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Graph Coloring and its Real Time Applications an Overview

Research Article

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Abstract: Graph coloring is one of the most important concepts in graph theory and it has huge number of applications in daily life. Various coloring methods are available and can be used on requirement basis. The proper coloring of a graph G is the coloring of the vertices and edges with minimal number of colors such that no two vertices should have the same color. The minimum number of color is called the chromatic number $\chi(G)$ and the graph G is called properly colored graph. This paper presents the applications of graph coloring and its importance in various fields.

Keywords: Chromatic number $\chi(G)$, Edge coloring, Vertex coloring.

1. Introduction

A Graph G consists of pair $(V(G), E(G))$ where $V(G)$ is a non-empty finite set whose element are vertices or points and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The number of vertices in a graph is called its order and the number of edges in a graph is called its size. We denote the order and size of a graph usually by n and m respectively. A graph of order n and size m is referred as a *(n, m)* graph. If $e = \{u, v\} \in E(G)$, the edge e is said to join u and v. We write $e = uv$ and we say that the vertex u and v are adjacent. We also say that the vertex u and the edge e are incident with each other. If two distinct edges incident with a common vertex then they are called adjacent edges.

2. Preliminaries

Definition 2.1. *The degree of a vertex of a graph is the number of edges incident to the vertex, with loop counted twice. The degree of vertex denoted by deg(v) or deg v. The maximum degree of a graph G denoted by* $\Delta(G)$, *and the minimum degree of graph G, denoted by d(G).*

Definition 2.2. *A graph with no self loop and no parallel edges is called a simple graph.*

Definition 2.3. *A graph whose edge set is empty is a null graph.*

Definition 2.4. *A graph in which any two distinct vertices are adjacent is called a complete graph. The complete graph with n vertices is denoted by* K_n.

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Definition 2.5. *A graph is called planar if one can draw the graph in the plane (or paper) such that no two edge cross.*

Definition 2.6. *A bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V. Vertex sets U and V are usually called the parts of the graph.*

Definition 2.7. *A clique is a subset of vertices of an undirected graph such that every two distinct vertices in the clique are adjacent; that is, its induced sub graph is complete.*

3. Graph Coloring

In graph theory, graph coloring is a special case of graph labeling; it is an assignment of labels traditionally called "colors" to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color; this is called a vertex coloring. Similarly, an edge coloring assigns a color to each edge so that no two adjacent edges share the same color, and a face coloring of a planar graph assigns a color to each face or region so that no two faces that share a boundary have the same color.

3.1. Vertex Coloring

- $k-$Vertex coloring: A $k-$vertex coloring of a graph G is an assignment of k colours, $1, 2, \ldots, k$, to the vertices of G. The coloring is proper if no two distinct adjacent vertices have the same color.
- $k-$Vertex colorable: A graph G is $k-$vertex colorable if G has a proper $k-$vertex coloring. $k-$Vertex colorable is also called as k-colorable. A graph is k-colorable if and only if its underlying simple graph is k-colorable.
- Chromatic number $\chi(G)$: The chromatic number $\chi(G)$ of a graph G is the minimum k for which, G is $k-$colorable.
- K- Chromatic: If $\chi(G) = k$, G is said to be k-chromatic.

3.2. Edge Coloring

- $k-$edge coloring: A $k-$edge coloring of a loop less graph G is an assignment of k colors, $1, 2, \ldots, k$, to the edges of G. The coloring is proper if no two distinct adjacent edges have the same color.
- $k-$edge colorable: A graph G is $k-$edge colorable if G has a proper $k-$edge coloring. Edge chromatic number $\chi'(G)$: The edge chromatic number $\chi'(G)$, of a loop less graph G is the minimum k for which G is $k-$edge colorable.
- $k-$edge chromatic: If $\chi'(G) = k$, G is said to be k-edge chromatic.

4. Some Properties and Theorems

4.1. Bounds on the Chromatic Number

Assigning distinct colors to distinct vertices always yields a proper coloring, so $1 \leq \chi(G) \leq n$. The only graphs that can be 1-colored are edgeless graphs. A complete graph K_n of n vertices requires $\chi(K_n) = n$ colors. In an optimal coloring there must be at least one of the graph's m edges between every pair of color classes, so $\chi(G)(\chi(G) - 1) \leq 2m$. If G contains a clique of size k, then at least k colors are needed to color that clique; in other words, the chromatic number is at least the clique number: $\chi(G) \geq \omega(G)$. For perfect graphs this bound is tight. The 2-colorable graphs are exactly the bipartite graphs, including trees and forests. By the four color theorem, every planar graph can be 4-colored. A greedy coloring shows that every graph can be colored with one more color than the maximum vertex degree, $\chi(G) \leq \Delta(G)+1$. Complete graphs have $X(G) = n$ and odd cycles ha $\chi(G)ve = 3$ and even cycle have $\chi(G) = 2$, so for these graphs this bound is be $\Delta(G) = n - 1$st possible. In all other cases, the bound can be slightly improved;

Theorem 4.1 (Brooks' Theorem)**.** *$\chi(G) \leq \Delta(G)$ for a connected, simple graph G, unless G is a complete graph or an odd cycle.*

4.2. Lower Bounds on the Chromatic Number

Several lower bounds for the chromatic bounds have been discovered over the years:

Definition 4.2 (Hoffman's bound)**.** *Let W be a real symmetric matrix such that $W_{ij} = 0$ Whenever (i,j) is not an edge in G. Define $\chi W(G) = 1 - \frac{\lambda_{\max}(W)}{\lambda_{\min}(W)}$, where $\lambda_{\min}(W)$, $\lambda_{\max}(W)$ are the largest and smallest Eigen values of W. Define $\chi H(G) = \max\limits_{w} \chi W(G)$, with W as above. Then: $\chi H(G) \leq \chi(G)$.*

Definition 4.3 (Vector Chromatic Number)**.** *Let W be a positive semi-definite matrix such that $W_{ij} \leq -\frac{1}{(k-1)}$ whenever (i,j) is an edge in G. Define $\chi V(G)$ to be the least k for which such a matrix W exists. Then $\chi V(G) \leq \chi(G)$.*

Definition 4.4 (Lovasz Number)**.** *The Lovasz number of a complementary graph, is also a lower bound on the chromatic number: $\vartheta\,(G) \leq \chi\,(G)$.*

Definition 4.5 (Fractional Chromatic Number)**.** *The Fractional chromatic number of a graph is a lower bound on the chromatic number as well: $\chi f(G) = \chi\,(G)$. These bounds are ordered as follows:*

$$\chi H\,(G) = \chi V\,(G) = \vartheta(G) = \chi f\,(G) = \chi\,(G)\,.$$

5. Graphs with High Chromatic Number

Graphs with large cliques have a high chromatic number, but the opposite is not true. The Grotzsch graph is an example of a 4-chromatic graph without a triangle, and the example can be generalized to the Mycielskians.

Theorem 5.1 (Mycielski's Theorem)**.** *There exist triangle-free graphs with arbitrarily high chromatic number.*

Theorem 5.2 (Erdos)**.** *There exist graphs of arbitrarily high girth and chromatic number.*

Definition 5.3 (Bounds on the Chromatic Index)**.** *An edge coloring of G is a vertex coloring of its line graph $L\,(G)$, and vice versa. Thus, $\chi'(G) = \chi(L(G))$. There is a strong relationship between edge color ability and the graph's maximum degree $\Delta(G)$. Since all edges incident to the same vertex need their own color, we have $\chi'\,(G) \geq \Delta\,(G)$.*

Theorem 5.4 (Konig's Theorem)**.** *Moreover, $\chi'(G) = \Delta(G)$, if G is bipartite.*

In general, the relationship is even stronger than what Brooks' theorem gives for vertex coloring:

Theorem 5.5 (Vizing's Theorem)**.** *A graph of maximal degree Δ has edge-chromatic number Δ or $\Delta + 1$.*

5.1. Applications of Graph Coloring

Graph coloring problem is to assign colors to certain elements of a graph subject to certain constraints. Here some problems that can be solved by concepts of graph coloring methodologies.

Sudoku: Sudoku is one of the most interested number placement-puzzle and it is also a variation of Graph-coloring problem.

3								4
		2		6		1		
	1		9		8		2	
		5				6		
	2						1	
		9				8		
	8		3		4		6	
		4		1		9		
5								7

Where, every cell represents a vertex. There is an edge between two vertices if they are in same row or same column or same block.

Register Allocation: A compiler is a computer program that translates one computer language into another. To improve the execution time of the resulting code, one of the techniques of compiler optimization is register allocation, where the most frequently used values of the compiled program are kept in the fast processor registers. Ideally, values are assigned to registers so that they can all reside in the registers when they are used. The textbook approach to this problem is to model it as a graph coloring problem. The compiler constructs an interference graph, where vertices are variables and an edge connects two vertices if they are needed at the same time. If the graph can be colored with k colors then any set of variables needed at the same time can be stored in at most k registers.

Scheduling: Vertex coloring models to a number of scheduling problems. In the cleanest form, a given set of jobs need to be assigned to time slots, each job requires one such slot. Jobs can be scheduled in any order, but pairs of jobs may be in conflict in the sense that they may not be assigned to the same time slot, for example because they both rely on a shared resource. The corresponding graph contains a vertex for every job and an edge for every conflicting pair of jobs. The chromatic number of the graph is exactly the minimum make span, the optimal time to finish all jobs without conflicts. Details of the scheduling problem define the structure of the graph. For example, when assigning aircraft to flights, the resulting conflict graph is an interval graph, so the coloring problem can be solved efficiently. In bandwidth allocation to radio stations, the resulting conflict graph is a unit disk graph, so the coloring problem is 3-approximable.

Job Scheduling: Here the jobs are assumed as the vertices of the graph and there is an edge between two jobs 1 they cannot be executed simultaneously. There is a 1-1 correspondence between the feasible scheduling of the jobs and the colorings of the graph.

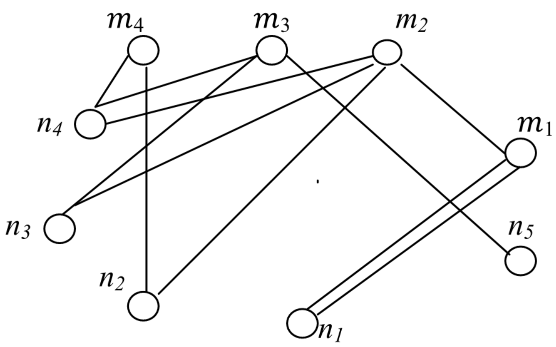
Aircraft Scheduling: Assuming that there are k Aircrafts and they have to be assigned n flights. The i^{th} flight should be during the time interval (a_i, b_i). If two flights overlap, then the same aircraft cannot be assigned to both the flights. The vertices of the graph correspond to the flights. Two vertices will be connected .If the corresponding time intervals overlap. Therefore, the graph is an interval graph that can be colored optimally in polynomial time.

Time Table Scheduling: Allocation of classes and subjects to the professors is one of the major issues if the constraints are complex. Graph theory plays an important role in this problem. For m Professors with n subjects the available number of p periods timetable has to be prepared. This is done as follows.

P	n_1	n_2	n_3	n_4	n_5
m_1	2	0	1	1	0
m_2	0	1	0	1	0
m_3	0	1	1	1	0
m_4	0	0	1	1	1

A bipartite graph G, where the vertices are the number of professors say $m_1, m_2 \ldots, m_k$ and n number of subjects say $n_1 n_2 \ldots n_m$ such that the vertices are connected by p_i edges. It is presumed that at any one period each professor can teach at most one subject and that each subject can be taught by maximum one professor. Consider the first period, the time table for this single period corresponds to a matching in the graph and conversely, each matching corresponds to a possible assignment of professors to subjects taught during that period. So, the solution for the time tabling problem will be obtained by partitioning the edges of graph G in to minimum number of matching. Also the edges have to be colored with minimum number of colors.

Bipartite Graph with 4 Professors and Five Subjects:



Finally the proper coloring of the above mentioned graph can be done by 4 colors using the vertex coloring algorithm which leads to the edge coloring of the bipartite multi graph G.

GSM Mobile Phone Networks: GSM (Groups special Mobile) is a mobile phone Network where the geographical area of this network is divided in to hexagonal regions or cells. Each cell has a communication tower which connects with mobile phones within the cells. All mobile phones connect to the GSM network by searching for cells in the neighbors. Since GSM operate only in four different frequency ranges, it is clear by the concept of graph theory that only four colors can be used to color the cellular regions. These four different colors are used for proper coloring of the regions. Therefore, the vertex coloring algorithm may be used to assign at most four different frequencies for any GSM mobile phone net work.

6. Conclusion

Graph coloring enjoys many practical applications as well as theoretical challenges. The main aim of this paper is to present the importance of various types of coloring. An overview is presented especially to project the applications of graph coloring in Graph theory.

References

[1] Dr.Natarajan Meghanathan, *Review of Graph Theory Algorithms*, Department of Computer Science, Jackson State University, Jackson, MS 39217.

[2] S.G.Shirinivas, S.Vetrivel and N.M.Elango, *Application of Graph theory in Computer Science an overview*, International Journal of Engineering Science and Technology, (2010).

[3] https://en.wikipedia.org/wiki/Graph_coloring.

[4] Preeti Guptha and Omprakash Sikhwa, *A study of vertex-edge coloring techniques with application*, International Journal of Core Engineering & Management, 1(2)(2014), 27-32.

[5] G.J.Chaitin, *Register allocation & spilling via graph coloring*, SIGPLAN Symposium on Compiler Construction, (1982), 98-105.