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Homeomorphism of Fuzzy Topological Graph

Research Article

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Abstract: This paper provides solid theoretical foundation for studying Fuzzy Topological Graph theory. The concepts such as subdivision, homeomorphism and connectedness of a fuzzy topological graph is established. In addition to this an attempt is made to study about planar and minimal non planar fuzzy topological graphs. These concepts are illustrated through examples.

Keywords: Homeomorphism, Fuzzy Topological Graph, Subdivision, Planar Fuzzy Topological Graph, Minimal Non Planar Fuzzy Topological Graph, Connectedness of Fuzzy Topological Graph.

1. Introduction and Preliminaries

L.A.Zadeh introduced Fuzzy set theory in 1965 describing fuzziness mathematically for the first time. The fuzzy graph theory was introduced by A.Rosenfeld using fuzzy relation representing the relationship between the objects by precisely indicating the level of the relationship between the objects of the given set. The notion of fuzzy topology was introduced by C.L.Chang in 1968. It is extension of the concepts of ordinary topological space where the family of all the fuzzy sets on the universe X takes I=[0,1] as the range by I^X. Substituting inclusive relation by the order relation in I^X, a topological structure is introduced naturally into I^X. Topological graph theory deals with ways to represents the geometric realization of graphs. In early 1987, the frontiers of topological graph theory are advancing in numerous different directions.This is the background to introduce the new concept fuzzy topological graph and some of its properties are discussed.

Definition 1.1 (Fuzzy graph [1])**.** *A fuzzy graph is a pair $G:(\sigma,\mu)$ where σ is a fuzzy subset of S, μ is a symmetric fuzzy relation on σ. The elements of S are called the nodes or vertices of G and the pair of vertices edges in G.*

Definition 1.2 (Path [2])**.** *A path P in a fuzzy graph $G:(\sigma,\mu)$ is a sequence of distinct nodes $v_0, v_1, v_2, \dots, v_n$ such that $\mu(v_{i-1}, v_i) > 0$, $1 < i < n$. Here n is called the length of the path. The consecutive pairs (v_{i-1}, v_i) are called arcs of the path.*

Definition 1.3 (Fuzzy connectedness [1])**.** *If u,v are nodes in G and if they are connected by means of a path the strength of that path is defined as $\overset{n}{\underset{i=1}{\wedge}} \mu\left(v_{i-1}, v_i\right)$. (i.e.) It is the strength of the weakest arc. If u,v are connected by means of path of length 'k' then $\mu^k(u,v)$ is defined as $\mu^k(u,v) = \sup\left\{\mu(u,v_1)\wedge\mu(v_1,v_2)\wedge\mu(v_2,v_3)\wedge\ldots\wedge\mu(v_{k-1},v)\,/u,v_1,v_2\ldots v_{k-1},v\in S\right\}$. If $u,v\in S$ the strength of connectedness between u and v is $\mu^\infty(u,v) = \sup\{\mu^k(u,v)\,/k = 1,2,3,..\}$.*

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Definition 1.4 (Fuzzy planar graph [6])**.** *Let $\psi = (v, \sigma, E)$ be a fuzzy planar graph with fuzzy planarity value 1 and $E = \{((x,y),(x,y)_{\mu^j}), j = 1,2,3...p_{xy}/(x,y) \in VXV\}$ and $p_{xy} = \max\{j/(x,y)_{\mu^j} \neq 0\}$. The membership value of the fuzzy face F is given by* $\min\left\{\frac{(x,y)_{\mu^j}}{\min\{\sigma(x),\sigma(y)\}}, j = 1,2,3...p_{xy}/(x,y) \in E'\right\}$.

Definition 1.5 (Intersecting value in fuzzy multi graph [6])**.** *Let in a fuzzy multigraph $\psi = (v, \sigma, E)$, E contains two edges $((a,b),(a,b)_{\mu_i^k})$ and $((c,d),(c,d)_{\mu_j^l})$ which are intersected at a point p, where k and l are fixed integers. Strength of the fuzzy edge (a,b) can be measured by the value $I_{(a,b)} = \frac{(a,b)_{\mu^k}}{\min\{\sigma(a),\sigma(b)\}}$. If $I_{(a,b)} \geq 0.5$ then the fuzzy edge is called strong otherwise weak. We define the intersecting value at the point P by $I_p = \frac{I_{(a,b)}+I_{(c,d)}}{2}$.*

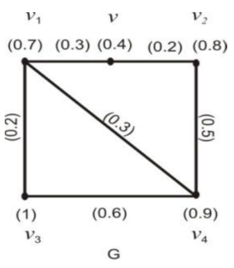
Definition 1.6 (Planarity value of fuzzy graph [6])**.** *Let ψ be a fuzzy multi graph and for a certain geometrical representation $p_1, p_2, \ldots, p_z$ be the points of intersection between the edges. ψ is said to be fuzzy planar graph with fuzzy planarity value f where $f = \frac{1}{1+\{I_{p_1}+I_{p_2}+I_{p_3}+\cdots+I_{p_z}\}}$.*

Definition 1.7 (Strong fuzzy planar graph [6])**.** *A fuzzy planar graph ψ is called strong fuzzy planar graph if the fuzzy planar value of the graph is greater than 0.5.*

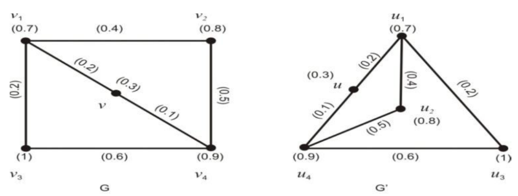
Definition 1.8 (Homomorphism of fuzzy graphs [5])**.** *A homomorphism of fuzzy graphs $h : G \to G'$ is a map $h : S \to S'$ satisfying $\sigma(x) \leq \sigma'(h(x)) \ \forall \ x \in S$ and $\mu(x,y) \leq \mu'(h(x),h(y)) \ \forall \ x,y \in S$.*

Definition 1.9 (Isomorphism of fuzzy graph [5])**.** *An isomorphism $h : G \to G'$ is a map $h : S \to S'$ which is bijective that satisfies$\sigma(x) = \sigma'(h(x)) \ \forall \ x \in S$, $\mu(x,y) = \mu'(h(x),h(y)) \ \forall \ x,y \in S$. If such a isomorphism from G to G' exists, then G is said to be isomorphic to G' and denoted as $G \cong G'$.*

Definition 1.10 (Subdivisions of fuzzy graph)**.** *Sub divide is an operation on fuzzy topological graph. The operation on a single edge $\mu(v_i, v_j)$ of a fuzzy topological graph G with end point set $\mu(v_1, v_2) = \{\sigma(v_1), \sigma(v_2)\}$ one subdivides the edge $\mu(v_1, v_2)$ into two new edges $\mu(v_1, v)$ and $\mu(v, v_2)$ by putting a new vertex $\sigma(v)$ anywhere in its interior. Therefore $\sigma(v) = \mu(v_1, v_2)$ and $\mu(v_1, v) \leq \{\sigma(v_1), \sigma(v)\}, \mu(v, v_2) \leq \{\sigma(v), \sigma(v_2)\}$.*



Definition 1.11 (Homeomorphism of fuzzy graph)**.** *In fuzzy topological graph theory two fuzzy graphs G and G' are homeomorphic if there is a fuzzy graph isomorphism from some subdivision of G to some subdivision of G'. If the edges of a fuzzy graph are regarded as lines drawn from one vertex to another then two fuzzy topological graphs are homeomorphic to each other.*



2. Subdivision and Homeomorphism of Fuzzy Graph

In this section planar, minimal nonplanar regular and complete fuzzy topological graph is given.Some of its properties on subdivision and homeomorphism of regular and irregular fuzzy topological graph is studied.

Theorem 2.1. *A fuzzy topological graph G is non planar, if and only if it contains a fuzzy topological sub graph that is homeomorphic from K_5 or $K_{3,3}$.*

Proof. Let G be fuzzy topological graph and H be a non empty fuzzy topological sub graph of G. Which is homeomorphic from K_5 or $K_{3,3}$. Therefore, H is isomorphic to subdivision of G. One edge of G can be subdivided into two edges by adding new vertex in its interior. Which implies G is non-planar. Assume that G is non planar. G cannot be drawn with crossing over of edges. There exist a sub division G' of G. Which implies G' is a sub graph of G which is homeomorphic from K_5 or $K_{3,3}$. □

Theorem 2.2. *The fuzzy topological graph G is planar if any only if it does not contain a fuzzy topological sub graph which is homeomorphic from K_5 or $K_{3,3}$.*

Proof. Let G be a fuzzy planar graph. Assume on the contrary that G contains a fuzzy topological sub graph H which is homeomorphic from K_5 or $K_{3,3}$. By Theorem 2.1, G is non planar. Hence, G does not contains a sub graph which is homeomorphic from K_5 or $K_{3,3}$.

Conversely Assume that, G does not contains a sub graph which is homeomorphic from K_5 or $K_{3,3}$. Hence, G does not contain any sub division. Therefore G is planar. □

Theorem 2.3. *Every minimal non planar fuzzy topological graph G is connected.*

Proof. Let G be a minimal non planar fuzzy topological graph. In order to prove G is connected. It is enough to prove $\mu^{\infty}(u,v) > 0$. For $\sigma(u_i), \sigma(v_j) \in G$ for the strength of the connectedness between $\sigma(u_i)$ and $\sigma(v_j)$ is

$$\mu^{\infty}(u_1, v_j) = \sup\{\mu^{n}(u_i, v_j)/n = 1,2,3,...\}$$
$$\mu^{\infty}(u_1, v_j) = \sup\{\mu^{n}(u_i, v_j)/n = 1,2,3,...\}$$

Therefore $\mu^{\infty}(u_i, v_j) > 0$. □

Theorem 2.4. *Every region of a fuzzy topological graph imbedding of a fuzzy topological graph G has a simple cycle for its boundary if and only if G is connected.*

Proof. Suppose that the boundary of some region m is not a cycle. Then there is a simple closed path in the plane that leaves from edge $\mu(u_i, v_j)$ between two vertices of the boundary of m, this closed path separates the plane into two pieces, both of which contains parts of the fuzzy topological graph G. It follows that $\mu(u_i, v_j)$ is a cut node of the fuzzy topological graph G. This removal of edge $\mu(u_i, v_j)$ reduces the strength of the connectedness of a fuzzy topological graph G. Which is a contradiction to the hypothesis. Therefore G is connected.

Conversely, Suppose that G has a cut node. Then G may be viewed as the amalgamation of two fuzzy topological graphs A and B at the edge $\mu(u_i, v_j)$ in any imbedding of G. Consider the region m with the corner at an edge $\mu(u_i, v_j)$ between the vertices $\sigma(u_i), \sigma(v_j)$. Since the vertex $\sigma(u_i)$ lies in fuzzy topological sub graph A and since $\mu(u_i, v_j)$ is a cut node no vertices of sub graph B are encountered until the boundary returns to $\mu(u_i, v_j)$. Since the vertex $\sigma(v_j)$ from B must occur somewhere on that boundary traversal, and since that closed walk must return to edge $\mu(u_i, v_j)$ the boundary of region m is not a simple cycle which is a contradiction. Therefore every region of a planar imbedding of a fuzzy graph G has a simple cycle for its boundary. □

Theorem 2.5. *If G is a connected fuzzy topological graph containing no homeomorph of K_5 or $K_{3,3}$. Then G is planar.*

Proof. Given that if G is a connected fuzzy topological graph containing no homeomorph of K_5 or $K_{3,3}$. Assume that G is a connected fuzzy topological graph containing homeomorph of K_5 or $K_{3,3}$. By Theorem 2.1, G is a non planar fuzzy topological graph if it contains a homeomorph of K_5 or $K_{3,3}$. Therefore, G is planar. □

Theorem 2.6. *Any planar connected fuzzy topological graph has planar imbedding such that every bounded region is convex.*

Proof. Let G be a connected planar fuzzy topological graph. Hence, G has a region of a planar imbedding.
To imagine polygon whose boundaries are cycles of the imbedded graph without any repeated vertices. Let us consider the boundary $\mu(v_i, v_j)\mu(v_j, v_k)\mu(v_k, v_i)$. The removal any edge from this boundary F_I is not a cycle. Which reduces the strength of the connectedness. Hence, the boundary of this region is a simple cycle and the unbound region is not convex. Therefore, the bounded region is convex. □

Theorem 2.7. *Let G be a fuzzy topological graph containing no homeomorph of K_5 or $K_{3,3}$. And if the addition of any edge of G creates a homeomorph then G is connected.*

Proof. Let G be a fuzzy topological graph containing no homeomorph of K_5 or $K_{3,3}$. Let $\mu(u_i, v_j)$ be an edge. The addition of G and an edge creates a new graph K homeomorphic to K_5 or $K_{3,3}$. Therefore, K is planar fuzzy topological graph and it is connected. This implies, G is also connected. This proof is by induction on the number m of vertices of G. Observe that the theorem is true for $m = 5$. Since $K_5 - \mu(u_i, v_j)$ is connected. Now assume that the theorem is true for all fuzzy topological graphs with fewer than m vertices. Where $m > 5$ assume that the fuzzy topological graph G has m vertices. The amalgamation of fuzzy topological graph G and edge $\mu(u_i, v_j)$ at two vertices $\sigma(u_i)$ and $\sigma(v_j)$. We that the vertices $\sigma(u_i)$ and $\sigma(v_j)$ and are adjacent in the fuzzy topological graph G. Then the fuzzy topological graph H obtained from G by adding an edge $\mu(u_i, v_j)$ from $\sigma(u_i) to \sigma(v_j)$. Also the addition of any edge would create a homeograph of K_5 or $K_{3,3}$ in original fuzzy topological graph G. Since the homeomorph cannot be disconnected by the addition of an edge. Therefore, H is connected and planar and G is also connected. □

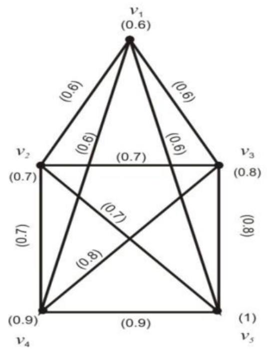
Theorem 2.8. *Let G be an imbedding of a connected simplicial fuzzy topological graph with at least three vertices into a plane then $\sum \mu_i \geq \sum F_i$.*

Proof. Let G be a fuzzy connected planar simplicical fuzzy topological graph with three vertices. The fuzzy topological graph G has two regions one is the bounded region closed with the edges $\{\mu(u_i, u_j), \mu(u_j, u_k), \mu(u_j, u_k)\}$ and other one is unbounded region. Now we calculate the face value of two regions using the membership value of the edges $\{\mu(u_i, u_j), \mu(u_j, u_k), \mu(u_j, u_k)\}$ by $\min\{\frac{(x,y)_{\mu j}}{\min\{\sigma(x),\sigma(y)\}},\ j = 1, 2, 3, ..., p_{xy}/(x, y) \in E/\}$ and adding these face values $F_1 + F_2$. Hence the sum of the member ship value of all the edges bounded by the region is greater than or equal to the sum of the face values. (i.e). $\sum \mu_i \geq \sum F_i$. □

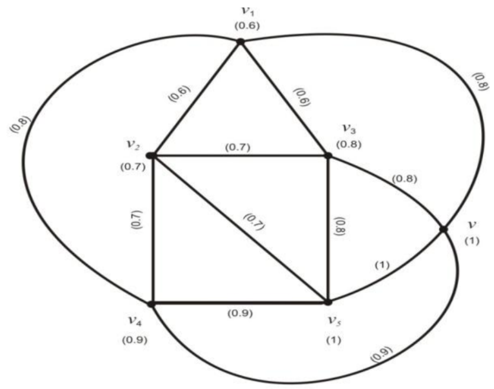
Example 2.9. *Strong fuzzy planar topological graph homeomorphic from K_5 or $K_{3,3}$.*

Solution. Let G be a fuzzy topological graph and let G_1 and G_2 are two non empty fuzzy topological sub graph homeomorphic form K_5 or $K_{3,3}$ let the membership value be

$$\mu(u_i, v_j) = \min(\sigma(u_i), \sigma(v_j))$$



K5



H

from the above figure two edges v_1v_5 and v_3v_4 intersect at a vertex v

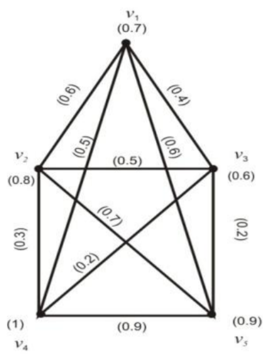
$$I_{(v_1,v_5)} = 1; \quad I_{(v_3,v_4)} = 1; \quad I_p = \frac{I_{(a,b)} + I_{(c,d)}}{2} = 1$$

$\sigma(v) = 1$, $\sigma(v_1, v) = 0.6$, $\sigma(v, v_4) = 0.9$, $\sigma(v_3, v) = 0.8$, $\sigma(v, v_5) = 1$. Membership value of the fuzzy face bounded region, $F_1 = 1$. Similarly, the member ship value of fuzzy face for all the region is 1. Hence, Fuzzy planarity value is $f = 1$ (i.e.) It is a fuzzy topological graph with fuzzy planarity value $1 \geq 0.5$. Hence it is a strong fuzzy planar graph homeomorphic from K_5 or $K_{3,3}$. □

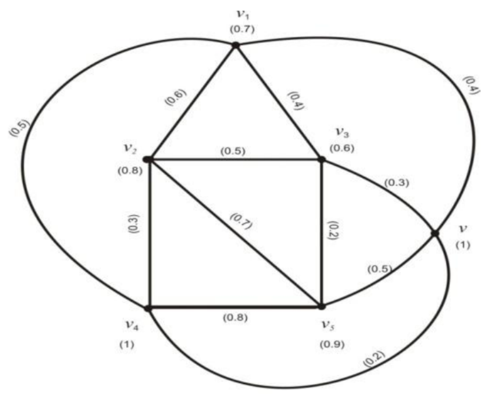
Example 2.10. *Weak fuzzy planar topological graph homeomorphic from K_5 or $K_{3,3}$.*

Solution. Let G be a fuzzy topological graph and let G_1 and G_2 are two non empty fuzzy topological sub graph homeomorphic form K_5 or $K_{3,3}$ let the membership value be

$$\mu(u_i, v_j) \leq (\sigma(u_i), \sigma(v_j))$$



K5



H

from the above figure, two edges v_1v_5 and v_3v_4 intersect at a vertex v. Intersecting value of fuzzy planar graph is $I_{(v_1,v_5)} = 0.8571$, $I_{(v_3,v_4)} = 0.333$, $I_p = 0.595$; $\sigma(v) = 0.595$, $\sigma(v_1, v) = 0.4$, $\sigma(v, v_4) = 0.2$, $\sigma(v_3, v) = 0.3$, $\sigma(v, v_4) = 0.5$. Membership value of the fuzzy face. Here, F_1, F_2, F_3, F_4, F_5, F_6, F_7 are bounded regions.

$$F_1(v_1, v_2, v_3) = 0.6667; \; F_2(v_2, v_3, v_5) = 0.333; \; F_3(v_2, v_4, v_5) = 0.375;$$
$$F_4(v_1, v_2, v_4) = 0.375; \; F_5(v_1, v_3, v) = 0.504; \; F_6(v, v_4, v_5) = 0.3361;$$
$$F_7(v, v_3, v_5) = 0.333.$$

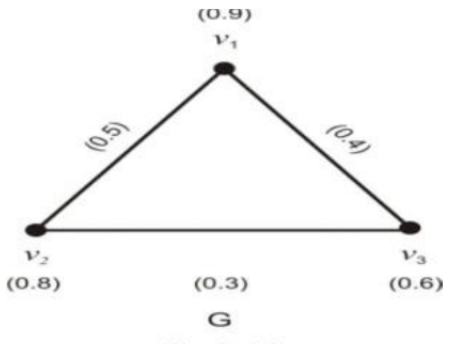
F_8 is not a bounded region.

$$F_8(v_1, v_4, v) = 0.3361$$

Hence, Fuzzy planarity value is $f = 0.2347$ (i.e.) It is a fuzzy topological graph with fuzzy planarity value $0.2347 \le 0.5$. Hence it is a weak fuzzy planar graph homeomorphic from K_5 or $K_{3,3}$. □

Example 2.11. *Finding the strength of the connectedness of a minimal non planar fuzzy topological graph.*

Solution.



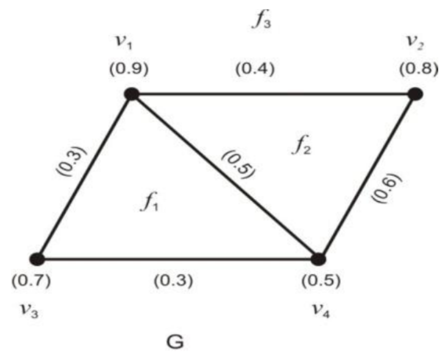
Let G be a fuzzy topological graph. Let the vertex set of g be $\{v_1, v_2, v_3\}$. The strength of the connectedness between the edges is. From the above figure,

$$\mu^{\infty}(u_1, v_j) = \sup\{\mu^n(u_i, v_j)/n = 1, 2, 3, ..\}$$
$$\mu(v_1, v_2) = \sup\{\mu(v_1, v_2) \wedge \mu(v_{1,}, v_3, v_2)\} = 0.5$$
$$\mu(v_1, v_3) = \sup\{\mu(v_1, v_3) \wedge \mu(v_{1,}, v_2, v_3)\} = 0.4$$
$$\mu(v_2, v_3) = \sup\{\mu(v_2, v_3) \wedge \mu(v_{2,}, v_1, v_3)\} = 0.3$$
$$\mu^{\infty}(u, v) = \sup\{\mu(v_1, v_2), \mu(v_2, v_3), \mu(v_3, v_1)\} = 0.5$$
$$\mu^{\infty}(u, v) = 0.5 > 0$$

Therefore, G is connected minimal non planar fuzzy graph. □ .

Example 2.12. *If a fuzzy topological graph has a planar imbedding then it is connected.*

Solution.



Let G be a fuzzy graph with vertex set $v = \{v_{1,} v_{2,} v_{3,} v_{4,}\}$. Given that G is planar then it has three regions F_1, F_2 and F_3. F_1 is bounded by the region with the vertices $\{v_1, v_3, v_4\}$. F_2 is bounded by the region with vertices $\{v_1, v_2, v_4\}$. And F_3 is an unbounded region. Here F_1, F_2 are simple cycles and it is convex. Now we find the strength of the connectedness of G.

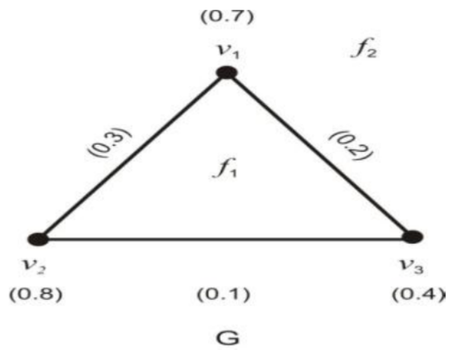
$$\mu(v_1, v_2) = 0.3;\ \mu(v_1, v_3) = 0.3;\ \mu(v_1, v_4) = 0.5;\ \mu(v_2, v_3) = 0.3;$$
$$\mu(v_2, v_4) = 0.6;\ \mu(v_3, v_4) = 0.3$$

$$\mu^{\infty}(u, v) = 0.6 > 0$$

□

Example 2.13. *Let G be a connected planar fuzzy topological graph with at least three vertices then $\sum \mu_i \geq \sum F_i$.*

Solution.



Let G a connected fuzzy planar simplicial graph with three vertices. From the above figure we have two regions F_1 and F_2. Here, F_1 is bounded region $F_1(v_1, v_2, v_3) = 0.25$. F_2 is a not a bounded region. $F_2 = 0.25$. $\sum \mu_i = 0.6$; $\sum F_i = 0.50$. Therefore $\sum \mu_i \geq \sum F_i$. □

Observations

- Two regular fuzzy topological graphs G and G' are homeomorphic if there is a fuzzy graph isomorphism from some subdivision of G to some subdivision of G'.
- Two irregular fuzzy topological graphs G and G' are homeomorphic if there is a fuzzy graph isomorphism from some subdivision of G to some subdivision of G'.
- Any planar connected fuzzy topological graph has a planar imbedding such that every edge $\mu(u_i, v_j)$ is a straight line segment.
- A minimal non planar fuzzy topological complete graph is a non planar fuzzy topological graph such that every proper fuzzy topological sub graph is planar.
- For any two positive integer m and n the m-cycle C_m and the n-cycle C_n are homeomorphic fuzzy topological graphs.
- For any two integers $m, n \geq 2$ then m-path P_m and the n-path P_n in are homeomorphic fuzzy topological graphs.

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