International Journal of Mathematics And its Applications
Volume 5, Issue 4–F (2017), 835–838.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

New Properties of Almost Finitely Cogenerated Module and New Result

Research Article

Abdul-Qawe Kaed[1]* and Salem Abdullah Basardah[2]

1 Department of Mathematics, Faculty of Applied Science, University of Thamar, Dhamar, Yemen.

2 Department of Mathematics, Faculty of Education, University of Aden, Aden, Yemen.

Abstract: In [6] The concept of almost finitely cogenerated is introduced and studied with some properties of it. recall that: Let R be a ring with identity. An R-module M is called almost finitely cogenerated if M is not finitely cogenerated but every factor of M is finitely cogenerated. In this paper we will introduce and study New properties of the almost finitely cogenerated R-module and prove a new result of it.

MSC: 13E05, 13D02, 13E15, 13E99, 16D50, 16D80.

Keywords: A finitely cogenerated module, Almost finitely cogenerated module, A Hopfian module, A Co-Hopfian module.

1. Introduction

Throughout this paper all rings are commutative with unit unless otherwise noted and all modules are assumed to be unitary. In 1983 W.D.Weakly [10] introduced and studied a concept of almost finitely generated as a module is not finitely generated but whose proper submodules are finitely generated. Recall that an R-module M is called finitely generated, if for any family $(M_i)_{i\in I}$ of submodules of M with $\sum\limits_{i\in I} M_i = 0$, there is a finite subset J of I such that $\sum\limits_{j\in J} M_j = 0$ (see [7, 9]). In [6] a concept of almost finitely cogenerated module is introduced and studied as a dual of a concept of almost finitely generated and it is defined as the following: Let R be a ring, An R-module M is called almost finitely cogenerated if it is not finitely cogenerated but all its factors are finitely cogenerated. And every submodule of an almost finitely cogenerated is also an almost finitely cogenerated. Recall that an R-module M is called finitely cogenerated if for any family $M_{i\in I}$ submodules of M with $\bigcap\limits_{i\in I} M_i = 0$, there is a finite subset J of I such that $\bigcap\limits_{j\in J} M_j = 0$ [9]. In [1] The following conceptes Hopfian (resp Co-Hopfian) are introduced and study such that An R- module M is called Hopfian (resp Co-Hopfian) if any surjective (resp injective) of endomorphism of M is an automorphism. See in [5] M is Hopfian if and only if M is Co-Hopfian. In this paper we will introduced and study New properties of an almost finitely cogenerated module. The theorem 2.2 is studied a behavior of the notion almost finitely cogenerated on short exact sequences. In 2.5 proved that a direct sum of almost finitely cogenerated modules is also almost finitely cogenerated. In 2.4 proved if N and K be almost finitely cogenerated module, then $N \cap K$ is almost finitely cogenerated module. And more than if $M_1, M_2 ......, M_n$ are almost finitely cogenerated, then $\bigcap\limits_{i=1}^{n} M_i$ is also an almost finitely cogenerated. In 3.2 we introduce the main result : Let M be an R- module, if M is an almost

* E-mail: kaed2012d@gmail.com

finitely cogenerated module, M is Hopfian. But the converse of this result is not true in general. where a counter example is proved this fact : If R is semisimple ring and let M be an R-module. then M is Hopfian if and only if M is Artinian see 3.11 [5] and by definition of almost finitely cogenerated module in [6] M is not almost finitely cogenerated module. Finally in 3.5 M is almost finitely cogenerated,then M is Co-Hopfian. It is clear that the inverse of the proposition above is also not true in general.

2. New Properties

The theorem 2.2 below is studied a behavior of the notion almost finitely cogenerated on short exact sequences. Before that we recall this definition as the following :

Definition 2.1 ([6])**.** *A module M is called almost finitely cogenerated if it is not finitely cogenerated but for any nonzero submodule N of M, M/N is finitely cogenerated. That is equivalent that M is not artinian, but M/N is artinian for any nonzero submodule N of M.*

Theorem 2.2. *Let N, M and K be R-modules and $0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$ be a short exact sequence, then*

(1). If K and N are almost finitely cogenerated modules, then M is almost finitely cogenerated module.

(2). If K and M are almost finitely cogenerated modules, then N is almost finitely cogenerated module.

(3). K and N are almost finitely cogenerated modules if and only if $K + M$ is almost finitely cogenerated module.

Proof.

(1). First M is not finitely cogenerated. Indeed, if we suppose that M is finitely cogenerated then, being a submodules of M, will be finitely cogenerated which is absurd. Now consider a short exact sequence $0 \longrightarrow T \longrightarrow M \longrightarrow M/T \longrightarrow 0$. We need to prove that M/T is finitely cogenerated. Consider the pullback diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & \grave{L} & \to & T & \to & K & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & N & \to & M & \to & K & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & M/T & = & M/T & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since N is almost finitely cogenerated and by the left vertical exact sequence, M/T is finitely cogenerated, as desired (see [5]).

(2). From [6] N is an almost finitely cogenerated module as isomorphism a submodule of M (Because $N/ker(f) \cong Im(f)$ and $ker(f) = 0$).

(3). Suppose that K and N are almost finitely cogenerated modules and from 2.2(1) $K + M$ is almost finitely cogenerated module. Conversally suppose that $K + M$ is almost finitely cogenerated module and K and M are submodules of $K + M$, then from [6], hence K and M are almost finitely cogenerated modules. ☐

Corollary 2.3. *Let R be a ring and $M_1, M_2, \ldots, M_n$ are R-modules. $M_1, M_2, \ldots, M_n$ are almost finitely cogenerated if and only if $\sum_{i=1}^{n} M_i$ is an almost finitely cogenerated.*

Proof. Let $M_1, M_2, \ldots, M_n$ be almost finitely cogenerated. We have a short exact sequence

$$0 \to M_n \to \sum_{i=1}^{n} M_i \longrightarrow \sum_{i=1}^{n-1} M_i \longrightarrow 0$$

and by induction if $n = 2$, then we get

$$0 \to M_2 \to \sum_{i=1}^{2} M_i \longrightarrow M_1 \longrightarrow 0$$

from Theorem 2.2 (4) the asseration is true. Now we suppose that $M_1, M_2, \ldots, M_n$ are almost finitely cogenerated if and only if $\sum_{i=1}^{n} M_i$ is an almost finitely cogenerated and we prove it when $n + 1$. The short exact

$$0 \to M_{n+1} \to \sum_{i=1}^{n+1} M_i \longrightarrow M_1 \longrightarrow 0$$

and from Theorem 2.2 (2) that is implies that M_{n+1} is an almost finitely cogenerated (because M_1 is almost finitely cogenerated). We have also

$$0 \to M_{n+1} \to \sum_{i=1}^{n+1} M_i \longrightarrow \sum_{i=1}^{n} M_i \longrightarrow 0,$$

then from Theorem 2.2 (4) $\sum_{i=1}^{n+1} M_i$ is an almost finitely cogenerated and it follows that $M_1, M_2, \ldots, M_n$ are almost finitely cogenerated if and only if $\sum_{i=1}^{n} M_i$ is an almost finitely cogenerated for every n. □

Proposition 2.4. *Let N and K be almost finitely cogenerated module, then $N \cap K$ is almost finitely cogenerated module.*

Proof. Since N and K are almost finitely cogenerated module. and $N \cap K$ is a submodule of N and K, then from [6] $N \cap K$ is almost finitely cogenerated module. □

Corollary 2.5. *Let R be a ring and $M_1, M_2, \ldots, M_n$ are R-modules. If $M_1, M_2, \ldots, M_n$ are almost finitely cogenerated, then $\bigcap_{i=1}^{n} M_i$ is an almost finitely cogenerated.*

Proof. Where $\cap_{i=1}^{n} M_i$ is a submodule of $M_i (1 \leq i \leq n)$ which are almost finitely cogenerated , then $\cap_{i=1}^{n} M_i$ is an almost finitely cogenerated (see [6]) Proposition 2.2. □

3. The Main Result

Before we introduce the main result we need to recall the following concepts : see [1, 5, 10]

Definition 3.1.

(1). Let M be a R-module, M is called Hopfian, if any surjective R-homomorphism $f : M \longrightarrow M$ is an isomorphism.

(2). M is called Co-Hopfian, if any injective R-homomorphism $f : M \longrightarrow M$ is an isomorphism.

Theorem 3.2. *Let M be an R- module,if M is an almost finitely cogenerated module, M is Hopfian.*

Before we prove the theorem we need to recall the proposition blew (see [6] Proportion 2.2).

Proposition 3.3. *For any $f \in End_R(M)$, either $f = 0$ or $ker(f) = 0$ and if $f(M) \neq M$, then $\bigcap_{n \geq 1} f^n(M) = 0$.*

Proof. Let f be an R-endomorphism of M. Since we have, $f(M) \cong M/Kerf, Kerf \neq 0$ implies $f(M) = 0$. On the other hand, if $N = \bigcap_{n\geq 1} f^n(M)$ and $N \neq 0$, then $kerf = 0$ and M/N is artinian and so there exists a nonzero integer n such that $f^{n+1}(M) = f^n(M)$, it follows That $f(M) = M$. □

Now we prove the theorem.

Proof. Suppose that M is almost finitely cogenerated and let for any $f \in End_R(M)$ surjective, then $f \neq 0$ and from proposition 2.2 in [6], then $ker(f) = 0$ and following that is injective that is implies that f is an automorphism, hence M is Hopfian. The converse of the theorem above is not true in general. we introduce a counter example as the following : If R is semisimple ring and let M be an R-module. then M is Hopfian if and only if M is Artinian see 3.11 [5] and by definition of almost finitely cogenerated module in [6] M is not almost finitely cogenerated module. □

Proposition 3.4. *Let M be R- module, M is Hopfian module if and only if M is Co-Hopfian.*

Proof. Suppose that M is Hopfian R-module. Then there exit an automorphism of M (i.e $f : M \longrightarrow M$) that is implies that M is Co-Hopfian. □

Proposition 3.5. *Let M be an R- module,if M is almost finitely cogenerated module,then M is Co-Hopfian.*

Proof. Suppose that M is an almost finitely cogenerated module, then it is Hopfian and from Proposition 3.4 M is Co-Hopfian. □

It is clear that the converse of the proposition above is also not true in general as before.

References

[1] A.Ghorbani and A.Haghany, *An intorduction to commutative Algebra*, 255(2002), 324-341.

[2] A.Hamaimou, A.Kaidi and E.Sanchez Campos, *Generalized Fitting Modules Rings*, Journal of Algebra, 308(1)(2007), 199-214.

[3] A.Haghany and M.R.Vedadi, *Modules whose injective endomorphisms are essential* , Journal of Algebra, 243(2001), 765-779.

[4] F.W.Andersen and K.R.Fuller, *Rings and Categories of Modules*, Spring-Verlag, Heidelberg, New York, (1974).

[5] H.Bouzraa and A.Q.Kaed, *Almost finitely Copresented*, J. of Algebra and Number Theory Academia, 2(5)(2012), 313-319.

[6] H.Essannouni and A.Q.Kaed, *Module of which all proper factor modules are finitely cogenerated*, Applied Mathematical Sciences, 4(1)(2010), 13-20.

[7] M.F.Atiyah and I.G.Macdonald, *introduction to commutative Algebra*, Addison-Wesley, (1969).

[8] M.X.Wei, *Hopfian and CO6Hopfian modules*, Comm. Algebra, 23(1995), 1219-1229.

[9] R.Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, (1991).

[10] W.D.Weakly, *Modules whose proper submodules are finitely generated*, Journal of Algebra, 84(1983), 189-219.