma(2) \neq 2 ... \sigma(n) \neq n$ must be zero, zero, one respectively, Consequently, $|A| = \left(\langle \bigwedge_{1 \leq i \leq n} a_{ii}^T, \bigwedge_{1 \leq i \leq n} a_{ii}^I, \bigvee_{1 \leq i \leq n} a_{ii}^F \rangle \right)$. □

The following theorem is evident from the definition of determinant of FNSM.

Theorem 3.9. *Let A and B be two FNSM. Then $|AB| \geq |A|\,|B|$.*

4. The Adjoint of FNSM

Definition 4.1. *The adjoint of an $n \times n$ FNSM A denoted by $adjA$, is defined as follows $b_{ij} = |A_{ji}|$ is the determinant of the $(n-1) \times (n-1)$ FNSM formed by deleting row j and column i from A and $B = adjA$.*

Remark 4.2. *We can write the element b_{ij} of $adjA = B = (b_{ij})$ as follows: $b_{ij} = \sum_{\pi \in S_{n_j n_i}} \prod_{t \in n_j} \langle a_{t\pi(t)}^T, a_{t\pi(t)}^I, a_{t\pi(t)}^F \rangle$, where $n_j = \{1, 2, 3.....n\} \setminus \{j\}$ and $S_{n_j n_i}$ is the set of all permutation of set n_j over the set n_i.*

Example 4.3. *Let $A = \begin{bmatrix} \langle 0.2, 0.5, 0 \rangle & \langle 0.2, 0.3, 0.5 \rangle \\ \langle 0.6, 0.2, 0.3 \rangle & \langle 0.6, 0.7, 0.3 \rangle \end{bmatrix}$, then*

$$adj\ A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$$

$$A_{11} = \langle 0.6, 0.7, 0.3 \rangle$$

$$A_{12} = \langle 0.6, 0.2, 0.3 \rangle$$

$$A_{21} = \langle 0.2, 0.3, 0.5 \rangle$$

$$A_{22} = \langle 0.2, 0.5, 0 \rangle$$

$$adj\ A = \begin{bmatrix} \langle 0.6, 0.7, 0.3 \rangle & \langle 0.2, 0.3, 0.5 \rangle \\ \langle 0.6, 0.2, 0.3 \rangle & \langle 0.2, 0.5, 0 \rangle \end{bmatrix}$$

Proposition 4.4. *For $n \times n$ FNSM A and B we have the following*

(1). $A \leq B$ implies $adjA \leq adj\ B$,

(2). $adj\ A + adj\ B \leq adj\ (A + B)$,

(3). $adj\ A^T = (adj\ A)^T$.

Proof.

(1). Let $C = adjA$ and $D = adjB$. That is

$$c_{ij} = \sum_{\pi \in S_{n_j n_i}} \prod_{t \in n_j} \langle a_{t\pi(t)}^T, a_{t\pi(t)}^I, a_{t\pi(t)}^F \rangle$$

$$d_{ij} = \sum_{\pi \in S_{n_j n_i}} \prod_{t \in n_j} \langle b_{t\pi(t)}^T, b_{t\pi(t)}^I, b_{t\pi(t)}^F \rangle.$$

It is clear that $c_{ij} \leq d_{ij}$ since

$$\begin{aligned} &a^T_{t\pi(t)} \leq b^T_{t\pi(t)} \\ &a^I_{t\pi(t)} \leq b^I_{t\pi(t)} \\ &a^F_{t\pi(t)} \geq b^F_{t\pi(t)} \quad \text{for every } t \neq j, \pi(t) \neq i. \end{aligned}$$

(2). Since A, $B \leq A+B$, it is clear that $adj\ A, adj\ B \leq adj\ (A+B)$ and so $adj\ A + adj\ B \leq adj\ (A+B)$.

(3). Let $B = adj\ A$ and $C = adj\ A^T$. Then

$$\begin{aligned} b_{ij} &= \sum_{\pi \in S_{n_j n_i}} \prod_{t \in n_j} \langle a^T_{t\pi(t)}, a^I_{t\pi(t)}, a^F_{t\pi(t)} \rangle \\ c_{ij} &= \sum_{\pi \in S_{n_j n_i}} \prod_{\pi(t) \in n_j} \langle a^T_{t\pi(t)}, a^I_{t\pi(t)}, a^F_{t\pi(t)} \rangle, \end{aligned}$$

which is the element b_{ji} . Hence $(adj\ A)^T = adj\ A^T$. □

Proposition 4.5. *Let A be an $n \times n$ FNSM. Then*

(1). $A\ adj\ A \geq |A|\, I_n$,

(2). $(adj\ A)A \geq |A|\, I_n$.

Proof.

(1). Let $C = A\ adj\ A$. The i-th row of A is $(\langle a^T_{i1}, a^I_{i1}, a^F_{i1} \rangle, \langle a^T_{i2}, a^I_{i2}, a^F_{i2} \rangle ..., \langle a^T_{in}, a^I_{in}, a^F_{in} \rangle)$. By the definition of $adjA$, the j-th column of $adjA$ is given by $(|A_{j1}|, |A_{j2}|, ..., |A_{jn}|)^T$. So that $\langle c^T_{ij}, c^I_{ij}, c^F_{ij} \rangle = \sum_{k=1}^{n} \langle a^T_{ik}, a^I_{ik}, a^F_{ik} \rangle |A_{jk}| \geq 0$ and hence $\langle c^T_{ii}, c^I_{ii}, c^F_{ii} \rangle = \sum_{k=1}^{n} \langle a^T_{ik}, a^I_{ik}, a^F_{ik} \rangle |A_{ik}|$ which is equal to $|A|$. Thus $C = AadjA \geq |A|\, I_n$.

(2). The proof is similar to (1). □

Proposition 4.6. *If a FNSM matrix A has a zero row then $(adjA)A = (\langle 0, 0, 1 \rangle)$ (the zero matrix).*

Proof. Let $H = (adj\ A)A$. That is, $h_{ij} = \sum_k |A_{ki}| \langle a^T_{kj} a^I_{kj} a^F_{kj} \rangle$. If the i-th row of A is zero, that means $(\langle 0, 0, 1 \rangle)$, then A_{ki} contains a zero row where $k \neq i$ and so $|A_{ki}| = \langle 0, 0, 1 \rangle$ (by the Theorem 3.7) for every $k \neq i$ and if $k = i$, then $a_{ij} = 0$ for every j and hence $\sum_k |A_{ki}| \langle a^T_{kj}, a^I_{kj}, a^F_{kj} \rangle = (\langle 0, 0, 1 \rangle)$. Thus $(adj\ A)A = (\langle 0, 0, 1 \rangle)$. □

Theorem 4.7. *For a FNSM A we have $|A| = |adjA|$.*

Proof. Since $adj\ A = \begin{bmatrix} |A_{11}| & |A_{21}| & \cdots & |A_{n1}| \\ |A_{12}| & |A_{22}| & \cdots & |A_{n2}| \\ \vdots & \vdots & \vdots & \vdots \\ |A_{1n}| & |A_{2n}| & \cdots & |A_{nn}| \end{bmatrix}$ we have

$$\begin{aligned} |adjA| &= \sum_{\pi \in S_n} \left|A_{1\pi(1)}\right| \left|A_{2\pi(2)}\right| ... \left|A_{n\pi(n)}\right| \\ &= \sum_{\pi \in S_n} \prod_{i=1}^{n} \left|A_{i\pi(i)}\right| \end{aligned}$$

$$= \sum_{\pi\in S_n} [\prod_{i=1}^{n} (\sum_{\pi\in S_{n_i} n_{\pi(i)}} \prod_{t\in n_i} \langle a^T_{t\theta(t)}, a^I_{t\theta(t)}, a^F_{t\theta(t)}\rangle)]$$

$$= \sum_{\pi\in S_n} [(\sum_{\pi\in S_{n_1} n_{\pi(1)}} \prod_{t\in n_1} \langle a^T_{t\theta(t)}, a^I_{t\theta(t)}, a^F_{t\theta(t)}\rangle)(\sum_{\pi\in S_{n_2} n_{\pi(2)}} \prod_{t\in n_2} \langle a^T_{t\theta(t)}, a^I_{t\theta(t)}, a^F_{t\theta(t)}\rangle)...(\sum_{\pi\in S_{n_n} n_{\pi(n)}} \prod_{t\in n_n} \langle a^T_{t\theta(t)}, a^I_{t\theta(t)}, a^F_{t\theta(t)}\rangle)]$$

$$= \sum_{\pi\in s_n} [(\prod_{t\in n_1} \langle a^T_{t\theta_1(t)}, a^I_{t\theta_1(t)}, a^F_{t\theta_n(t)}\rangle)(\prod_{t\in n_1} \langle a^T_{t\theta_2(t)}, a^I_{t\theta_2(t)}, a^F_{t\pi_2(t)}\rangle)(\prod_{t\in n_1} \langle a^T_{t\theta_n(t)}, a^I_{t\theta_n(t)}, a^F_{t\theta_n(t)}\rangle)]$$

for some $\theta_1 \in S_{n_1 n_{\pi(1)}}, \theta_2 \in S_{n_2 n_{\pi(1)}}, ...\theta_n \in S_{n_n n_{\pi(n)}}$

$$= \sum_{\pi\in S_n} [(a_{2\theta_1(2)} a_{3\theta_1(3)} ... a_{n\theta_1(n)})(a_{1\theta_2(1)} a_{3\theta_2(3)} ... a_{n\theta_2(n)})...(a_{1\theta_n(1)} a_{2\theta_n(2)} ... a_{n-1\theta_n(n-1)})]$$

$$= \sum_{\pi\in S_n} [(a_{1\theta_2(1)} a_{1\theta_3(1)} ... a_{1\theta_n(1)})(a_{2\theta_1(2)} a_{2\theta_1(2)} ... a_{2\theta_n(2)})(a_{3\theta_1(3)} a_{3\theta_2(3)} a_{3\theta_4(3)} ... a_{3\theta_n(3)})...(a_{n\theta_1(n)} a_{n\theta_2(n)} ... a_{n\theta_n(n)})]$$

$$= \sum_{\pi\in S_n} [(a_{1\theta_{f_1(1)}} a_{2\theta_{f_2(2)}} ... a_{n\theta_{f_n(n)}}]$$

for some $f_h \in \{1, 2, ..., n\} \setminus \{h\}, h = 1, 2, ..., n$. However because $a_{h\theta_{f_h}}(h) \neq a_{h\sigma}(f_h)$. We can see that $a_h \theta_{f_h} h = a_{h\pi}(f_h)$ therefore, $|adjA| = \sum_{\sigma\in S_n} a_{1\sigma(1)} a_{2\sigma(2)} ... a_{n\sigma(n)}$, which is the expansion of $|A|$. This complete the proof. □

Definition 4.8. *An $m \times n$ FNSM A is said to be constant if $\langle a^T_{ik}, a^I_{ik}, a^F_{ik}\rangle = \langle a^T_{jk}, a^I_{jk}, a^F_{jk}\rangle$ for all i, j, k, that is its row are equal to each other.*

Proposition 4.9. *Let A be an $n \times n$ constant FNSM Then we have:*

(1). $(adj\ A)^T$ is constant,

(2). $C = A(adj\ A)$ is constant and $C_{ij} = |A|$ which is the least element in A.

Proof.

(1). Let $B = adj\ A$. Then $b_{ij} = \sum_{\pi\in S_{n_j n_i}} \prod_{t\in n_j} (\langle a^T_{t\pi(t)}, a^I_{t\pi(t)}, a^F_{t\pi(t)}\rangle)$ and $b_{ik} = \sum_{\pi\in S_{n_k n_i}} \prod_{t\in n_k} (\langle a^T_{t\pi(t)}, a^I_{t\pi(t)}, a^F_{t\pi(t)}\rangle)$. We notice that $b_{ij} = b_{ik}$ since the numbers $\pi(t)$ of columns cannot be changed in the two expansion of b_{ij} and b_{ik}. So that $(adjA)^T$ is constant.

(2). Since A is constant we can see that $A_{jk} = A_{ik}$ and so $|A_{jk}| = |A_{ik}|$ for every $i, \ j \in \{1, 2, ..., n\}$. Thus

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n} (\langle a^T_{ik}, a^I_{ik}, a^F_{ik}\rangle) |A_{jk}| \\ &= \sum_{k=1}^{n} (\langle a^T_{ik}, a^I_{ik}, a^F_{ik}\rangle) |A_{ik}| \\ &= |A| . \end{aligned}$$

Now, $|A| = \sum_{\pi\in S_n} a_{1\pi(1)} a_{2\pi(2)} ... a_{n\pi(n)} = a_{2\pi(2)} a_{3\pi(3)} ... a_{n\pi(n)}$ for any $\pi \in S_n$ (since A is constant). Taking π as the identity permutation we get $|A| = a_{11} a_{22} ... a_{nn}$ which is the least element in A. □

Definition 4.10. *For a FNSM $A \in \mathcal{F}_{n\times n}$ we have the following*

(1). If $A \geq I_n$, then A is called reflexive.

(2). If $a_{ii} \geq a_{ij}$, then A is called weakely reflexive for all $i, j \in \{1, 2, ..., n\}$ where $A = (a_{ij}) = (\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle)$

(3). If $A = A^T$, then A is called symmetric

(4). If $A = A^2$, then A is called idempotent

(5). If $A^2 \leq A$, then A is called transitive.

Proposition 4.11. *Let A be an $n \times n$ reflexive FNSM. Then $adjA = A^k$ where A^k is idempotent and $k \leq n-1$.*

Proof. The proof of the proposition is similar to fuzzy matrices refer [11]. □

Proposition 4.12. *Let A be an $n \times n$ reflexive FNSM. Then we have the following:*

(1). $adj\ A^2 = (adj\ A)^2 = adj\ A$,

(2). If A is idempotent, then $adj\ A = A$,

(3). $adj\ A$ is reflexive,

(4). $adj\ (adj\ A) = adj\ A$,

(5). $adj\ A \geq A$,

(6). $A(adj\ A) = (adj\ A)A = adj\ A$.

Proof.

(1). Since A is reflexive, we get A^2 is also reflexive and $adjA^2 = (A^2)^k = (A^k)^2 = (adj\ A)^2$. But since A^k is idempotent, we have $(adj\ A)^2 = (adj\ A)$.

(2). We have by Proposition 2.10 that $adjA = A^k (k \leq n-1)$. But we have also that A is idempotent. So $A^k = A$. Thus $adjA = A$.

(3). Let $B = adjA$. That is, $(\langle b^T_{ii} b^I_{ii} b^F_{ii}\rangle) = \sum_{\pi \in S_{n_i}} \prod_{t \in n_i} \langle a^T_{t\pi(t)}, a^I_{t\pi(t)}, a^F_{t\pi(t)}\rangle$. Taking the identity permutation $\pi(t) = t$ we get

$$(\langle b^T_{ii}, b^T_{ii}, b^T_{ii}\rangle) \geq \langle a^T_{11} a^T_{22} ... a^T_{i-1i-1} a^T_{i+1i+1} ... a^T_{nn},\ a^I_{11} a^I_{22} ... a^I_{i-1i-1} a^I_{i+1i+1} ... a^I_{nn}, a^F_{11} a^F_{22} ... a^F_{i-1i-1} a^F_{i+1i+1} ... a^F_{nn}\rangle = (\langle 1, 1, 0\rangle)$$

that is $\langle b^T_{ii}, b^I_{ii}, b^F_{ii}\rangle = \langle 1, 1, 0\rangle$ and $adjA$ is thus reflexive.

(4). Since A is reflexive, we get $adjA$ is idempotent by the above proposition and reflexive by (3). So that by (2) $adj\ (adj\ A) = adj\ A$.

(5). Let $B = adj\ A$. That is $(\langle b^T_{ij} b^I_{ij} b^F_{ij}\rangle) = \sum_{\pi \in S_{n_j n_i}} \prod_{t \in n_j} \langle a^T_{t\pi(t)}, a^I_{t\pi(t)}, a^F_{t\pi(t)}\rangle$. Taking the identity permutation $\pi(h) = h$, $\pi(i) = j$, $h \neq i$, that is the permutation $\begin{bmatrix} 1 & 2 & 3 & \cdots & i & \cdots j-1 & j+1 \cdots n \\ 1 & 2 & 3 & \cdots & j & \cdots j-1 & j+1 \cdots n \end{bmatrix}$ then $\langle a^T_{11} a^T_{22} ... a^T_{i-1i-1} a^T_{i+1i+1} ... a^T_{nn}, a^I_{11} a^I_{22} ... a^I_{i-1i-1} a^I_{i+1i+1} ... a^I_{nn}, a^F_{11} a^F_{22} ... a^F_{i-1i-1} a^F_{i+1i+1} ... a^F_{nn}\rangle$ is a term of $\langle b^T_{ij}, b^I_{ij}, b^F_{ij}\rangle$. So that $\langle b^T_{ij}, b^I_{ij}, b^F_{ij}\rangle \geq \langle a^T_{11}, a^T_{22} ... a^T_{i-1i-1} a^T_{i+1i+1} ... a^T_{nn} = \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle$. Therefore $B = adjA \geq A$.

(6). Let $C = A(adjA)$ and $D = (adjA)A$. Then

$$\begin{aligned} \langle c^T_{ij}, c^I_{ij}, c^F_{ij}\rangle &= \sum_{k=1}^{n} \langle a^T_{ik}, a^I_{ik}, a^F_{ik}\rangle\, |A_{jk}| \\ &\geq \langle a^T_{ii}, a^I_{ii}, a^F_{ii}\rangle\, |A_{ji}| = |A_{ji}| = (\langle b^T_{ij}, b^I_{ij}, b^F_{ij}\rangle) \quad and \quad d_{ij} = \sum_{k=1}^{n} |A_{ki}|\, \langle a^T_{kj}, a^I_{kj}, a^F_{kj}\rangle \end{aligned}$$

$$\geq |A_{ji}| \langle a_{jj}^T, a_{jj}^I, a_{jj}^F \rangle$$
$$= |A_{ji}| = (\langle b_{ij}^T, b_{ij}^I, b_{ij}^F \rangle).$$

Thus we have $A(adj\ A) \geq adj\ A$ and $(adj\ A)A \geq adj\ A$. But by (1) and (5) and Proposition 4.11, we see that $adj A = (adj\ A)(adj\ A) \geq Aadj\ A$. So that $A(adj\ A) = adj\ A$. Also $adj\ A = (adj\ A)(adj\ A) \geq (adj\ A)A$ so that $(adj\ A)A = adj\ A$. Thus we get $A(adj\ A) = (adj\ A)A = adj\ A$. □

Example 4.13. *Let* $A = \begin{bmatrix} \langle 1,1,0\rangle & \langle 0.2,0.3,0.5\rangle \\ \langle 0.6,0.2,0.3\rangle & \langle 1,1,0\rangle \end{bmatrix}$. *Then*

$$A_{11} = \langle 1,1,0\rangle,$$
$$A_{12} = \langle 0.6,0.2,0.3\rangle$$
$$A_{21} = \langle 0.2,0.3,0.5\rangle,$$
$$A_{22} = \langle 1,1,0\rangle$$
$$A^2 = \begin{bmatrix} \langle 1,1,0\rangle & \langle 0.2,0.3,0.5\rangle \\ \langle 0.6,0.2,0.3\rangle & \langle 1,1,0\rangle \end{bmatrix}\begin{bmatrix} \langle 1,1,0\rangle & \langle 0.2,0.3,0.5\rangle \\ \langle 0.6,0.2,0.3\rangle & \langle 1,1,0\rangle \end{bmatrix}$$
$$= \begin{bmatrix} \langle 1,1,0\rangle & \langle 0.2,0.3,0.5\rangle \\ \langle 0.6,0.2,0.3\rangle & \langle 1,1,0\rangle \end{bmatrix}$$
$$A^2 \leq A \textit{ is transitive.}$$
$$adj\ A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$$
$$adj\ A = \begin{bmatrix} \langle 1,1,0\rangle & \langle 0.2,0.3,0.5\rangle \\ \langle 0.6,0.2,0.3\rangle & \langle 1,1,0\rangle \end{bmatrix}$$
$$A(adj\ A) = \begin{bmatrix} \langle 1,1,0\rangle & \langle 0.2,0.3,0.5\rangle \\ \langle 0.6,0.2,0.3\rangle & \langle 1,1,0\rangle \end{bmatrix}\begin{bmatrix} \langle 1,1,0\rangle & \langle 0.2,0.3,0.5\rangle \\ \langle 0.6,0.2,0.3\rangle & \langle 1,1,0\rangle \end{bmatrix}$$
$$= \begin{bmatrix} \langle 1,1,0\rangle & \langle 0.2,0.3,0.5\rangle \\ \langle 0.6,0.2,0.3\rangle & \langle 1,1,0\rangle \end{bmatrix}$$
$$(adj\ A)A = \begin{bmatrix} \langle 1,1,0\rangle & \langle 0.2,0.3,0.5\rangle \\ \langle 0.6,0.2,0.3\rangle & \langle 1,1,0\rangle \end{bmatrix}\begin{bmatrix} \langle 1,1,0\rangle & \langle 0.2,0.3,0.5\rangle \\ \langle 0.6,0.2,0.3\rangle & \langle 1,1,0\rangle \end{bmatrix}$$
$$= \begin{bmatrix} \langle 1,1,0\rangle & \langle 0.2,0.3,0.5\rangle \\ \langle 0.6,0.2,0.3\rangle & \langle 1,1,0\rangle \end{bmatrix}$$
$$(adj\ A)A = A\ adj\ A.$$

It is clear that the above example satisfies of the above Theorem.

Definition 4.14. *An* $n \times n$ *FNSM* A *is called circular if and only if* $(A^2)^T \leq A$, *or more explicitly,* $\langle a_{jk}^T, a_{jk}^I, a_{jk}^F\rangle\langle a_{ki}^T, a_{ki}^I, a_{ki}^F\rangle \leq \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle$ *for every* $k = 1, 2, ...n$.

Theorem 4.15. *For an* $n \times n$ *FNSM* A *we have the following:*

(1). If A is symmetric, then adj A is symmetric,

(2). If A is transitive, then adj A is transitive,

(3). If A is circular, then adj A is circular.

Proof.

(1). Let $B = adjA$. Then

$$\langle b^T_{ij}, b^I_{ij}, b^F_{ij}\rangle = \sum_{\pi\in S_{n_jn_i}}\prod_{t\in n_j}\langle a^T_{t\pi(t)}, a^I_{t\pi(t)}, a^F_{t\pi(t)}\rangle \sum_{\pi\in S_{n_in_j}}\prod_{t\in n_i}\langle a^T_{\pi(t)t}, a^I_{\pi(t)t}, a^F_{\pi(t)t}\rangle = (\langle b^T_{ij}, b^I_{ij}, b^F_{ij}\rangle).$$

(since A is symmetric).

(2). Let $D = A_{ij}$. We can determine the elements of D in terms of the elements of A as follows:

$$\langle d^T_{hk}, d^I_{hk}, d^k_{hk}\rangle = \begin{cases} \langle a^T_{hk}, a^I_{hk}, a^F_{hk}\rangle & \text{if } \ h < i, k < j, \\ \langle a^T_{(h+1)k}, a^I_{(h+1)k}, a^F_{(h+1)k}\rangle & \text{if } \ h \geq i, k < j, \\ \langle a^T_{h(k+1)}, a^I_{h(k+1)}, a^F_{h(k+1)}\rangle & \text{if } \ h < i, k \geq j, \\ \langle a^T_{(h+1)(k+1)}, a^I_{(h+1)(k+1)}, a^F_{(h+1)(k+1)}\rangle & \text{if } h \geq i, k \geq j, \end{cases}$$

where A_{ij} denotes the $(n-1)\times(n-1)$ FNSM obtained from A by deleting the $i-th$ row and column j. Now we show that $A_{st}A_{tu} \leq A_{su}$ for every $t \in \{1, 2, ...n\}$. Let $R = A_{st}, C = A_{tu}, F = A_{su}$ and $W = A_{st}A_{tu}$. Note that A is transitive. Then

$$\begin{aligned}
\langle w^T_{ij}, w^I_{ij}, w^F_{ij}\rangle &= \sum_{k=1}^{n-1}\langle r^T_{ik}, r^I_{ik}, r^F_{ik}\rangle\langle c^T_{kj}, c^I_{kj}, c^F_{kj}\rangle \\
&= \sum_{k=1}^{n-1}\langle a^T_{ik}a^T_{kj}, a^I_{ik}a^I_{kj}, a^F_{ik}a^F_{kj}\rangle \leq \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \\
&= \langle f^T_{ij}, f^I_{ij}, f^F_{ij}\rangle \ \text{ if } i < s, k < t, j < u, \\
&= \sum_{k=1}^{n-1}\langle a^T_{ik}a^T_{k(j+1)}, a^I_{ik}a^I_{k(j+1)}, a^F_{ik}a^F_{k(j+1)}\rangle \leq \langle a^T_{i(j+1)}, a^I_{i(j+1)}, a^F_{i(j+1)}\rangle \\
&= \langle f^T_{ij}, f^I_{ij}, f^F_{ij}\rangle \quad \text{if } i < s, k < t, j \geq u, \\
&= \sum_{k=1}^{n-1}\langle a^T_{i(k+1)}a^T_{(k+1)j}, a^I_{i(k+1)}a^I_{(k+1)j}, a^F_{i(k+1)}a^F_{(k+1)j}\rangle \leq \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \\
&= \langle f^T_{ij}, f^I_{ij}, f^F_{ij}\rangle \ \text{ if } i < s, k \geq t, j < u, \\
&= \sum_{k=1}^{n-1}\langle a^T_{i(k+1)}a^T_{(k+1)(j+1)}, a^I_{i(k+1)}a^I_{(k+1)(j+1)}, a^F_{i(k+1)}a^F_{(k+1)(j+1)}\rangle \\
&\leq \langle a^T_{i(j+1)}, a^I_{i(j+1)}, a^F_{i(j+1)}\rangle = \langle f^T_{ij}, f^I_{ij}, f^F_{ij}\rangle \quad \text{if } i < s, k \geq t, j \geq u, \\
&= \sum_{k=1}^{n-1}\langle a^T_{(i+1)k}a^T_{kj}, a^I_{(i+1)k}a^I_{kj}, a^F_{(i+1)k}a^F_{kj}\rangle \\
&\leq \langle a^T_{(i+1)j}, a^I_{(i+1)j}, a^F_{(i+1)j}\rangle = \langle f^T_{ij}, f^I_{ij}, f^F_{ij}\rangle \quad \text{if } i \geq s, k < t, j < u, \\
&= \sum_{k=1}^{n-1}\langle a^T_{(i+1)(k+1)}a^T_{(k+1)j}, a^I_{(i+1)(k+1)}a^I_{(k+1)j}, a^F_{(i+1)(k+1)}a^F_{(k+1)j}\rangle \\
&\leq \langle a^T_{(i+1)j}, a^I_{(i+1)j}, a^F_{(i+1)j}\rangle = \langle f^T_{ij}, f^I_{ij}, f^F_{ij}\rangle \quad \text{if } i \geq s, k \geq t, j < u,
\end{aligned}$$

$$= \sum_{k=1}^{n-1} \langle a^T_{(i+1)(k+1)} a^T_{(k+1)(j+1)}, a^I_{(i+1)(k+1)} a^I_{(k+1)(j+1)}, a^F_{(i+1)(k+1)} a^F_{(k+1)(j+1)} \rangle$$
$$\leq \langle a^T_{(i+1)(j+1)}, a^I_{(i+1)(j+1)}, a^F_{(i+1)(j+1)} \rangle = \langle f^T_{ij}, f^I_{ij}, f^F_{ij} \rangle \quad \text{if } i \geq s, k \geq t, j < u,$$
$$= \sum_{k=1}^{n-1} \langle a^T_{(i+1)k} a^T_{k(j+1)}, a^I_{(i+1)k} a^I_{k(j+1)}, a^F_{(i+1)k} a^F_{k(j+1)} \rangle$$
$$\leq \langle a^T_{(i+1)(j+1)}, a^I_{(i+1)(j+1)}, a^F_{(i+1)(j+1)} \rangle = \langle f^T_{ij}, f^I_{ij}, f^F_{ij} \rangle \quad \text{if } i \geq s, k < t, j \geq u,$$

Thus $w_{ij} \leq f_{ij}$ in every case and therefore $\langle a^T_{st}, a^I_{st,} a^F_{st} \rangle |A_{tu}| \leq |A_{su}|$ for every $t \in \{1, 2, ...n\}$. By Theorem 3.9, we get $|A_{st}| |A_{tu}| \leq |A_{su}|$. This means that

$$\langle b^T_{ts}, b^I_{ts}, b^F_{ts} \rangle \langle b^T_{ut}, b^I_{ut}, b^F_{ut} \rangle \leq \langle b^T_{us}, b^I_{us}, b^F_{us} \rangle,$$

that is $\langle b^T_{ut}, b^I_{ut}, b^F_{ut} \rangle \langle b^T_{ts}, b^I_{ts}, b^F_{ts} \rangle \leq \langle b^T_{us}, b^I_{us}, b^F_{us} \rangle$, for every $t \in 1, 2, ..., n$. Hence $B = adj A$ is transitive.

(3). Similarly, as in (2) we can show that $A_{st} A_{tu} \leq A_{us}$. For every $t \in 1, 2, ..., n$ so that $|A_{st}| |A_{tu}| \leq \left| A^T_{us} \right| = |A_{us}|$. Thus $\langle b^T_{st}, b^I_{st}, b^F_{st} \rangle \langle b^T_{tu}, b^I_{tu}, b^F_{tu} \rangle \leq \langle b^T_{us}, b^I_{us}, b^F_{us} \rangle$, and $B = adj A$ is circular. □

Corollary 4.16. *If a FNSM A is similarity then AadjA is also Similarity.*

Theorem 4.17. *For any $n \times n$ FNSM A , the FNSM AadjA is transitive.*

Proof. Let $C = Aadj\ A$, that is

$$c_{ij} = \sum_{k=1}^{n} \langle a^T_{ik}, a^I_{ik}, a^F_{ik} \rangle |A_{jk}|$$
$$= \langle a^T_{if}, a^I_{if}, a^F_{if} \rangle |A_{jf}| \quad for some \quad f \in \{1, 2, 3, ..n\}, \text{ and}$$
$$c^2_{ij} = \sum_{s=1}^{n} c_{is} c_{sj}$$
$$= \sum_{s=1}^{n} (\sum_{l=1}^{n} (\langle a^T_{il}, a^I_{il}, a^F_{il} \rangle |A_{sl}|) \sum_{t=1}^{n} \langle a^T_{st}, a^I_{st}, a^F_{st} \rangle |A_{jt}|$$
$$= \sum_{s=1}^{n} \langle a^T_{ih}, a^I_{ih}, a^F_{ih} \rangle |A_{sh}| \langle a^T_{su}, a^I_{su}, a^F_{su} \rangle |A_{ju}|$$
$$\leq \langle a^T_{ih}, a^I_{ih}, a^F_{ih} \rangle |A_{ju}|$$
$$\leq \langle a^T_{if}, a^I_{jf}, a^F_{jf} \rangle |A_{jf}|,$$

for some $h, u \in \{1, 2, ..., n\}$. Thus $(A\ adj\ A)^2 \leq A\ adj\ A$. □

Example 4.18. *LetA be a FNSM* $\begin{bmatrix} \langle 0.1, 0, 0.3 \rangle & \langle 0.5, 0.6, 0.7 \rangle \\ \langle 0.2, 0.5, 0.3 \rangle & \langle 0.6, 0.2, 0.3 \rangle \end{bmatrix}$. *Then*

$$(adj\ A) = \begin{bmatrix} \langle 0.6, 0.2, 0.3 \rangle & \langle 0.5, 0.6, 0.7 \rangle \\ \langle 0.2, 0.5, 0.3 \rangle & \langle 0.1, 0, 0.3 \rangle \end{bmatrix}$$
$$A(adj\ A) = \begin{bmatrix} \langle 0.1, 0, 0.3 \rangle & \langle 0.5, 0.6, 0.7 \rangle \\ \langle 0.2, 0.5, 0.3 \rangle & \langle .6, .2, 0.3 \rangle \end{bmatrix} \begin{bmatrix} \langle 0.6, 0.2, 0.3 \rangle & \langle 0.5, 0.6, 0.7 \rangle \\ \langle 0.2, 0.5, 0.3 \rangle & \langle 0.1, 0, 0.3 \rangle \end{bmatrix}$$
$$= \begin{bmatrix} \langle 0.2, 0.5, 0.3 \rangle & \langle 0.1, 0, 0.7 \rangle \\ \langle 0.2, 0.2, 0.3 \rangle & \langle 0.2, 0.5, 0.3 \rangle \end{bmatrix}$$

$$(A\ adj\ A)^2 = \begin{bmatrix} \langle 0.2,0.5,0.3\rangle & \langle 0.1,0.0,0.7\rangle \\ \langle 0.2,0.2,0.3\rangle & \langle 0.2,0.5,0.3\rangle \end{bmatrix} \begin{bmatrix} \langle 0.2,0.5,0.3\rangle & \langle 0.1,0,0.7\rangle \\ \langle 0.2,0.2,0.3\rangle & \langle 0.2,0.5,0.3\rangle \end{bmatrix}$$

$$(A\ adj\ A)^2 = \begin{bmatrix} \langle 0.2,0.5,0.3\rangle & \langle 0.1,0,0.7\rangle \\ \langle 0.2,0.2,0.3\rangle & \langle 0.2,0.5,0.3\rangle \end{bmatrix}$$

$$(A\ adj\ A)^2 \leq (A\ adj\ A) \quad \textit{is also transitive.}$$

We omit the proofs for type-II FNSM as the proofs are analogous to type-I FNSM.

5. Conclusion

In this paper we have introduced determinant and adjoint of two types of FNSMs and discussed some of its properties.

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