International Journal of Mathematics And its Applications

Volume 5, Issue 4–F (2017), 923–938.

ISSN: 2347-1557

Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

# Fixed Point Theorems for Generalized $(\psi, \phi)$-contractive Mappings in a Complete Strong Fuzzy Metric Space

**Research Article**

**K.K.M. Sarma[1] and Yohannes Gebru[1]***

1 Department of Mathematics, Andhra University, Viskapatnam, (A.P), India.

**Abstract:** In this paper, we introduce generalized $(\psi, \phi)$-contractive mapping in strong fuzzy metric spaces and prove fixed point theorems to this class of maps. Further we introduce a generalized $(\psi, \phi)$-contractive mapping $f$ with respect a mapping $g$ and prove common fixed point theorems. We provide examples in support of our results.

**MSC:** Primary 47H10; Secondary 54H25.

**Keywords:** Strong fuzzy metric space, generalized $(\psi, \phi)-$contractive mappings, generalized $(\psi, \phi)-$ contractive pair of mappings.



## 1. Introduction

The concept of fuzzy metric space was introduced in different ways by various authors (see [5, 9]) and the fixed point theory in these spaces has been intensively studied . The notion of fuzzy metric space, introduced by Kramosil and Michálek [9] was modified by George and Veeramani [3] and that obtained a Hausdorff topology for this section of fuzzy metric spaces. Gregori and Sapena [5] have introduced a kind of contractive mappings in fuzzy metric spaces in the sense of George and Veeramani and proved a fuzzy Banach contraction theorem using a strong condition for completeness, which is Completeness in the sense of Grabiec, or G-completeness. Subsequently, deeper and significant research in fuzzy metric spaces was undertaken by various researchers (see [1, 14, 15]). In 2010, Gregori et al. [11] introduced Strong fuzzy metric space and proved a fixed point theorem. Motivated from Azizollah et al. [2], we have developed this paper. In this paper, we first introduce generalized contractive conditions of maps and also prove some fixed point theorems for generalized $(\psi, \phi)$-contractive mapping in strong fuzzy metric spaces.

## 2. Preliminaries

We begin with some basic definitions and results which will be used in the main part of our paper.

**Definition 2.1** ([16])**.** *A binary operation* $* : [0, 1] \times [0, 1] \to [0, 1]$ *is said to be a continuous t-norm if it satisfies the following conditions :*

*(T1)* $*$ *is associative and commutative,*

* *E-mail: yohannesgebru2005@gmail.com*

*(T2)* $*$ *is continuous,*

*(T3)* $a * 1 = a$ *for all* $a \in [0, 1]$,

*(T4)* $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0, 1]$.

**Remark 2.2.** *A t-norm* $*$ *is called positive, if* $a * b > 0$ *for all* $a, b \in (0, 1)$.

The Lukasievicz t-norm, i.e, $a *_L b = \max\{a + b - 1, 0\}$, product t-norm, i.e, $a * b = ab$ and minimum t-norm, *i.e.,* $a *_M b = \min\{a, b\}$, for $a, b \in [0, 1]$ are some examples of t-norms. The concept of fuzzy metric space as defined by George and Veeramani [3] is as follows.

**Definition 2.3** ([3])**.** *Let* $X$ *be a nonempty set,* $*$ *be a continuous t-norm. Assume that a fuzzy set* $M : X \times X \times (0, \infty) \to [0, 1]$ *satisfies the following conditions, for each* $x, y, z \in X$ *and* $t, s > 0$,

*(M1)* $M(x, y, t) > 0$,

*(M2)* $M(x, y, t) = 1$ *if and only if* $x = y$,

*(M3)* $M(x, y, t) = M(y, x, t)$,

*(M4)* $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

*(M5)* $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ *is continuous,*

*then we call* $M$ *a fuzzy metric on* $X$, *and we call the 3-tuple* $(X, M, *)$ *a fuzzy metric space.*

**Definition 2.4** ([6])**.** *Let* $(X, M, *)$ *be a fuzzy metric space. The fuzzy metric* $M$ *is said to be strong (non-Archimedean) if it satisfies*

$$(M4^{'}) : M(x, z, t) \geq M(x, y, t) * M(y, z, t), \quad \text{for each } x, y, z \in X \text{ and each } t > 0.$$

**Remark 2.5.** *Axiom* $(M4^{'})$ *can not replace axiom* $(M4)$ *in the above definition of fuzzy metric, since in that case,* $M$ *could not be a fuzzy metric on* $X$ *(See Example 8 in [13]).*

Note that it is possible to define a strong fuzzy metric by replacing $(M4)$ by $(M4^{'})$ and demanding in $(M5)$ that the function $M(x, y, \cdot)$ be an increasing continuous function on $t$, for each $x, y \in X$. (In fact, in such a case we have that $M(x, z, t + s) \geq M(x, y, t + s) * M(y, z, t + s) \geq M(x, y, t) * M(y, z, s)$).

**Remark 2.6.** *Not every fuzzy metric space is a strong fuzzy metric space.*

The following example shows that there exists non -strong fuzzy metric spaces.

**Example 2.7** ([8])**.** *Let* $X = \{x, y, z\}$, $* = \cdot$ *and* $M : X \times X \times (0, \infty) \to [0, 1]$ *be defined for each* $t > 0$ *as* $M(x, x, t) = M(y, y, t) = M(z, z, t) = 1$, $M(x, z, t) = M(z, x, t) = M(y, z, t) = M(z, y, t) = \frac{t}{t+1}$, $M(x, y, t) = M(y, x, t) = \frac{t^2}{(t+2)^2}$. *Then* $(X, M, *)$ *is non-strong fuzzy metric space.*

**Lemma 2.8** ([4])**.** *Let* $(X, M, *)$ *be a fuzzy metric space. For all* $x, y \in X$, $M(x, y, \cdot)$ *is a non-decreasing function on* $(0, \infty)$.

**Remark 2.9.** *We observe that* $0 < M(x, y, t) < 1$, *provided* $x \neq y$, *for all* $t > 0$ *(see [10]). Let* $(X, M, *)$ *be a fuzzy metric space. For* $t > 0$, *the open ball* $B(x, r, t)$ *with a center* $x \in X$ *and radius* $0 < r < 1$ *is defined by* $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$. *A subset* $A \subset X$ *is called open, if for each* $x \in A$, *there exists* $t > 0$ *and* $0 < r < 1$ *such that* $B(x, r, t) \subset A$. *Let* $\tau$ *denote the family of all open subsets of* $X$. *Then* $\tau$ *is a topology on* $X$, *called the topology induced by the fuzzy metric* $M$. *This topology is metrizable (see [7]).*

**Definition 2.10** ([3]). *Let $(X, M, *)$ be a fuzzy metric space.*

1. *A sequence$\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.*
2. *A sequence$\{x_n\}$ in $X$ is called a Cauchy sequence if, for each $0 < \epsilon < 1$ and $t > 0$, there exits $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for each $n, m \geq n_0$.*
3. *A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*
4. *A fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.*

**Remark 2.11.** *In a fuzzy metric space the limit of a convergent sequence is unique.*

**Definition 2.12** ([17]). *Let $(X, M, *)$ be a fuzzy metric space. Then the mapping $M$ is said to be continuous on $X \times X \times (0, \infty)$ if*

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t),$$

*when $\{(x_n, y_n, t_n)\}$ is a sequence in $X \times X \times (0, \infty)$ which converges to a point $(x, y, t) \in X \times X \times (0, \infty)$, i.e.,*

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

**Lemma 2.13** ([12]). *If $(X, M, *)$ is a fuzzy metric space, then $M$ is a continuous function on $X \times X \times (0, \infty)$.*

In section 3, we prove the existence of fixed points to generalized $(\psi, \phi)-$ contractive mappings of a complete strong fuzzy metric space.

## 3. Main results

We begin our main results with the following definition

**Definition 3.1.** *Let $\psi : (0, 1] \to [1, \infty)$ be a function which satisfies the following conditions.*

*(1). $\psi$ is continuous and non-increasing, and*

*(2). $\psi(x) = 1$ if and only if $x = 1$.*

*We denote by $\Psi$ the class of all functions which satisfies the above conditions. Note that $\Psi \neq \emptyset$, in fact the map $\psi : (0, 1] \to [1, \infty)$ defined by $\psi(t) = \frac{1}{t}$ is in $\Psi$.*

**Definition 3.2.** *Let $\phi : (0, 1] \times (0, 1] \to (0, 1]$ be a function which satisfies the following conditions.*

*(1). $\phi$ is upper semi continuous and non-decreasing, and*

*(2). $\phi(s, t) = 1$ if and only if $s = t = 1$.*

*We denote by $\Phi$ the class of all functions which satisfies the above conditions. Note that $\Phi \neq \emptyset$, in fact the map $\phi : (0, 1] \times (0, 1] \to (0, 1]$ defined by $\phi(s, t) = st$ is in $\Phi$.*

Now, we introduce generalized $(\psi, \phi)$ contractive mapping in fuzzy metric space.

**Definition 3.3.** *Let $(X, M, *)$ be a fuzzy metric space. We say that a mapping $T : X \to X$ is a generalized $(\psi, \phi)$- contractive mapping if there exists $(\psi, \phi) \in \Psi \times \Phi$ such that,*

$$\psi(M(Tx, Ty, t)) \leq \psi(N(x, y, t))\phi(N^{'}(x, y, t), N^{''}(x, y, t)), \tag{1}$$

*for all $x, y \in X$, and for all $t > 0$, where*

$$N(x, y, t) = \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\},$$
$$N^{'}(x, y, t) = \min\{M(x, y, t), M(x, Tx, t), M(x, Ty, t)\},$$
$$N^{''}(x, y, t) = \min\{M(x, y, t), M(y, Ty, t), M(y, Tx, t)\}.$$

**Definition 3.4.** *Let $(X, M, *)$ be a fuzzy metric space and let $f, g$ be two self mappings on $X$. A mapping $f$ is said to be generalized $(\psi, \phi)$- contractive with respect to $g$ if there exist $(\psi, \phi) \in \Psi \times \Phi$ such that,*

$$\psi(M(fx, gy, t)) \leq \psi(N(x, y, t))\phi(N^{'}(x, y, t), N^{''}(x, y, t)), \tag{2}$$

*for all $x, y \in X$, and for all $t > 0$, where*

$$N(x, y, t) = \min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\},$$
$$N^{'}(x, y, t) = \min\{M(x, y, t), M(x, fx, t), M(x, gy, t)\},$$
$$N^{''}(x, y, t) = \min\{M(x, y, t), M(y, Ty, t), M(y, Tx, t)\}.$$

The following propositions are useful to prove our main results.

**Proposition 3.5.** *Let $(X, M, *)$ be a strong fuzzy metric space. Let $T : X \to X$ be a generalized $(\psi, \phi)$- contractive mapping. Fix $x_0 \in X$. Define a sequence $\{x_n\}$ in $X$ by $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \cdots$. If $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ for all $t > 0$ then $\{x_n\}$ is a Cauchy sequence.*

*Proof.* Since the mapping $T$ is generalized $(\psi, \phi)$- contractive there exists $(\psi, \phi) \in \Psi \times \Phi$ such that

$$\psi(M(Tx, Ty, t)) \leq \psi(N(x, y, t))\phi(N^{'}(x, y, t), N^{''}(x, y, t)) \quad \forall x, y \in X.$$

Suppose that sequence $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon \in (0, 1)$ and $t_0 > 0$ such that for all $k \geq 1$, there are positive integers $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and

$$M(x_{n(k)}, x_{m(k)}, t_0) \leq 1 - \epsilon. \tag{3}$$

We assume that $m(k)$ is the least integer exceeding $n(k)$ and satisfying the above inequality, that is equivalently,

$$M(x_{n(k)}, x_{m(k)-1}, t_0) > 1 - \epsilon \text{ and } M(x_{n(k)}, x_{m(k)}, t_0) \leq 1 - \epsilon.$$

Now, we have

$$\begin{aligned} 1 - \epsilon \geq M(x_{n(k)}, x_{m(k)}, t_0) &\geq M(x_{n(k)}, x_{m(k)-1}, t_0) * M(x_{m(k)-1}, x_{m(k)}, t_0) \\ &> (1 - \epsilon) * M(x_{m(k)-1}, x_{m(k)}, t_0). \end{aligned}$$

$\lim_{k\to\infty} (1 - \epsilon) * M(x_{m(k)-1}, x_{m(k)}, t_0) = 1 - \epsilon$. It follows that $\lim_{k\to\infty} M(x_{n(k)}, x_{m(k)}, t_0)$ exists and equal to $1 - \epsilon$. First we prove that (i). $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t_0) = 1-\epsilon$,

(ii). $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0) = 1-\epsilon$,

(iii). $\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0) = 1-\epsilon$.

We have

$$M(x_{m(k)}, x_{n(k)}, t_0) \geq M(x_{m(k)}, x_{m(k)-1}, t_0) * M(x_{m(k)-1}, x_{n(k)-1}, t_0) * M(x_{n(k)-1}, x_{n(k)}, t_0), \tag{4}$$

$$M(x_{m(k)-1}, x_{n(k)-1}, t_0) \geq M(x_{m(k)-1}, x_{m(k)}, t_0) * M(x_{m(k)}, x_{n(k)}, t_0) * M(x_{n(k)}, x_{n(k)-1}, t_0). \tag{5}$$

Taking limit superior in (4) and limit inferior in (5) we get,

$$1-\epsilon \geq \limsup_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t_0) \tag{6}$$

and

$$\liminf_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t_0) \geq 1-\epsilon. \tag{7}$$

Since limit superior is always greater than or equal to limit inferior, from (6) and (7), we obtain

$$\limsup_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t_0) = 1-\epsilon$$

and

$$\liminf_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t_0) = 1-\epsilon.$$

Thus, $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t_0)$ exists and equal to $1-\epsilon$. Thus (i) holds.
We now prove (ii). By condition $(M4^{'})$ of strong fuzzy metric space, we have

$$M(x_{m(k)-1}, x_{n(k)}, t_0) \geq M(x_{m(k)-1}, x_{m(k)}, t_0) * M(x_{m(k)}, x_{n(k)}, t_0), \tag{8}$$

and

$$M(x_{m(k)}, x_{n(k)}, t_0) \geq M(x_{m(k)}, x_{m(k)-1}, t_0) * M(x_{m(k)-1}, x_{n(k)}, t_0). \tag{9}$$

Taking limit inferior in (8) and limit superior in (9) as $n \to \infty$, we have

$$\liminf_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0) \geq 1-\epsilon,$$

and

$$1-\epsilon \geq \limsup_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0).$$

This implies that

$$1-\epsilon \geq \limsup_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0) \geq \liminf_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0) \geq 1-\epsilon.$$

Thus,

$$\limsup_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0) = \liminf_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0) = 1-\epsilon.$$

Hence $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0)$ exists and $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, t_0) = 1-\epsilon$. Thus (ii) holds.

We now prove (iii). By Condition $(M^{'}4)$ in a strong fuzzy metric space, we have

$$M(x_{n(k)-1}, x_{m(k)}, t_0) \geq M(x_{n(k)-1}, x_{n(k)}, t_0) * M(x_{n(k)}, x_{m(k)}, t_0), \tag{10}$$

and

$$M(x_{n(k)}, x_{m(k)}, t_0) \geq M(x_{n(k)}, x_{n(k)-1}, t_0) * M(x_{n(k)-1}, x_{m(k)}, t_0). \tag{11}$$

Taking limit inferior in (10) and limit superior in (11) as $n \to \infty$, we obtain

$$\liminf_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0) \geq 1 - \epsilon,$$

and

$$1 - \epsilon \geq \limsup_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0).$$

This implies that

$$1 - \epsilon \geq \limsup_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0) \geq \liminf_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0) \geq 1 - \epsilon.$$

Thus,

$$\limsup_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0) = \liminf_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0) = 1 - \epsilon.$$

Hence $\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0)$ exists and $\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)}, t_0) = 1 - \epsilon$. Hence (iii) holds. Now, from the inequality (1), we have

$$\begin{aligned}\psi(M(x_{m(k)}, x_{n(k)}, t_0)) &= \psi(M(Tx_{m(k)-1}, Tx_{n(k)-1}, t_0)) \\ &\leq \psi(N(x_{m(k)-1}, x_{n(k)-1}, t_0))\phi(N^{'}(x_{m(k)-1}, x_{n(k)-1}, t_0), N^{''}(x_{m(k)-1}, x_{n(k)-1)}, t_0))\end{aligned}$$

where

$$N(x_{m(k)-1}, x_{n(k)-1}, t_0) = \min\{M(x_{m(k)-1}, x_{n(k)-1}, t_0), M(x_{m(k)-1}, x_{m(k)}, t_0), M(x_{n(k)-1}, x_{n(k)}, t_0)\},$$

$$N^{'}(x_{m(k)-1}, x_{n(k)-1}, t_0) = \min\{M(x_{m(k)-1}, x_{n(k)-1}, t_0), M(x_{m(k)-1}, x_{m(k)}, t_0), M(x_{m(k)-1}, x_{m(k)}, t_0)\},$$

$$N^{''}(x_{m(k)-1}, x_{n(k)-1}, t_0) = \min\{M(x_{m(k)-1}, x_{n(k)-1}, t_0), M(x_{n(k)-1}, x_{n(k)}, t_0), M(x_{n(k)-1}, x_{m(k)}, t_0)\}.$$

Hence, it follows that

$$\lim_{k\to\infty} N(x_{m(k)-1}, x_{n(k)-1}, t_0) = 1 - \epsilon, \tag{12}$$

$$\lim_{k\to\infty} N^{'}(x_{m(k)-1}, x_{n(k)-1}, t_0) = 1 - \epsilon, \tag{13}$$

$$\lim_{k\to\infty} N^{''}(x_{m(k)-1}, x_{n(k)-1}, t_0) = 1 - \epsilon. \tag{14}$$

Since $\psi$ is continuous and $\phi$ is upper semi continuous with respect to both components, by taking limit superior as $k \to \infty$ in (12), and by using (12), (13) and (14), we get

$$\psi(1 - \epsilon) \leq \psi(1 - \epsilon)\phi(1 - \epsilon, 1 - \epsilon).$$

it follows that, $\phi(1 - \epsilon, 1 - \epsilon) = 1$. Hence from the property of $\phi$, we have $\epsilon = 0$, which contradicts that $0 < \epsilon < 1$. Therefore, $\{x_n\}$ is a Cauchy sequence in $X$. □

**Proposition 3.6.** *Let $(X, M, *)$ be a strong fuzzy metric space. Let $f, g$ be two self maps on $X$ and let $f$ be a generalized $(\psi, \phi)$- contractive mapping with respect to $g$. Fix $x_0 \in X$. Define a sequence $\{x_n\}$ in $X$ by $x_{2n} = fx_{2n-1}$ and $x_{2n+1} = gx_{2n}$ for all $n = 0, 1, 2, \cdots$. If $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1, \quad \forall t > 0$. Then $\{x_n\}$ is a Cauchy sequence.*

*Proof.* Suppose that the sequence $\{x_n\}$ is not a Cauchy sequence. Since $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ for all $t > 0$, the sequence $\{x_{2n}\}$ is not Cauchy. Then there exist $\epsilon \in (0, 1)$ and $t_0 > 0$ such that for all $k \geq 1$, there are positive integers $m(k), n(k) \in \mathbb{N}$ with $n(k) > m(k) \geq k$ and

$$M(x_{2n(k)}, x_{2m(k)}, t_0) \leq 1 - \epsilon. \tag{15}$$

We assume that $2n(k)$ is the least positive even integer exceeding $2m(k)$ and satisfying the above inequality, that is equivalently,

$$M(x_{2n(k)}, x_{2m(k)}, t_0) \leq 1 - \epsilon, \quad \textit{and} \quad M(x_{2m(k)}, x_{2n(k)-2}, t_0) > 1 - \epsilon.$$

By condition $(M4')$ in a strong fuzzy metric space, we have

$$\begin{aligned} 1 - \epsilon \geq M(x_{2n(k)}, x_{2m(k)}, t_0) &\geq M(x_{2n(k)}, x_{2n(k)-2}, t_0) * M(x_{2n(k)-2}, x_{2m(k)}, t_0) \\ &\geq M(x_{2n(k)-2}, x_{2n(k)}, t_0) * (1 - \epsilon) \quad \forall k \in \mathbb{N}. \end{aligned} \tag{16}$$

Since $\{M(x_{2n(k)}, x_{2n(k)}, t)\}$ is a sub sequence of $\{M(x_n, x_{n+1}, t)\}$ by taking limit as $k \to \infty$ on both sides of (16) we get ,

$$\lim_{k\to\infty} M(x_{2n(k)}, x_{2m(k)}, t_0) = 1 - \epsilon. \tag{17}$$

From the condition $(M^{'}4)$ of strong fuzzy metric space, we have
$M(x_{2m(k)}, x_{2n(k)}, t_0) \geq M(x_{2m(k)}, x_{2m(k)+1}, t_0) * M(x_{2m(k)+1}, x_{2n(k)}, t_0) \geq M(x_{2m(k)}, x_{2m(k)+1}, t_0) * M(x_{2m(k)+1}, x_{2n(k)+1}, t_0) * M(x_{2n(k)+1}, x_{2n(k)}, t_0) \geq M(x_{2m(k)}, x_{2m(k)+1}, t_0) * M(x_{2m(k)+1}, x_{2m(k)+2}, t_0) * M(x_{2m(k)+2}, x_{2n(k)+1}, t_0) * M(x_{2n(k)+1}, x_{2n(k)}, t_0) \geq M(x_{2m(k)}, x_{2m(k)+1}, t_0) * M(x_{2m(k)+1}, x_{2m(k)+2}, t_0) * M(x_{2m(k)+2}, x_{2n(k)}, t_0) * M(x_{2n(k)}, x_{2n(k)+1}, t_0) * M(x_{2n(k)}, x_{2n(k)+1}, t_0) \geq M(x_{2m(k)}, x_{2m(k)+1}, t_0) * M(x_{2m(k)+1}, x_{2m(k)+2}, t_0) * M(x_{2m(k)+2}, x_{2m(k)+1}, t_0) * M(x_{2m(k)+1}, x_{2m(k)}, t_0) * M(x_{2m(k)}, x_{2n(k)}, t_0) * M(x_{2n(k)}, x_{2n(k)+1}, t_0) * M(x_{2n(k)}, x_{2n(k)+1}, t_0)$. By taking limits on both sides of the above inequality we obtain, $\lim_{k\to\infty} M(x_{2n(k)}, x_{2m(k)+1}, t_0) = 1 - \epsilon$, $\lim_{k\to\infty} M(x_{2m(k)+2}, x_{2n(k)}, t_0) = 1 - \epsilon$ and $\lim_{k\to\infty} M(x_{2m(k)+1}, x_{2n(k)+1}, t_0) = 1 - \epsilon$. Since $f$ is a generalized $(\psi, \phi)$- contractive mapping with respect to $g$ by substituting $x$ with $x_{2m(k)+1}$ and $y$ with $x_{2n(k)}$ in (2), we get that

$$\begin{aligned} \psi(M(x_{2m(k)+2}, x_{2n(k)+1}, t_0)) &= \psi(M(fx_{2m(k)+1}), gx_{2n(k)}, t_0) \\ &\leq \psi(N(x_{2m(k)+1}, x_{2n(k)}, t_0))\phi(N^{'}(x_{2m(k)+1}, x_{2n(k)}, t_0), N^{''}(x_{2m(k)+1}, x_{2n(k)}, t_0)), \end{aligned}$$

where

$$N(x_{2m(k)+1}, x_{2n(k)}, t_0) = \min\{M(x_{2m(k)+1}, x_{2n(k)}, t_0), M(x_{2m(k)+1}, fx_{2m(k)+1}, t_0), M(x_{2n(k)}, gx_{2n(k)}, t_0)\}, \tag{18}$$

$$N^{'}(x_{2m(k)+1}, x_{2n(k)}, t_0) = \min\{M(x_{2m(k)+1}, x_{2n(k)}, t_0), M(x_{2m(k)+1}, fx_{2m(k)+1}, t_0), M(x_{2m(k)}, gx_{2n(k)}, t_0)\} \tag{19}$$

and

$$N^{''}(x_{2m(k)+1}, x_{2n(k)}, t_0) = \min\{M(x_{2m(k)+1}, x_{2n(k)}, t_0), M(x_{2n(k)}, gx_{2n(k)}, t_0), M(x_{2n(k)}, fx_{2m(k)+1}, t_0)\}. \tag{20}$$

Thus, letting as $k \to \infty$ in (18), (19) and (20), it follows that

$$\lim_{k\to\infty} N(x_{2m(k)+1}, x_{2n(k)}, t_0) = 1 - \epsilon, \tag{21}$$

$$\lim_{k\to\infty} N^{'}(x_{2m(k)+1}, x_{2n(k)}, t_0) = 1 - \epsilon, \tag{22}$$

$$\lim_{k\to\infty} N(x_{2m(k)+1}, x_{2n(k)}, t_0) = 1 - \epsilon. \tag{23}$$

On taking limit as $k \to \infty$ in (18) and by using (21), (22) and (23), it follows that

$$\psi(1 - \epsilon) \leq \psi(1 - \epsilon)\phi(1 - \epsilon, 1 - \epsilon). \tag{24}$$

Which implies $\epsilon = 0$, a contradiction. Therefore $\{x_n\}$ is a Cauchy Sequence. □

We now prove our main theorems and draw some corollaries.

**Theorem 3.7.** *Let $(X, M, *)$ be a strong fuzzy metric space and $T : X \to X$ be continuous and generalized $(\psi, \phi)$-contractive mapping. Then $T$ has a unique fixed point.*

*Proof.* Let $x_0 \in X$ be arbitrary element of $X$. We define a sequence $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \cdots$. If there exist $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then $x_0$ is the fixed point of $T$. Assume that $x_n \neq x_{n+1}$, for all $n = 1, 2, 3, \ldots$. Since $T$ is a generalized $(\psi, \phi)$-contractive mapping there exists $(\psi, \phi) \in \Psi \times \Phi$ such that

$$\psi(M(Tx, Ty, t)) \leq \psi(N(x, y, t))\phi(N^{'}(x, y, t), N^{''}(x, y, t)) \text{ for all } x, y \in X \text{ and for each } t > 0.$$

Thus, for $x_{n-1} \neq x_n$ and $t > 0$, we have

$$\psi(M(Tx_{n-1}, Tx_n, t)) \leq \psi(N(x_{n-1}, x_n, t))\phi(N^{'}(x_{n-1}, x_n, t), N^{''}(x_{n-1}, x_n, t)).$$

This implies,

$$\psi(M(x_n, x_{n+1}, t)) \leq \psi(N(x_{n-1}, x_n, t))\phi(N^{'}(x_{n-1}, x_n, t), N^{''}(x_{n-1}, x_n, t)),$$

where

$$N(x_{n-1}, x_n, t) = \min\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\},$$

$$N^{'}(x_{n-1}, x_n, t) = \min\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_{n-1}, x_{n+1}, t)\},$$

$$N^{''}(x_{n-1}, x_n, t) = \min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), M(x_n, x_n, t)\}.$$

Since $\phi(N^{'}(x_{n-1}, x_n, t), N^{''}(x_{n-1}, x_n, t)) < 1$, we conclude that

$$\psi(M(x_n, x_{n+1}, t)) < \psi(\min\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t)\})$$

Again $\psi$ is non-increasing imply that

$$M(x_n, x_{n+1}, t) > \min\{M(x_n, x_{n+1}, t), M(x_{n-1}, x_n, t)\}.$$

This implies,

$$\min\{M(x_n,x_{n+1},t),M(x_{n-1},x_n,t)\}=M(x_{n-1},x_n,t).$$

Thus, $M(x_n,x_{n+1},t)>M(x_{n-1},x_n,t)$. Therefore, for every $t>0$, $\{M(x_n,x_{n+1},t)\}$ is an increasing sequence of real numbers in $(0,1]$. Since every bounded and monotone sequence is convergent, the sequence $\{M(x_n,x_{n+1},t)\}$ converges to some number in $(0,1]$. Let $\lim_{n\to\infty} M(x_n,x_{n+1},t)=l_t$.

**Claim:** $l_t=1,\forall t>0$. We have that $T$ is a generalized contractive mapping, so for all $n\in\mathbb{N}$ and $t>0$

$$\psi(M(Tx_{n-1},Tx_n,t))\le\psi(N(x_{n-1},x_n,t))\phi(N^{'}(x_{n-1},x_n,t),N^{''}(x_{n-1},x_n,t)).$$

Since $\phi$ is non-decreasing with respect to both variables, we get that

$$\psi(M(x_n,x_{n+1},t))\le\psi(N(x_{n-1},x_n,t))\phi(M(x_{n-1},x_n,t_0),\min\{M(x_{n-1},x_n,t_0),M(x_n,x_{n+1},t_0))\}. \tag{25}$$

Taking limit superior as $k\to\infty$ in the inequality (25), the continuity of $\psi$ and the upper semi continuity of $\phi$, shows $\psi(l_t)\le\psi(l_t)\phi(l_t,l_t)$. Which implies $\phi(l_t,l_t)=1$. Hence $l_t=1$. Now by Proposition (3.5) the sequence $\{x_n\}$ is Cauchy. Since $X$ is a complete strong fuzzy metric space there exists $x\in X$ such that $x_n\to x$ as $n\to\infty$. The continuity of $T$ implies that $Tx_n\to Tx$ as $n\to\infty$. Since the limit of a convergent sequence in fuzzy metric space is unique, we have that $Tx=x$. Therefore $x$ is a fixed point of $T$. We show the uniqueness of fixed points of $T$. Let $u$ and $v$ be two fixed points of $T$. Then $Tu=u$ and $Tv=v$. Since $T$ is a generalized $(\psi,\phi)$-contractive map, for $u,v\in X$, and $t>0$ we have

$$\begin{aligned}\psi(M(u,v,t)) &= \psi(M(Tu,Tv,t)) \qquad (26)\\ &\le \psi(N(u,v,t))\phi(N^{'}(u,v,t),N^{''}(u,v,t)),\end{aligned}$$

where

$$\begin{aligned}N(u,v,t) &= \min\{M(u,v,t),M(u,Tu,t),M(v,Tv,t)\}\\ &= \min\{M(u,v,t),1,1\} \qquad (27)\\ &= M(u,v,t),\\ N^{'}(u,v,t) &= \min\{M(u,v,t),M(u,Tu,t),M(u,Tv,t)\}\\ &= \min\{M(u,v,t),1,M(u,v,t)\}=M(u,v,t), \qquad (28)\\ N^{''}(u,v,t) &= \min\{M(u,v,t),M(v,Tv,t),M(v,Tu,t)\}\\ &= \min\{M(u,v,t),1,M(v,u,t)\}=M(u,v,t). \qquad (29)\end{aligned}$$

From (26)-(29) we have observed that

$$\psi(M(u,v,t))\le\psi(M(u,v,t))\phi((M(u,v,t),M(u,v,t))).$$

This implies, $\phi((M(u,v,t),M(u,v,t)))=1$, thus $M(u,v,t)=1$, which implies $u=v$. Therefore, the fixed point of $T$ is unique. □

If we take $\psi(t)=\frac{1}{t}$ and $\phi(s,t)=st$ in Theorem 3.7 we get the following corollary.

**Corollary 3.8.** *Let $(X, M, *)$ be a strong fuzzy metric space and $T$ be a self map of $X$ which satisfies*

$$N(x, y, t) \leq M(Tx, Ty, t)N^{'}(x, y, t)N^{''}(x, y, t),$$

*where*

$$N(x, y, t) = \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\},$$
$$N^{'}(x, y, t) = \min\{M(x, y, t), M(x, Tx, t), M(x, Ty, t)\},$$
$$N^{''}(x, y, t) = \min\{M(x, y, t), M(y, Ty, t), M(y, Tx, t)\}.$$

*Then $T$ has a unique fixed point.*

For the next result we use the following notation: Let $f : X \to X$, $g : X \to X$ be maps, we denote the set of all fixed points of $f$ by $F(f) = \{x \in X | f(x) = x\}$ and the set of all common fixed points of $f$ and $g$ by $F(f, g) = \{x \in X | f(x) = g(x) = x\}$.

**Theorem 3.9.** *Let $(X, M, *)$ be a strong complete fuzzy metric space. Let $f, g : X \to X$ be two mappings and $f$ is generalized $(\psi, \phi)$- contractive mapping with respect $g$ then $F(f) = F(g)$. Further if either $f$ or $g$ is continuous then $f$ and $g$ have a unique common fixed point.*

*Proof.* By our assumption there exists $(\psi, \phi) \in \Psi \times \Phi$ , for all $x, y$ in $X$ and $t > 0$ such that

$$\psi(M(fx, gy, t)) \leq \psi(N(x, y, t))\phi(N^{'}(x, y, t), N^{''}(x, y, t)), \tag{30}$$

where

$$N(x, y, t) = \min\{M(x, y, t), M(x, fx, t), M(y, gy, t)\},$$
$$N^{'}(x, y, t) = \min\{M(x, y, t), M(x, fx, t), M(x, gy, t)\},$$
$$N^{''}(x, y, t) = \min\{M(x, y, t), M(y, gy, t), M(y, fx, t)\}.$$

We now show that $F(f) = F(g) = F(f, g)$. Let $z \in F(f)$, so $fz = z$. Thus, for $x = y = z$, we have

$$\psi(M(z, gz, t)) \leq \psi(N(z, z, t))\phi(N^{'}(z, z, t), N^{''}(z, z, t)) \text{ for all } t > 0, \tag{31}$$

where

$$N(z, z, t) = M(z, gz, t), N^{'}(z, z, t) = M(z, gz, t) \text{ and } N^{''}(z, z, t) = M(z, gz, t).$$

Thus from (31) and the above result we get the following inequality;

$$\psi(M(z, gz, t)) \leq \psi(M(z, gz, t))\phi(M(z, gz, t), M(z, gz, t)), \text{ for all } t > 0. \tag{32}$$

Which yields $\phi(M(z, gz, t), M(z, gz, t)) = 1$, for all $t > 0$. Since $\phi \in \Phi$, we obtain $M(z, gz, t) = 1, \forall t > 0$. Hence $gz = z$, that is $z \in F(g)$. Thus $F(f) \subset F(g)$. Similarly we can show that $F(g) \subset F(f)$. Therefore, we have $F(f, g) = F(f) = F(g)$. Now let $x_0 \in X$, we define a sequence $\{x_n\}$ by $x_1 = x_0$ and

$$x_{2n} = fx_{2n-1}, \quad x_{2n+1} = gx_{2n} \quad \text{for } n = 1, 2, 3, \cdots . \tag{33}$$

If there exist $m \in \mathbb{N}$ such that either $x_{2m} = x_{2m-1}$ or $x_{2m+1} = x_{2m}$ holds then $F(f)$ is nonempty. Since if $x_{2m} = x_{2m-1}$, then $fx_{2m-1} = x_{2m} = x_{2m-1}$, so $x_{2m-1} \in F(f)$. Hence $x_{2m} \in F(g) = F(f)$. Therefore we may suppose that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Suppose $n$ arbitrary. For each $t > 0$, we have

$$\begin{aligned} \psi(M(x_{2n}, x_{2n+1}, t)) &= \psi(M(fx_{2n-1}, gx_{2n}, t)) \\ &\leq \psi(N(x_{2n-1}, x_{2n}, t))\phi(N^{'}(x_{2n-1}, x_{2n}, t), N^{''}(x_{2n-1}, x_{2n}, t), \end{aligned} \tag{34}$$

where

$$N(x_{2n-1}, x_{2n}, t) = \min\{M(x_{2n-1}, x_{2n}, t), M(x_{2n-1}, fx_{2n-1}, t), M(x_{2n}, gx_{2n}, t)\}, \tag{35}$$

$$N^{'}(x_{2n-1}, x_{2n}, t) = \min\{M(x_{2n-1}, x_{2n}, t), M(x_{2n-1}, fx_{2n-1}, t), M(x_{2n-1}, gx_{2n}, t)\}, \tag{36}$$

$$N^{''}(x_{2n-1}, x_{2n}, t) = \min\{M(x_{2n-1}, x_{2n}, t), M(x_{2n}, gx_{2n}, t), M(x_{2n}, fx_{2n-1}, t)\}. \tag{37}$$

From (34), (35), (36) and (37) we have

$$\begin{aligned} \psi(M(x_{2n}, x_{2n+1}, t)) &\leq \psi(\min\{M(x_{2n-1}, x_{2n}, t), M(x_{2n}, x_{2n+1}, t)\}) \\ &\quad \times \phi(\min\{M(x_{2n-1}, x_{2n}, t), M(x_{2n-1}, x_{2n+1}, t)\}, \min\{M(x_{2n-1}, x_{2n}, t), M(x_{2n}, x_{2n+1}, t)\}. \end{aligned}$$

Since $\phi$ is non-decreasing with respect to both components it follows that

$$\begin{aligned} \psi(M(x_{2n}, x_{2n+1}, t)) \leq \psi(\min\{M(x_{2n-1}, x_{2n}, t), M(x_{2n}, x_{2n+1}, t)\})\phi(M(x_{2n-1}, x_{2n}, t), \\ \min\{M(x_{2n-1}, x_{2n}, t), M(x_{2n}, x_{2n+1}, t)\}). \end{aligned} \tag{38}$$

If $\phi(M(x_{2n-1}, x_{2n}, t), \min\{M(x_{2n-1}, x_{2n}, t), M(x_{2n}, x_{2n+1}, t)\}) = 1$ then

$$M(x_{2n-1}, x_{2n}, t) = \min\{M(x_{2n-1}, x_{2n}, t), M(x_{2n}, x_{2n+1}, t)\},$$

which implies $M(x_{2n-1}, x_{2n}, t) \leq M(x_{2n}, x_{2n+1}, t)$. On the hand, if

$$\phi(M(x_{2n-1}, x_{2n}, t), \min\{M(x_{2n-1}, x_{2n}, t), M(x_{2n}, x_{2n+1}, t)\}) < 1$$

then from (38) we have

$$\psi(M(x_{2n}, x_{2n+1}, t)) < \psi(\min\{M(x_{2n-1}, x_{2n}, t), M(x_{2n}, x_{2n+1}, t)\}). \tag{39}$$

Combining (39) with the non-increasing property of $\psi$, we get

$$M(x_{2n}, x_{2n+1}, t) > \min\{M(x_{2n-1}, x_{2n}, t), M(x_{2n}, x_{2n+1}, t)\}.$$

Which implies $\min\{M(x_{2n-1}, x_{2n}, t), M(x_{2n}, x_{2n+1}, t)\} = M(x_{2n-1}, x_{2n}, t)$. Thus, $M(x_{2n}, x_{2n+1}, t) > M(x_{2n-1}, x_{2n}, t)$. Hence $\{M(x_{2n}, x_{2n+1}, t)\}$ is an increasing sequence in $(0, 1]$. Consequently there exist $l_t \in (0, 1]$ such that $\lim_{n\to\infty} M(x_{2n}, x_{2n+1}, t) = l_t$, $\forall t > 0$. We now prove that $l_t = 1$ for all $t > 0$. Let $t > 0$, from (38) we have

$$\psi(M(x_{2n}, x_{2n+1}, t)) = \psi(M(fx_{2n-1}, gx_{2n}, t)) \leq \psi(N(x_{2n-1}, x_{2n}, t))\phi(M(x_{2n-1}, x_{2n}, t), N^{''}(x_{2n-1}, x_{2n}, t)). \tag{40}$$

Since $\psi$ is continuous and $\phi$ is upper semi continuous with respect to both variables on taking limit superior in (40) we get

$$\psi(l_t) \leq \psi(l_t)\phi(l_t, l_t). \tag{41}$$

Which implies $\phi(l_t, l_t) = 1$. By property of $\phi$, $l_t = 1$. Hence by Proposition 2 it follows that $\{x_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is a complete strong fuzzy metric space there exist $u \in X$ such that $x_n \to u$. Without loss of generality we assume that $f$ is Continuous. As $x_{2n-1} \to u$ as $n \to \infty$, the continuity of $f$ implies that $fx_{2n-1} = x_{2n} \to fu$ as $n \to \infty$, by uniqueness of the limit , we obtain $fu = u$. Therefore $u \in F(f) = F(g)$. We will show $u$ is unique. Suppose that $v \in F(f, g) = F(f) = F(g)$. For each $t > 0$, we have

$$\begin{aligned} \psi(M(u, v, t)) &= \psi(M(fu, gv, t)) \\ &\leq \psi(N(u, v, t))\phi(N^{'}(u, v, t), N^{''}(u, v, t)), \end{aligned} \tag{42}$$

where

$$N(u, v, t) = \min\{M(u, v, t), M(u, fu, t), M(v, gv, t)\} = \min\{M(u, v, t), 1, 1\} = M(u, v, t), \tag{43}$$

$$N^{'}(u, v, t) = \min\{M(u, v, t), M(u, fu, t), M(u, gv, t)\} = \min\{M(u, v, t), 1, M(u, v, t)\} = M(u, v, t), \tag{44}$$

$$N^{''}(u, v, t) = \min\{M(u, v, t), M(v, gv, t), M(v, fu, t)\} = \min\{M(u, v, t), 1, M(u, v, t)\} = M(u, v, t). \tag{45}$$

From (42), (43), (44) and (45), we have

$$\psi(M(u, v, t)) = \psi(M(fu, gv, t)) \leq \psi(M(u, v, t))\phi(M(u, v, t), M(u, v, t)).$$

Thus, $\phi(M(u, v, t), M(u, v, t)) = 1$. Which implies $M(u, v, t) = 1$. Therefore $u = v$. □

By taking $\psi(t) = \frac{1}{t}$ and $\phi(s, t) = st$ in Theorem 3.9 we draw the following corollary:

**Corollary 3.10.** *Let $(X, M, *)$ be a strong complete fuzzy metric space. Let $f, g : X \to X$ be two mappings such that for each $x, y \in X$ and $t > 0$*

$$\frac{M(x, y, z)}{M(fx, gy, t)} \leq N^{'}(x, y, t)N^{''}(x, y, t),$$

*where*

$$N^{'}(x, y, t) = \min\{M(x, y, t), M(x, fx, t), M(x, gy, t)\}$$

$$N^{''}(x, y, t) = \min\{M(x, y, t), M(y, gy, t), M(y, fx, t)\}.$$

*Then $F(f) = F(g)$. Further if either $f$ or $g$ is continuous then $f$ and $g$ have a unique common fixed point.*

# 4. Examples

In this section we provide examples in support of the main results of section 3. The following example is in support of Theorem 3.7.

**Example 4.1.** *Let* $X = [0,\infty)$ *and* $M(x,y,t) = (\frac{t}{t+1})^{d(x,y)}$*, where* $d(x,y) = |x-y|$*,* $*$ *be product continuous t-norm. Here* $(X, M, *)$ *is complete strong fuzzy metric space. Let* $T : X \to X$ *be a map defined by*

$$Tx = \begin{cases} \frac{x}{2}, & if\ x \in [0,1) \\ \frac{1}{2}, & if\ x \in [1,\infty) \end{cases}$$

***Claim*** . *T is a generalized* $(\psi,\phi)$*-contractive map for* $\psi(t) = \frac{1}{t^6}$ *and* $\phi(s,t) = \sqrt{st}$*. clearly* $(\psi,\phi) \in \Psi \times \Phi$*. Now we wish to show for all* $x, y$ *in* $X$ *and* $t > 0$

$$\psi(M(Tx,Ty,t)) \le \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)). \tag{46}$$

***Case 1:*** *Let* $x, y \in [0,1], Tx = \frac{x}{2}, Ty = \frac{y}{2}$ *and let* $a = \frac{t}{t+1}$*.*
***Sub case 1;*** *if* $x \ge y$ *then* $M(Tx,Ty,t) = a^{\frac{x-y}{2}}, M(x,y,t) = a^{x-y}, M(x,Tx,t) = a^{\frac{x}{2}}, M(y,Ty,t) = a^{\frac{y}{2}}, M(x,Ty,t) = a^{x-\frac{y}{2}}$ *and*

$$M(y,Tx,t) = \begin{cases} a^{y-\frac{x}{2}}, & if\ y \ge \frac{x}{2} \\ a^{\frac{x}{2}-y}, & if\ y < \frac{x}{2} \end{cases}$$

*When* $y \ge \frac{x}{2}$,

$$N(x,y,t) = \min\{a^{x-y}, a^{\frac{x}{2}}, a^{\frac{y}{2}}\} = a^{\frac{x}{2}},\ \ N^{'}(x,y,t) = \min\{a^{x-y}, a^{\frac{x}{2}}, a^{x-\frac{y}{2}}\} = a^{x-\frac{y}{2}} \quad and$$

$$N^{''}(x,y,t) = \min\{a^{x-y}, a^{\frac{y}{2}}, a^{y-\frac{x}{2}}\} = \begin{cases} a^{x-y}, & if\ x \ge \frac{3y}{2}, \\ a^{\frac{y}{2}}, & if\ x < \frac{3y}{2}. \end{cases}$$

*For* $x \ge \frac{3y}{2}$*, we have* $\psi(M(Tx,Ty,t)) = a^{-3x+3y}, \psi(N(x,y,t)) = a^{-3x}, \phi(N^{'}(x,y,t), N^{''}(x,y,t)) = a^{x-\frac{3y}{4}}$*. Since* $-3x+3y \ge -3x + x - \frac{3y}{4},\ \forall\ y \ge \frac{x}{2}$*, we have*

$$\psi(M(Tx,Ty,t)) \le \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)).$$

*If* $x < \frac{3y}{2}$*, then* $\phi(N^{'}(x,y,t), N^{''}(x,y,t)) = a^{\frac{x}{2}}$*. Again since* $-3x + 3y \ge -3x + \frac{x}{2}, \forall y \ge \frac{x}{2}$*, we have*

$$\psi(M(Tx,Ty,t)) \le \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)).$$

*If* $y < \frac{x}{2}$ *then* $N(x,y,t) = a^{x-y},\ N^{'}(x,y,t) = a^{x-\frac{y}{2}}$*, and* $N^{''}(x,y,t) = a^{x-y}$*. Thus we have*

$$\psi(M(Tx,Ty,t)) = a^{-3x+3y},\ \ \psi(N(x,y,t)) = a^{-6x+6y}\ and\ \phi(N^{'}(x,y,t), N^{''}(x,y,t)) = a^{x-\frac{3y}{4}}.$$

*Here,* $-3x + 3y \ge -6x + 6y + x - \frac{3y}{4}, \forall y < \frac{x}{2}$*, so* $\psi(M(Tx,Ty,t)) \le \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t))$*.*
***Sub case 2:*** *if* $y > x$ *then* $M(Tx,Ty,t) = a^{\frac{y-x}{2}}$*,* $M(x,y,t) = a^{y-x}, M(x,Tx,t) = a^{\frac{x}{2}}$*,* $M(y,Ty,t) = a^{\frac{y}{2}}$*,* $M(y,Tx,t) = a^{y-\frac{x}{2}}$*, and*

$$M(x,Ty,t) = \begin{cases} a^{x-\frac{y}{2}}, & if\ x \ge \frac{y}{2}, \\ a^{\frac{y}{2}-x}, & if\ x < \frac{y}{2}. \end{cases}$$

*When* $x \ge \frac{y}{2}$ *we have* $N(x,y,t) = a^{\frac{y}{2}}$*,* $N^{''}(x,y,t) = a^{y-\frac{x}{2}}$ *and* $N^{'}(x,y,t) = \begin{cases} a^{\frac{x}{2}}, & if\ y \le \frac{3x}{2} \\ a^{y-x}, & if\ y > \frac{3x}{2} \end{cases}$ *If* $y \le \frac{3x}{2}$ *then* $\psi(M(Tx,Ty,t)) = a^{-3y+3x}$*,* $\psi(N(x,y,t)) = a^{-3y}$*,* $\phi(N^{'}(x,y,t), N^{''}(x,y,t)) = a^{\frac{y}{2}}$*. Since* $-3y+3x \ge -3y+y-\frac{y}{2},\ \forall\ y \le \frac{3x}{2}$*, we have*

$$\psi(M(Tx,Ty,t)) \le \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)).$$

*If* $y > \frac{3x}{2}$, $\phi(N^{'}(x,y,t), N^{''}(x,y,t)) = a^{y-\frac{3x}{4}}$. *Again since* $-3y+3x \geq -3y+y-\frac{3x}{4}$, $\forall\ x \geq \frac{y}{2}$, *so we have*

$$\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)), \forall t > 0.$$

*If* $x < \frac{y}{2}$ *then* $N(x,y,t) = a^{y-x}$, $N^{'}(x,y,t) = a^{y-x}$, $N^{''}(x,y,t) = a^{y-\frac{x}{2}}$.

$$\psi(M(Tx,Ty,t)) = a^{-3y+3x}, \psi(N(x,y,t)) = a^{-6x+6y}, \phi(N^{'}(x,y,t) \ and \ N^{''}(x,y,t)) = a^{y-\frac{3x}{4}}.$$

*Here* $-3y+3x \geq -6y+6x+y-\frac{3x}{4}, \forall y > 2x$, *imply that*

$$\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)),\ t > 0.$$

*Thus, for all* $x, y \in [0,1)$

$$\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)).$$

***Case 2:*** $x \in [0,1)$ *and* $y \in [1,\infty)$

***Sub case 1:*** $\frac{y}{2} \leq x < y$; $M(Tx,Ty,t) = a^{\frac{1}{2}-\frac{x}{2}}$, $M(x,y,t) = a^{y-x}, M(x,Tx,t) = a^{\frac{x}{2}}$, $M(y,Ty,t) = a^{y-\frac{1}{2}}$, $M(y,Tx,t) = a^{y-\frac{x}{2}}$, $M(x,Ty,t) = a^{x-\frac{1}{2}}$. *We observe that*

$$N(x,y,t) = a^{y-\frac{1}{2}}$$

$$N^{'}(x,y,t) = \begin{cases} a^{y-x}, & if\ y \leq \frac{3x}{2} \\ a^{\frac{x}{2}}, & if\ y > \frac{3x}{2} \end{cases}$$

$$N^{''}(x,y,t) = a^{y-\frac{x}{2}}$$

*When* $y \leq \frac{3x}{2}$, *we get that* $\psi(M(Tx,Ty,t)) = a^{-3+3x}$, $\psi(N(x,y,t)) = a^{-6y+3}$, $\phi(N^{'}(x,y,t), N^{''}(x,y,t)) = a^{y-\frac{3x}{4}}$. *It can easily be observed that* $-3+3x \geq -6y+3+y-\frac{3x}{4}, \forall x \geq \frac{y}{2}$. *Thus, for* $y \leq \frac{3x}{2}$

$$\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)).$$

*When* $y > \frac{3x}{2}$, $\phi(N^{'}(x,y,t), N^{''}(x,y,t)) = a^{\frac{y}{2}}$. *Similarly, we observe that,* $-3+3x \geq -6y+3+\frac{y}{2}$. *Thus,* $\forall\ y > \frac{3x}{2}$,

$$\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)).$$

***Sub case 2*** *If* $x < \frac{y}{2}$ *then*

$M(Tx,Ty,t) = a^{\frac{1}{2}-\frac{x}{2}}$, $M(x,y,t) = a^{y-x}$, $M(x,Tx,t) = a^{\frac{x}{2}}$, $M(y,Ty,t) = a^{y-\frac{1}{2}}$, $M(y,Tx,t) = a^{y-\frac{x}{2}}$ *and*

$$M(x,Ty,t) = \begin{cases} a^{x-\frac{1}{2}}, & if\ x \geq \frac{1}{2} \\ a^{\frac{1}{2}-x}, & if\ x < \frac{1}{2}. \end{cases}$$

*When* $x \geq \frac{1}{2}$, *we get that*

$$N(x,y,t) = a^{y-\frac{1}{2}}.$$

$$N^{'}(x,y,t) = a^{y-x}.$$

$$N^{''}(x,y,t) = a^{y-\frac{x}{2}}.$$

*Hence*

$$\psi(M(Tx,Ty,t)) = a^{-3+3x}, \psi(N(x,y,t)) = a^{-6y+3} \text{ and } \phi(N^{'}(x,y,t), N^{''}(x,y,t)) = a^{y-\frac{3x}{4}}.$$

*Since* $-3+3x \geq -6y+3+y-\frac{3x}{4}, \forall x \geq \frac{1}{2}$ *and* $y > 1$, *we have*

$$\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)).$$

*When* $x < \frac{1}{2}$, *we get that* $N(x,y,t) = a^{y-x}$, $N^{'}(x,y,t) = a^{y-x}$ *and* $N^{''}(x,y,t) = a^{y-\frac{x}{2}}$. *Here we can easily observe that* $\psi(M(Tx,Ty,t)) = a^{-3+3x}$, $\psi(N(x,y,t)) = a^{-6y+6x}$, $\phi(N^{'}(x,y,t), N^{''}(x,y,t)) = a^{y-\frac{3x}{4}}$. *Since* $-3+3x \geq -6y+6x+y-\frac{3x}{4}$, $\forall\ x < \frac{1}{2}$ *and* $y > 1$, *we get that*

$$\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)).$$

*Therefore, for all* $x \in [0,1)$ , $y \in [1,\infty)$ *and* $t > 0$,

$$\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)).$$

*In a similar way, we can show that* $\forall y \in [0,1), \forall x \in [1,\infty)$ *and* $t > 0$,

$$\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)).$$

***Case 3.*** *Let* $x, y \in [1,\infty)$ *and* $x \geq y$.

***Sub case 1****: if* $x > 2y$ *then* $M(Tx,Ty,t) = 1$, $M(x,y,t) = a^{x-y}$, $M(x,Tx,t) = a^{x-\frac{1}{2}}$, $M(y,Ty,t) = a^{y-\frac{1}{2}}$, $M(y,Tx,t) = a^{y-\frac{1}{2}}$ *and* $M(x,Ty,t) = a^{x-\frac{1}{2}}$. *Now, we have* $N(x,y,t) = a^{x-\frac{1}{2}}$, $N^{'}(x,y,t) = a^{x-\frac{1}{2}}$ *and*

$$N^{''}(x,y,t) = \min\{a^{y-x}, a^{y-\frac{1}{2}}\} = \begin{cases} a^{x-y}, & if\ x \geq 2y - \frac{1}{2} \\ a^{y-\frac{1}{2}}, & if\ x < 2y - \frac{1}{2} \end{cases}.$$

*When* $x < 2y - \frac{1}{2}$, *we get that*

$$\psi(M(Tx,Ty,t)) = 1, \psi(N(x,y,t)) = a^{-6x+3}, \phi(N^{'}(x,y,t) \text{ and } N^{''}(x,y,t)) = a^{\frac{x}{2}+\frac{y}{2}-\frac{1}{2}}.$$

*Since* $-6x+3+\frac{x}{2}+\frac{y}{2}-\frac{1}{2} \leq 0$, $\forall\ x \geq y$ *and* $y > 1$, *we have*

$$\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)).$$

*When* $x \geq 2y - \frac{1}{2}$, *we get that*

$$\psi(M(Tx,Ty,t)) = 1, \psi(N(x,y,t)) = a^{-6x+3} \text{ and } \phi(N^{'}(x,y,t), N^{''}(x,y,t)) = a^{x-\frac{y}{2}-\frac{1}{4}}$$

*Since* $-6x+3+x-\frac{y}{2}-\frac{1}{4} \leq 0, \forall x \geq 1$ *and* $y \geq 1$,

$$\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)).$$

*In a similar way we Show that (46) is true when* $x, y \in [1,\infty)$ *and* $y \geq x$. *Hence, from all the cases we conclude that,*

$$\psi(M(Tx,Ty,t)) \leq \psi(N(x,y,t))\phi(N^{'}(x,y,t), N^{''}(x,y,t)),\ \forall\ x, y \in [0,\infty)$$

*and* $t > 0$. *Therefore,* $T$ *is generalized* $(\psi,\phi)$ *contractive mapping. By theorem (3.7)* $T$ *has a unique fixed point. Indeed,* $0$ *is the unique fixed point of* $T$.

The following example is in support of Theorem 3.9.

**Example 4.2.** *Let $X = [0, \infty)$ and $f, g : X \to X$ defined by $fx = \frac{x}{2}$ and $g(x) = \frac{x}{3}$. Let $M$ be a strong fuzzy metric space define by $M(x, y, t) = (\frac{t}{t+1})^{d(x,y)}$, where $d(x, y) = |x - y|$. Let $\psi : (0, 1] \to [1, \infty)$ and $\phi : (0, 1] \times (0, 1] \to (0, 1]$ define by $\psi(t) = \frac{1}{t^8}$ and $\phi(s, t) = st$. we prove that $f$ is $(\psi, \phi)-$ generalized contractive mapping with respect to $g$. Thus, by Theorem 3.9 we conclude that $f$ and $g$ have a unique common fixed point in $X$, in fact 0 is a common fixed point for $f$ and $g$ in $X$.*

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