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Minimum Dominating Partition Energy of a Graph

Research Article

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Abstract: In this paper we compute minimum dominating partition energies of a star graph, complete graph, crown graph and cocktail party graphs. We also establish upper and lower bounds for minimum dominating partition energy $PE_D(G)$ of a graph G.

MSC: 05C50, 05C69.

Keywords: Minimum dominating set, Minimum dominating partition matrix, Minimum dominating partition eigenvalues, Minimum dominating partition energy of a graph.

1. Introduction

The concept of energy of a graph was introduced by I. Gutman [7] in the year 1978. Let G be a graph with n vertices and m edges and let $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A, assumed in non increasing order, are the eigenvalues of the graph G. As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy $E(G)$ of G is defined to be the sum of the absolute eigenvalues of G. i.e., $E(G) = \sum_{i=1}^{n} |\lambda_i|$. For details on the mathematical aspects of the theory of graph energy see the reviews[8], papers [4, 5, 9] and the references cited therein. The basic properties including various upper and lower bounds for energy of a graph have been established in [11, 12], and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [6, 10]. Also in the year 2012 C. Adiga et al. [1] defined the minimum covering energy, $E_C(G)$ of a graph which depends on its particular minimum cover C. Motivated by this, M. R. Rajesh Kanna et al. [13] introduced minimum dominating energy of a graph $E_D(G)$. Recently E. Sampathkumar et al. [14] defined partition energy of a graph. Motivated by these two definitions, we now introduce minimum dominating partition energy $PE_D(G)$ of a graph G. In this paper we have computed minimum dominating partition energies of a star graph, complete graph, crown graph and cocktail party graphs. We also establish upper and lower bounds for $PE_D(G)$.

1.1. Partition Energy

Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set E. Let $P_k = \{V_1, V_2, V_3, ..., V_k\}$ be a partition of a vertex set V. The partition matrix of G is the $n \times n$ matrix defined by $A(G) = (a_{ij})$, where

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$$a_{ij} = \begin{cases} 2 & \text{if } v_i \text{ and } v_j \text{ are adjacent where } v_i, v_j \in V_r \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non adjacent where } v_i, v_j \in V_r \\ 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent between the sets } V_r \text{ and } V_s \text{ for } r \neq s \text{ where } v_i \in V_r \text{ and } v_j \in V_s \\ 0 & \text{otherwise} \end{cases}$$

The eigenvalues of this matrix are called k-partition eigenvalues of G. The k-partition energy $P_kE(G)$ is defined as the sum of the absolute values of k-partition eigenvalues of G [14].

1.2. Minimum Dominating Partition Energy

Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. Let $P_k = \{V_1, V_2, V_3, ..., V_k\}$ be a partition of a vertex set V. A subset D of V is called a dominating set of G if every vertex of $V - D$ is adjacent to some vertex in D. Any dominating set with minimum cardinality is called a minimum dominating set. Let D be a minimum dominating set of a graph G. The minimum dominating k - partition matrix of G is the $n \times n$ matrix defined by $P_kA_D(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 2 & \text{if } v_i \text{ and } v_j \text{ are adjacent where } v_i, v_j \in V_r \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non adjacent where } v_i, v_j \in V_r \\ 1 & \text{if } i = j,\ v_i \in D \text{ or } v_i \text{ and } v_j \text{ are adjacent between the sets } V_r \text{ and } V_s \text{ for } r \neq s \text{ where } v_i \in V_r \text{ and } v_j \in V_s \\ 0 & \text{otherwise} \end{cases}$$

The characteristic polynomial of $A_D(G)$ is denoted by $f_n(G, \lambda)= det(\lambda I - A_D(G))$. The minimum dominating k - partition eigenvalues of the graph G are the eigenvalues of $A_D(G)$. Since $A_D(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. The minimum dominating k - partition energy of G is defined as $P_kE_D(G) = \sum_{i=1}^{n} |\lambda_i|$. Note that the trace of $A_D(G) = |D|$.

2. Minimum Dominating Partition Energy of Some Standard Graphs

Theorem 2.1. *For $n \geq 2$, the minimum dominating 2-partition energy of star graph $K_{1,n-1}$ in which the vertex of degree $n-1$ is in one partition and vertices of degree 1 are in another partition is $(n-2) + \sqrt{n^2 + 2n - 3}$.*

Proof. Consider the star graph $K_{1,n-1}$ with vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$. The minimum dominating set $D = \{v_1\}$. Then 2-partition minimum dominating adjacency matrix is

$$P_2A_D(K_{1,n-1}) = \begin{pmatrix} & v_1 & v_2 & v_3 & \ldots & v_{n-2} & v_{n-1} & v_n \\ v_1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ v_2 & 1 & 0 & -1 & \cdots & -1 & -1 & -1 \\ v_3 & 1 & -1 & 0 & \cdots & -1 & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ v_{n-2} & 1 & -1 & -1 & \cdots & 0 & -1 & -1 \\ v_{n-1} & 1 & -1 & -1 & \cdots & -1 & 0 & -1 \\ v_n & 1 & -1 & -1 & \cdots & -1 & -1 & 0 \end{pmatrix}_{n \times n}$$

characteristic equation is $(-1)^n(\lambda-1)^{n-1}[\lambda^2-(n-3)\lambda-(2n-3)]=0$. Minimum dominating 2-partition eigenvalues of $K_{1,n-1}$ are $\lambda=1$ [$(n-2)$ times], $\lambda=\dfrac{-(n-3)\pm\sqrt{n^2+2n-3}}{2}$ [one time each]. The minimum dominating 2-partition spectrum of $K_{1,n-1}$ is

$$\begin{pmatrix} 1 & \frac{-(n-3)+\sqrt{n^2+2n-3}}{2} & \frac{-(n-3)-\sqrt{n^2+2n-3}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Minimum dominating 2-partition energy of $K_{1,n-1}$ is

$$P_2E_D(K_{1,n-1}) = |1|(n-2) + \left|\frac{-(n-3)+\sqrt{n^2+2n-3}}{2}\right| + \left|\frac{-(n-3)-\sqrt{n^2+2n-3}}{2}\right|$$

$$P_2E_D(K_{1,n-1}) = (n-2)+\sqrt{n^2+2n-3}.$$

□

Definition 2.2. *The cocktail party graph, denoted by $K_{n\times2}$, is a graph having the vertex set $V=\bigcup_{i=1}^{n}\{u_i,v_i\}$ and the edge set $E=\{u_iu_j, v_iv_j : i\neq j\}\bigcup\{u_iv_j, v_iu_j : 1\leq i<j\leq n\}$.*

Theorem 2.3. *For $n\geq 2$, the minimum dominating 2-partition energy of cocktail party graph $K_{n\times2}$ in which $2n$ vertices are partitioned into $U_n=\{u_1,u_2,u_3,\dots,u_n\}$ and $V_n=\{v_1,v_2,v_3,\dots,v_n\}$ is $4(n-2)+\sqrt{9n^2-6n+13}+\sqrt{n^2-2n+5}$.*

Proof. Let $K_{n\times2}$ be the cocktail party graph with vertex set $V=\bigcup_{i=1}^{n}\{u_i,v_i\}$ and the edge set $E=\{u_iu_j, v_iv_j; i\neq j\}\cup\{u_iu_j, v_iv_j; 1\leq i<j\leq n\}$. The minimum dominating set is $D=\{u_1,v_1\}$. Then the minimum dominating 2-partition matrix of cocktail party graph is

$$P_2A_D(K_{n\times2}) = \left(\begin{array}{c|ccccc|ccccc} & u_1 & u_2 & u_3 & \dots & u_n & v_1 & v_2 & v_3 & \dots & v_n \\ \hline u_1 & 1 & 2 & 2 & \dots & 2 & 0 & 1 & 1 & \dots & 1 \\ u_2 & 2 & 0 & 2 & \dots & 2 & 1 & 0 & 1 & \dots & 1 \\ u_3 & 2 & 2 & 0 & \dots & 2 & 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_n & 2 & 2 & 2 & \dots & 0 & 1 & 1 & 1 & \dots & 0 \\ \hline v_1 & 0 & 1 & 1 & \dots & 1 & 1 & 2 & 2 & \dots & 2 \\ v_2 & 1 & 0 & 1 & \dots & 1 & 2 & 0 & 2 & \dots & 2 \\ v_3 & 1 & 1 & 0 & \dots & 1 & 2 & 2 & 0 & \dots & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n & 1 & 1 & 1 & \dots & 0 & 2 & 2 & 2 & \dots & 0 \end{array}\right)_{2n\times2n}$$

characteristic equation is $(\lambda+1)^{n-2}(\lambda+3)^{n-2}[\lambda^2-(3n-5)\lambda-(6n-3)][\lambda^2-(n-1)\lambda-1]=0$. The minimum dominating 2-partition eigen values of $K_{n\times2}$ is $\lambda=-1[(n-2)$ times], $\lambda=-3[(n-2)$ times], $\lambda=\dfrac{(3n-5)\pm\sqrt{9n^2-6n+13}}{2}$ (one time each), $\lambda=\dfrac{(n-1)\pm\sqrt{n^2-2n+5}}{2}$ (one time each). The minimum dominating 2-partition spectrum of $K_{n\times2}$ is

$$\begin{pmatrix} -1 & -3 & \frac{(3n-5)\pm\sqrt{9n^2-6n+13}}{2} & \frac{(n-1)\pm\sqrt{n^2-2n+5}}{2} \\ n-2 & n-2 & 1 & 1 \end{pmatrix}.$$

Minimum dominating 2-partition energy of $K_{n\times2}$ is

$$P_2E_D(K_{n\times2}) = |-1|(n-2)+|-3|(n-2)+\left|\frac{(3n-5)+\sqrt{9n^2-6n+13}}{2}\right|+\left|\frac{(3n-5)-\sqrt{9n^2-6n+13}}{2}\right| + \left|\frac{(n-1)+\sqrt{n^2-2n+5}}{2}\right|+\left|\frac{(n-1)-\sqrt{n^2-2n+5}}{2}\right|$$

$$P_2E_D(K_{n\times2}) = 4(n-2)+\sqrt{9n^2-6n+13}+\sqrt{n^2-2n+5}.$$

□

Theorem 2.4. *For $n \geq 2$, the minimum dominating 1-partition energy of complete graph K_n is equal to $2(n-2) + \sqrt{4n^2 - 4n + 9}$.*

Proof. Let K_n be the complete graph with vertex set $V = \{v_i, v_2, v_3, \ldots, v_n\}$. The minimum dominating set is $D = \{v_1\}$. Then the minimum dominating 1-partition matrix of complete graph is

$$P_1A_D(K_n) = \left(\begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & \ldots & v_{n-2} & v_{n-1} & v_n \\ \hline v_1 & 1 & 2 & 2 & \ldots & 2 & 2 & 2 \\ v_2 & 2 & 0 & 2 & \ldots & 2 & 2 & 2 \\ v_3 & 2 & 2 & 0 & \ldots & 2 & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ v_{n-2} & 2 & 2 & 2 & \ldots & 0 & 2 & 2 \\ v_{n-1} & 2 & 2 & 2 & \ldots & 2 & 0 & 2 \\ v_n & 2 & 2 & 2 & \ldots & 2 & 2 & 0 \end{array}\right)_{n\times n}$$

Characteristic equation is $(-1)^n(\lambda+2)^{n-2}[\lambda^2 - (2n-3)\lambda - 2n] = 0$. The minimum dominating 1-partition eigen values of K_n is $\lambda = -2[(n-2)$ times$]$, $\lambda = \dfrac{(2n-3) \pm \sqrt{4n^2-4n+9}}{2}$ (one time each). Minimum dominating 1-partition spectrum of K_n is

$$\begin{pmatrix} -2 & \dfrac{(2n-3)+\sqrt{4n^2-4n+9}}{2} & \dfrac{(2n-3)-\sqrt{4n^2-4n+9}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Minimum dominating 1-partition energy of K_n is

$$P_1E_D(K_n) = |-2|(n-2) + \left|\frac{(2n-3)+\sqrt{4n^2-4n+9}}{2}\right| + \left|\frac{(2n-3)-\sqrt{4n^2-4n+9}}{2}\right|$$

$$P_1E_D(K_n) = 2(n-2) + \sqrt{4n^2-4n+9}.$$

□

Theorem 2.5. *For $n \geq 2$, the minimum dominating n-partition energy of complete graph K_n is equal to $(n-2) + \sqrt{n^2 - 2n + 5}$.*

Proof. Consider the complete graph K_n with vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$. The minimum dominating set is $D = \{v_1\}$. Then the minimum dominating n-partition matrix of complete graph is

$$P_nA_D(K_n) = \left(\begin{array}{c|ccccccc} & v_1 & v_2 & v_3 & \ldots & v_{n-2} & v_{n-1} & v_n \\ \hline v_1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & \ldots & 1 & 1 & 1 \\ v_3 & 1 & 1 & 0 & \ldots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ v_{n-2} & 1 & 1 & 1 & \ldots & 0 & 1 & 1 \\ v_{n-1} & 1 & 1 & 1 & \ldots & 1 & 0 & 1 \\ v_n & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \end{array}\right)_{n\times n}$$

characteristic equation is $(-1)^n(\lambda+1)^{n-2}[\lambda^2 - (n-1)\lambda - 1] = 0$. Minimum dominating n-partition eigenvalues of K_n is $\lambda = -1[(n-2)$ times$]$, $\lambda = \dfrac{(n-1) \pm \sqrt{n^2-2n+5}}{2}$ (one time each). The minimum dominating n-partition spectrum of K_n is

$$\begin{pmatrix} -1 & \dfrac{(n-1)+\sqrt{n^2-2n+5}}{2} & \dfrac{(n-1)-\sqrt{n^2-n+5}}{2} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Minimum dominating n-partition energy of K_n is

$$P_nE_D(K_n) = |-1|(n-2) + \left|\frac{(n-1)+\sqrt{n^2-n+5}}{2}\right| + \left|\frac{(n-1)-\sqrt{n^2-n+5}}{2}\right|$$
$$P_nE_D(K_n) = (n-2) + \sqrt{n^2-2n+5}.$$

□

Definition 2.6. *The crown graph S_n^0 for an integer $n \geq 2$ is the graph with vertex set $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and edge set $\{u_iv_j : 1 \leq i, j \leq n, i \neq j\}$. $\therefore$ S_n^0 coincides with the complete bipartite graph $K_{n,\ n}$ with horizontal edges removed.*

Theorem 2.7. *For $n \geq 2$, the minimum dominating 2-partitions energy of Crown Graph $S_n^{(0)}$ in which $2n$ vertices are partitioned into $U_n = \{u_1, u_2, u_3, \ldots, u_n\}$ and $V_n = \{v_1, v_2, v_3, \ldots, v_n\}$ is equal to $(2n-1) + \sqrt{4n^2+4n-7}$.*

Proof. Consider the crown graph $S_n^{(0)}$ with vertex set $V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$. The minimum dominating set is $D = \{u_1, v_1\}$. Then the minimum dominating 2-partition matrix of crown graph is

$$MD(S_n^0) = \left(\begin{array}{c|ccccc|ccccc} & u_1 & u_2 & u_3 & \ldots & u_n & v_1 & v_2 & v_3 & \ldots & v_n \\ \hline u_1 & 1 & -1 & -1 & \ldots & -1 & 0 & 1 & 1 & \ldots & 1 \\ u_2 & -1 & 0 & -1 & \ldots & -1 & 1 & 0 & 1 & \ldots & 1 \\ u_3 & -1 & -1 & 0 & \ldots & -1 & 1 & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_n & -1 & -1 & -1 & \ldots & 0 & 1 & 1 & 1 & \ldots & 0 \\ \hline v_1 & 0 & 1 & 1 & \ldots & 1 & 1 & -1 & -1 & \ldots & -1 \\ v_2 & 1 & 0 & 1 & \ldots & 1 & -1 & 0 & -1 & \ldots & -1 \\ v_3 & 1 & 1 & 0 & \ldots & 1 & -1 & -1 & 0 & \ldots & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n & 1 & 1 & 1 & \ldots & 0 & -1 & -1 & -1 & \ldots & 0 \end{array}\right)_{2n\times 2n}$$

Characteristic equation is $\lambda^{n-1}(\lambda-1)(\lambda-2)^{n-1}[\lambda^2+(2n-5)\lambda-(6n-8)] = 0$. The minimum dominating 2-partition eigen values are $\lambda = 1$ (one time), $\lambda = 0[(n-1)$ times], $\lambda = 2[(n-1)$ times], $\lambda = \dfrac{-(2n-5) \pm \sqrt{4n^2+4n-7}}{2}$ (one time each). The minimum dominating 2-partition spectrum of $S_n^{(0)}$ is

$$\begin{pmatrix} 1 & 0 & 2 & \dfrac{-(2n-5)+\sqrt{4n^2+4n-7}}{2} & \dfrac{-(2n-5)-\sqrt{4n^2+4n-7}}{2} \\ 1 & (n-1) & (n-1) & 1 & 1 \end{pmatrix}.$$

The minimum dominating 2-partition spectrum of $S_n^{(0)}$ is

$$P_2E_D(S_n^{(0)}) = |1|(1) + |0|(n-1) + |2|(n-1) + \left|\frac{-(2n-5)+\sqrt{4n^2+4n-7}}{2}\right|(1) + \left|\frac{-(2n-5)-\sqrt{4n^2+4n-7}}{2}\right|(1)$$
$$P_2E_D(S_n^{(0)}) = (2n-1) + \sqrt{4n^2+4n-7}.$$

□

3. Properties of Minimum Dominating Partition Eigenvalues

Let $G = (V, E)$ be a graph with n vertices and $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of V. For $1 \leq i \leq k$, let b_i denote the total number of edges joining the vertices of V_i and c_i be the total number of edges joining the vertices from V_i to V_j for $i \neq j$, $1 \leq j \leq k$ and d_i be the number of non-adjacent pairs of vertices within V_i. Let $m_1 = \sum_{i=1}^{k} b_i$, $m_2 = \sum_{i=1}^{k} c_i$ and $m_3 = \sum_{i=1}^{k} d_i$.

Theorem 3.1. *Let G be a simple graph with vertex set $V = \{v_1, v_2, ..., v_n\}$, edge set E. $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of V and D be a minimum dominating set. If $\lambda_1, \lambda_2, ..., \lambda_n$ are the minimum dominating k-partition eigenvalues of minimum dominating k-partition matrix $A_D(G)$ then*

(1). $\sum_{i=1}^{n} \lambda_i = |D|$

(2). $\sum_{i=1}^{n} \lambda_i^2 = |D| + 2(4m_1 + m_2 + m_3)$.

Proof.

(1). We know that the sum of the k-partition eigenvalues of $A_D(G)$ is the trace of $A_D(G)$. Therefore $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = |D|$.

(2). Similarly the sum of squares of the k- partition eigenvalues of $A_D(G)$ is trace of $[A_D(G)]^2$. Therefore

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n}(a_{ii})^2 + \sum_{i\neq j} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n}(a_{ii})^2 + 2\sum_{i<j}(a_{ij})^2 \\ &=| D | +2(4m_1 + m_2 + m_3).\end{aligned}$$

□

4. Bounds For Minimum Dominating Partition Energy

In this section we find bounds for $P_kE_D(G)$ which are in sequal to the work of McClelland's [12].

Theorem 4.1. *Let $G = (V, E)$ be a graph with n vertices and $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of V. Then*

$$P_kE_D(G) \leq \sqrt{n[|D| + 2(4m_1 + m_2 + m_3)]}$$

where m_1, m_2, m_3 are as defined above for G.

Proof. Cauchy-Schwartz inequality is $\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_j^2\right)$. If $a_i = 1, b_i =| \lambda_i |$ then

$$\begin{aligned}\left(\sum_{i=1}^{n} | \lambda_i |\right)^2 &= \sum_{i=1}^{n} 1 \sum_{i=1}^{n} |\lambda_i|^2 \\ \Rightarrow [PE_D(G)]^2 &\leq n[|D| + 2(4m_1 + m_2 + m_3)]).\end{aligned}$$

Therefore $P_kE_D(G) \leq \sqrt{n[|D| + 2(4m_1 + m_2 + m_3)]}$ which is an upper bound. □

Theorem 4.2. *Let G be a simple graph with n vertices and m edges.$P_k = \{V_1, V_2, ..., V_k\}$ be a partition of V. If D is the minimum dominating set and $P = |detA_D(G)|$ then*

$$\sqrt{|D| + 2(4m_1 + m_2 + m_3) + n(n-1)P^{\frac{2}{n}}} \leq P_kE_D(G) \leq \sqrt{n[|D| + 2(4m_1 + m_2 + m_3)]}.$$

Proof. Cauchy Schwartz inequality is $\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big)$. If $a_i = 1, b_i =\mid \lambda_i \mid$ then

$$\Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2 \leq \Big(\sum_{i=1}^{n} 1\Big)\Big(\sum_{i=1}^{n} \lambda_i^2\Big)$$

$$[P_kE_D(G)]^2 \leq n[|D| + 2(4m_1 + m_2 + m_3)] \quad \text{[Theorem 4.1]}$$

$$\Rightarrow P_kE_D(G) \leq \sqrt{n[|D| + 2(4m_1 + m_2 + m_3)]}$$

Now by arithmetic mean and geometric mean inequality we have

$$\frac{1}{n(n-1)}\sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid \geq \Big[\prod_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid\Big]^{\frac{1}{n(n-1)}}$$

$$= \Big[\prod_{i=1}^{n} \mid \lambda_i \mid^{2(n-1)}\Big]^{\frac{1}{n(n-1)}}$$

$$= \Big[\prod_{i=1}^{n} \mid \lambda_i \mid\Big]^{\frac{2}{n}}$$

$$= \Big|\prod_{i=1}^{n} \lambda_i\Big|^{\frac{2}{n}}$$

$$= |detA_D(G)|^{\frac{2}{n}} = P^{\frac{2}{n}}$$

$$\sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid \geq n(n-1)P^{\frac{2}{n}} \qquad (1)$$

Now consider,

$$[P_kE_D(G)]^2 = \Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2$$

$$= \sum_{i=1}^{n} \mid \lambda_i \mid^2 + \sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid$$

$$\therefore \quad [P_kE_D(G)]^2 \geq |D| + 2(4m_1 + m_2 + m_3) + n(n-1)P^{\frac{2}{n}} \quad \text{[From (5.1)]}$$

$$i.e., \ P_kE_D(G) \geq \sqrt{|D| + 2(4m_1 + m_2 + m_3) + n(n-1)P^{\frac{2}{n}}}$$

□

Theorem 4.3. *If $\lambda_1(G)$ is the largest minimum dominating k-eigenvalue of $A_D(G)$, then* $\lambda_1(G) \geq \frac{2(2m_1 + m_2 - m_3) + |D|}{n}$.

Proof. Let X be any nonzero vector. Then by [2] ,We have $\lambda_1(A) = \max_{X\neq 0}\Big\{\frac{X'AX}{X'X}\Big\}$. Therefore $\lambda_1(A) \geq \frac{J'AJ}{J'J} = \frac{2(2m_1 + m_2 - m_3) + |D|}{n}$ where J is a unit column matrix. □

Upper bound for $P_kE_D(G)$ are computed in the following theorem which is in sequal to Koolen and Moulton [15].

Theorem 4.4. *Let G be a graph with n vertices and m edges with* $2(2m_1 + m_2 - m_3) + |D|) \geq n$ *and* $(4m_1 + 2m_2 - 2m_3 + |D|)^2 - n(8m_1 + 2m_2 + 2m_3 + |D|) \geq 0$ *then*

$$P_kE_D(G) \leq \frac{2(2m_1 + m_2 - m_3) + |D|}{n} + \sqrt{(n-1)\Big[(2(2m_1 + m_2 - m_3) + |D|) - \Big(\frac{2(2m_1 + m_2 - m_3) + |D|}{n}\Big)^2\Big]}.$$

Proof. Cauchy-Schwartz inequality is $\Big[\sum_{i=2}^{n} a_i b_i\Big]^2 \leq \Big(\sum_{i=2}^{n} a_i^2\Big)\Big(\sum_{i=2}^{n} b_j^2\Big)$. Put $a_i = 1$, $b_i = \mid \lambda_i \mid$ then

$$\Big(\sum_{i=2}^{n} \mid \lambda_i \mid\Big)^2 = \sum_{i=2}^{n} 1 \sum_{i=2}^{n} \lambda_i^2$$
$$\Rightarrow [P_k E_D(G) - \lambda_1]^2 \leq (n-1)(2(4m_1 + m_2 + m_3) + |D| - \lambda_1^2)$$
$$\Rightarrow P_k E_D(G) \leq \lambda_1 + \sqrt{(n-1)(2(4m_1 + m_2 + m_3) + |D| - \lambda_1^2)}$$

Let $f(x) = x + \sqrt{(n-1)(2(4m_1 + m_2 + m_3) + |D| - x^2)}$. For decreasing function

$$f'(x) \leq 0 \Rightarrow 1 - \frac{x(n-1)}{\sqrt{(n-1)(2(4m_1 + m_2 + m_3) + |D| - x^2)}} \leq 0$$
$$\Rightarrow x \geq \sqrt{\frac{2(2m_1 + m_2 - m_3) + |D|}{n}}$$

Since $(4m_1 + 2m_2 - 2m_3 + |D|)^2 - n(8m_1 + 2m_2 + 2m_3 + |D|) \geq 0$, we have

$$\sqrt{\frac{2(4m_1 + m_2 + m_3) + |D|}{n}} \leq \frac{2(2m_1 + m_2 - m_3) + |D|}{n}.$$

Since $(2(2m_1 + m_2 - m_3) + |D|) \geq n$, we have

$$\sqrt{\frac{2(2m_1 + m_2 - m_3) + |D|}{n}} \leq \frac{2(2m_1 + m_2 - m_3) + |D|}{n} \leq \lambda_1.$$

Therefore

$$f(\lambda_1) \leq f\Big(\frac{2(2m_1 + m_2 - m_3) + |D|}{n}\Big)$$
$$i.e.,\ P_k E_D(G) \leq f(\lambda_1) \leq f\Big(\frac{2(2m_1 + m_2 - m_3) + |D|}{n}\Big)$$
$$i.e.,\ P_k E_D(G) \leq f\Big(\frac{2(2m_1 + m_2 - m_3) + |D|}{n}\Big)$$
$$i.e.,\ P_k E_D(G) \leq \frac{2(2m_1 + m_2 - m_3) + |D|}{n} + \sqrt{(n-1)\Big[2(2m_1 + m_2 - m_3) + |D| - \Big(\frac{2(2m_1 + m_2 - m_3) + |D|}{n}\Big)^2\Big]}.$$

□

R. B. Bapat and S. Pati [3] proved that if the graph energy is a rational number then it is an even integer. Similar result for minimum dominating energy is given in the following theorem.

Theorem 4.5. *Let G be a graph with a minimum dominating set D and $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of V. If the minimum dominating k- partition energy $P_k E_D(G)$ is a rational number, then $P_k E_D(G) \equiv \mid D \mid (mod\ 2)$.*

Proof. Proof is similar to Theorem 5.4 of [13]. □

References

[1] C.Adiga, A.Bayad, I.Gutman and S.A.Srinivas, *The minimum covering energy of a graph*, Kragujevac J. Sci., 34(2012), 39-56.

[2] R.B.Bapat, *Graphs and Matrices*, Hindustan Book Agency, (2011).

[3] R.B.Bapat and S.Pati, *Energy of a graph is never an odd integer*, Bull. Kerala Math. Assoc., 1(2011), 129-132.

[4] D.Cvetković and I.Gutman, *Applications of Graph Spectra*, Mathematical Institution, Belgrade, (2009).

[5] D.Cvetković and I.Gutman, *Selected Topics on Applications of Graph Spectra*, Mathematical Institute Belgrade, (2011).

[6] A.Graovac, I.Gutman and N.Trinajstić, *Topological Approach to the Chemistry of Conjugated Molecules*, Springer, Berlin, (1977).

[7] I.Gutman, The energy of a graph, Ber. Math-Statist. Sekt. Forschungsz. Graz, 103(1978), 1-22.

[8] I.Gutman, X.Li, J.Zhang, in Graph Energy, ed. by M.Dehmer, F.Emmert - Streib, *Analysis of Complex Networks. From Biology to Linguistics* Wiley-VCH, Weinheim, (2009), 145-174.

[9] I.Gutman, *The energy of a graph : Old and New Results*, ed. by A. Betten, A. Kohnert, R. Laue, A. Wassermann, Algebraic Combinatorics and Applications, Springer, Berlin, (2001), 196-211.

[10] I.Gutman and O.E.Polansky, *Mathematical Concepts in Organic Chemistry*, Springer, Berlin, (1986).

[11] Huiqing Liu, Mei Lu and Feng Tian, *Some upper bounds for the energy of graphs*, Journal of Mathematical Chemistry, 41(1)(2007).

[12] B.J.McClelland, *Properties of the latent roots of a matrix : The estimation of π-electron energies*, J. Chem. Phys., 54(1971), 640-643.

[13] M.R.Rajesh Kanna, B.N.Dharmendra and G.Sridhara, *Minimum dominating energy of a graph*, International Journal of Pure and Applied Mathematics, 85(4)(2013), 707-718.

[14] E.Sampathkumar, S.V.Roopa, K.A.Vidya and M.A.Sriraj, *Partition energy of a graph*, Proc. Jangjeon Math. Soc., 16(3)(2015), 335-351

[15] J.H.Koolen and V.Moulton, *Maximal energy graphs*, Adv. Appl. Math., 26(2001), 47-52.