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On Schur Complements in Range Quaternion Hermitian Matrices

Research Article

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Abstract: It is established that under contain conditions a schur complement in a q-EP matrix is as well as q-EP matrix. As an application a decomposition of a partitioned matrix into a sum of q-EP matrices is given.

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1. Introduction

Throughout we shall deal with $n \times n$ quaternion matrices: Let A^*denote thje conjugate transpose of A. Any matrix $A \in H_{n\times n}$ is called q-EP. If $R(A) = R(A^*)$ and is called $q - Ep_r$, if A is q-EP and $rk(A) = r$, where $N(A)$, $R(A)$ and $rk(A)$ denote the null space, range space and rank of A respectively. It is well known that sum and product of q-EP, Generalized Inverse Group Inverse and Reverse order law for q-EP and Bicomplex representation methods and application of q-EP matrices. In this section, Schur complements in a q-EP matrices.

Lemma 1.1. *If X and Y are generalized inverse of A, then $CXB = CYB$ if and only if $N(A) \subseteq M(C)$ and $N(A^*) \subseteq N(B^*)$ or, equivalently if and only if*

$$C = CA^-A \text{ and } B = AA^-B \quad \text{for every } A^- \tag{1}$$

Throughout this paper, we are concerned with $n \times n$ quaternion matrices M partitioned in the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2}$$

Where A and D are square matrices with respect to this partitioning a Schur complements of A in M is a matrix at the form $(M/A) = D - CA^-B$. For entries of Schur complements one may refer to [2, 3, 5]. On account of Lemma 1.1 it is obvious that under certain conditions (M/A) is independent of the choice of A^-. However in the sequel we shall always assume that (M/A) is given in terms of specific choice of A^-.

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In [9] necessary and sufficient conditions are derived for a matrix of the (2) with $B = 0$ and $C = 0$ to be q-EP. The results are here extended for general matrices of the form (2). If a partitioned matrix of the form (2) is q-EP, then in general (M/A) is not q-EP. Here we determine necessary and sufficient conditions for M/A to be q-EP. In particular, when $rk(M) = rk(A)$our results include as special cases the results of paper [13]. In [5] we have given conditions for a sum of q-EP matrices to be q-EP.

Theorem 1.2. *Let M be a matrix of the form (2) with $N(A) \subseteq N(C)$ and $N(M/A) \subseteq N(B)$, then the following are equivalent.*

(1). M is a q-EP matrix

(2). A and M/A are q-EP, $N(A^*) \subseteq N(B^*)$ and $N((M/A)^*) \subseteq M(C^*)$;

(3). Both the matrices $\begin{pmatrix} A & 0 \\ C & M/A \end{pmatrix}$ and $\begin{pmatrix} A & B \\ 0 & M/A \end{pmatrix}$ are q-EP.

Proof.

(1)⇒(2) Let us consider the matrices

$$p = \begin{pmatrix} I & 0 \\ CA^- & I \end{pmatrix}, \ Q = \begin{pmatrix} I & B(M/A)^- \\ 0 & I \end{pmatrix}, \ L = \begin{pmatrix} A & 0 \\ 0 & M/A \end{pmatrix}$$

Clearly P and Q are non-singular. By assumption $N(A) \subseteq N(C)$ and $N(M/A) \subseteq N(B)$ and by using Lemma 1.1 it is obvious that M can be factorized as $M = PQL$. Hence $rk(M) = rk(L)$ and $N(M) = N(L)$. But $M$is q-EP, e.g. $N(M^*) = N(M) = N(L)$. Therefore by using Lemma 1.1 again $M^* = M^*L^-L$ holds for every L^-. One choice of L^- is

$$L^- = \begin{pmatrix} A^- & 0 \\ 0 & (M/A)^- \end{pmatrix},$$

which gives

$$M^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} A^-A & 0 \\ 0 & (M/A)^-(M/A) \end{pmatrix}$$

$A^* = A^*A^-A$ implies $N(A^*) \supseteq N(A)$, and since $rk(A^*) = rk(A)$ these imply $N(A^*) = N(A)$. Hence A is q-EP. From $B^* = B^*A^-A$ it follows that $N(B) \supseteq N(A) = N(A^*)$. After substituting $D = M/A + BA^-C$and using $C^* = C^*(M/A)^-M/A$ in $D^* = D^*(M/A)^-M/A$ we get $(M/A)^* = (M/A)^*(M/A)^-M/A$. This implies that $N((M/A)^*) \supseteq N(M/A)$ and since

$$rk((M/A)^*) = rk(M/A)$$

we get $N((M/A)^*) = N(M/A)$. Thus M/A is q-EP. Further

$$N(C^*) \supseteq N(M/A) = N((M/A)^*)$$

Hence (2) holds.

(1)⇒(2) Since $N(A) \subseteq N(C)$, $N(A^*) \subseteq N(B^*)$, $N(M/A) \subseteq N(B)$ and $N((M/A)^*) \subseteq N(C^*)$ hold according to the assumption. So $M^\dagger$ is given buy the formula

$$M^\dagger = \begin{pmatrix} A^\dagger + A^\dagger B(M/A)^\dagger CA^\dagger & -A^\dagger B(M/A)^\dagger \\ -(M/A)^\dagger CA^\dagger & (M/A)^\dagger \end{pmatrix}$$

According to Lemma 1.1 the assumptions $N(A) \subseteq N(C)$ and $N(A^*) \subseteq N(B^*)$ imply that M/A is invariant for every choice of A^-. Hence $M/A = D - CA^\dagger B$. Further, using $C = M/A(M/A)^\dagger C$ and $B = AA^\dagger B$, $MM^\dagger$ is reduced to the form

$$M^\dagger M = \begin{pmatrix} AA^\dagger & 0 \\ 0 & (M/A)(M/A)^\dagger \end{pmatrix}$$

The relations $AA^\dagger = A^\dagger A$ and $(M/A)(M/A)^\dagger = (M/A)^\dagger (M/A)$ result $MM^\dagger = M^\dagger M$, e.g., M is q-EP. Thus (1) holds.
(2)⇒(3) By Corollary 8 in [9]

$$\begin{pmatrix} A & 0 \\ C & M/A \end{pmatrix}$$

is q-EP, iff A and (M/A) are q-EP, further $N(A) \subseteq N(C)$ and $N((M/A)^*) \subseteq N(C^*)$

$$\begin{pmatrix} A & B \\ 0 & M/A \end{pmatrix}$$

Is q-EP iff A and M/A are q-EP, further $N(A^*) \subseteq N(B^*)$ and $N(M/A) \subseteq N(B)$. This proves the equivalence of (2) and (3). The proof is complete.

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

Theorem 1.3. *Let M be a matrix of the form (2) with $N(A^*) \subseteq N(B^*)$ and $N((M/A)^*) \subseteq N(C^*)$, then the following are equivalent.*

(1). M is an q-EP matrix

(2). A and (M/A) are q-EP, further $N(A) \subseteq N(C)$ and $N(M/A) \subseteq N(B)$;

(3). Both the matrices $\begin{pmatrix} A & 0 \\ C & M/A \end{pmatrix}$ and $\begin{pmatrix} A & B \\ 0 & M/A \end{pmatrix}$ are q-EP.

Proof. Theorem 1.3 follows immediately from Theorem 1.2 and from the fact that M is q-EP iff M^* is q-EP. If and only if M^* is q-EP. □

In this special case when $B = C^*$ we get the following.

Corollary 1.4. *Let $M = \begin{pmatrix} A & C^* \\ C & D \end{pmatrix}$ with $N(A) \subseteq N(C)$ and $N(M/A) \subseteq N(C^*)$, then the following are equivalent.*

(1). M is an q-EP matrix

(2). A and (M/A) are q-EP matrices.

(3). the matrix $\begin{pmatrix} A & 0 \\ C & M/A \end{pmatrix}$ is q-EP.

Remark 1.5. *The conditions that taken on M in the previous theorems are essential. This is illustrated in the following example. Let*

$$M = \begin{bmatrix} 1 & 1 & 1 & 1+i+j+k \\ 1 & 1 & 1-i-j-k & 1 \\ 1 & 1+i+j+k & 1 & 1 \\ 1-i-j-k & 1 & 1 & 0 \end{bmatrix}$$

M is symmetric and

$$B = C = \begin{pmatrix} 1 & 1+i+j+k \\ 1-i-j-k & 1 \end{pmatrix}$$

$$(M/A) = D - CA^{\dagger}B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Clearly A and (M/A) are q-EP, $N(A) \subseteq N(C)$and $N(A^*) \subseteq N(B^*)$, but $N(M/A) \subseteq N(B)$ and $N((M/A)^*) \not\subseteq N(C^*)$, further $\begin{pmatrix} A & 0 \\ C & M/A \end{pmatrix}$ and $\begin{pmatrix} A & B \\ 0 & M/A \end{pmatrix}$ Or not q-EP. Thus Theorem 1.2 and 1.3 as well as Corollary 1.4 fail.

Remark 1.6. *We conclude from Theorem 1.2 and Theorem 1.3 that for an q-EP matrix M of the form equation (2) the following are equivalent*

$$N(A) \subseteq N(C), N(M/A) \subseteq N(B) \tag{3}$$

$$N(A^*) \subseteq N(B^*), N((M/A)^*) \subseteq N(C^*) \tag{4}$$

However this fails if we omit the condition that M is q-EP. For example Let

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

M is not q-EP. Here

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = C^* = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

A is q-EP, $N(A) \subseteq N(C)$and $N(A^*) \subseteq N(B^*)$. Hence (M/A) is independent of the choice of A^- and so

$$(M/A) = D - CA^{\dagger}B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(M/A) is not q-EP, $N((M/A)^*) \subseteq N(C^*)$, but $N(A) \subseteq N(B)$. Thus Equation (4) holds, while Equation (4) fails.

Remark 1.7. *It has been proved is [2] that for any matrix Aits Moore-Penrose inverse. $M^{\dagger}$ is given by the formula Equation (**??**) iff both Equation (3) and Equation (4) holds. However it is clear by the previous Remark 1.6 that for an q-EP matrix formula (**??**) gives $M^{\dagger}$ iff either (3) or (4) holds.*

Theorem 1.8. *Let M be of the form Equation (2) with $rk(M) = rk(A) = r$. Then M is an q-EP$_r$, matrix if and only if A is q-EP, and $CA^\dagger = (A^\dagger B)^*$.*

Proof. Since $rk(M) = rk(A) = r$, we have by reason of the corollary of Theorem 1 in [3] that $N(A) \subseteq N(C)$, $N(A^*) \subseteq N(B^*)$, and $M/A = D - CA^\dagger B = 0$. According to Theorem 1.1 these relation are equivalent $C = CA^\dagger A$, $B = AA^\dagger B$ and $D = CA^\dagger B$. Let us consider the matrices

$$P = \begin{pmatrix} I & 0 \\ CA^\dagger & I \end{pmatrix}, \quad Q = \begin{pmatrix} I & A^\dagger B \\ 0 & I \end{pmatrix}, \quad L = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

P and Q are non-singular and by assumption $CA^\dagger = (A^\dagger B)^*$ it holds $P = Q^*$. Therefore M can be factorized as $M = PLP^*$. Since A is q-EP$_r$ consequently L is as well q-EP$_r$. Hence $N(L) = N(L^*)$ and so we have according to Lemma 3 of [1] that $N(M) = N(PLP^*) = N(PL^*P^*) = N(M^*)$. This shows that M is q-EP$_r$.

Conversely, let us assume that M is q-EP$_r$. Since $M = PLQ$, one choice of A^- is

$$M^- = Q^{-1} \begin{pmatrix} A^\dagger & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$$

We know that $N(M) = N(M^*)$, therefore by Lemma 1.1 $M^* = M^*M^-M$ holds, e.g

$$M^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} A^\dagger A & A^\dagger B \\ 0 & 0 \end{pmatrix}$$

or equivalently, $A^* = A^*A^\dagger A$and $C^* = C^*A^\dagger B$. From $A^* = A^*A^\dagger A$ it follows $N(A^*) = N(A)$, i.e., A is q-EP$_r$ and therefore $AA^\dagger = A^\dagger A$ taking into account $C^* = C^*A^\dagger B$, we have

$$\begin{aligned} CA^\dagger &= B^*(A^\dagger)^*(A^\dagger A) \\ &= B^*(A^\dagger AA^\dagger)^* \\ &= B^*(A^\dagger)^* \\ &= (A^\dagger B)^* \end{aligned}$$

□

Corollary 1.9. *Let M of the form (2) with A non-singular matrix and $rk(M) = rk(A)$. Then M is q-EP if and only if $CA^\dagger = (A^\dagger B)^*$.*

Corollary 1.10. *Let M be an $n \times n$ matrix f rank r. Then M is q-EP$_r$ if and only if every principal sub matrix of rank r is q-EP$_r$.*

Proof. Suppose M is an q-EP$_r$ matrix. Let A be any principal submatrix of M such that $rk(M) = rk(A) = r$. Then there exists a permutation matrix such that $\widehat{M} = PMP^T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $rk(A) = r$. According to Lemma 3 in [1], is q-EP$_r$. Now, we conclude from Theorem 1.3 that A q-EP$_r$ as well. Since A was arbitrary, it follows that very principal submatrix of rank r is q-EP$_r$. The converse is obvious. □

Remark 1.11. *Theorem 1.8 fails if we relax the condition on rank of M.*

2. Application

We give conditions under which a partitioned matrix is decomposed into complementary summands of q-EP matrices. M_1 and M_2 are called complementary summand of M if $M = M_1 + M_2$ and $rk(M) = rk(M_1) + rk(M_2)$.

Theorem 2.1. *Let M of the form (2) with $rk(M) = rk(A) = rk(M/A)$, where $(M/A) = D - CA^{\dagger}B$. If $A$and (M/A) are q-EP matrices such that $CA^{\dagger} = (A+B)^{*}$and $B(M/A)^{\dagger} = ((M/A)^{\dagger}C^{*})$ then M can be decomposed into complementary summands of q-EP matrices.*

Proof. Let us consider the matrices

$$M_1 = \begin{pmatrix} A & AA^{\dagger}B \\ CA^{\dagger}A & CA^{\dagger}B \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 0 & (I - AA^{\dagger})B \\ C(I - A^{\dagger}A) & M/A \end{pmatrix}$$

Taking into account that $N(A) \subseteq N(CA^{\dagger}A)$, $N(A^{*}) \subseteq N(AA^{\dagger}B)^{*}$ and

$$M_1/A = CA^{\dagger}B - ((CA^{\dagger}A)A - (AA^{\dagger}B) = CA^{\dagger}B - CA^{\dagger}B = 0$$

we obtain by the corollary after Theorem 1 in [5], that $rk(M_1) = rk(A)$. Since A is q-EP and $(CA^{\dagger}A)A^{\dagger} = CA^{\dagger} = (A^{\dagger}B)^{*} = (A^{\dagger}AA^{\dagger}B)^{*}$. We have from Theorem 1.8 that M_1 is q-EP. Since $rk(M) = rk(A) + rk(M/A)$, Theorem 1 of [5] gives $N(M/A) \subseteq N(I - AA^{\dagger})B$, $N(M/A) \subseteq N((I - A^{\dagger})C)^{*}$ and $(I - AA^{\dagger})M(M/A)^{\dagger}C(I - A^{\dagger}A) = 0$. Thus by the corollary of the just applied Theorem 1.1 in [5], we have $rk(M_2) = rk(M/A)$. Further, using $AA^{\dagger} = A^{\dagger}A$, we obtain

$$\begin{aligned}(I - AA^{\dagger})B(M/A)^{\dagger} &= (I - AA^{\dagger})((M/A)^{\dagger})^{*} \\ &= ((M/A)^{\dagger}C(I - AA))^{*} \\ &= ((M/A)^{\dagger}C(I - A^{\dagger}A))^{*}\end{aligned}$$

Thus by Theorem 1.8, M_2 is also q-EP. Clearly $M = M_1 + M_2$, where both M_1 and M_2 are q-EP matrices and

$$rk(M) = rk(A) + rk(M/A) = rk(M_1) + rk(M_2).$$

Hence M_1 and M_2 are complementary summands of q-EP matrices. □

Remark 2.2. *Any matrix that is represented as the sum of complementary summands of q-EP matrices is itself q-EP. For if $M = \sum_{i=1}^{k} M_i$ such that each M_i is q-EP and $rk(M) = \sum rk(M_i)$, then*

$$N(M) = \bigcap_{i=1}^{k} N(M_i) = \bigcap_{i=1}^{k} N(M_i^{*}) = N(M_i^{*}).$$

References

[1] T.S.Baskett and I.J.Katz, *Theorems on products of EP matrices*, Linear Algebra and its Appl., 2(1969), 87-103.

[2] F.Burns, D.Carlson, E.Haynsworth and TH.Markham, *Generalized inverse formulas using the Schur Complement*, SIAM J. Appl. Math., 26(1974), 254-259.

[3] D.Carlson, E.Haynsworth and T.H.Markham, *A generalization of the Schur complement by means of the Moore-Penrose inverse*, SIAM J. Appl. Math., 26(1974), 169-175.

[4] K.Gunasekaran and S.Sridevi, *On Range Quaternion Hermitian Matrices*, International Journal of Mathematical Archive, 6(8)(2015), 159-163.

[5] K.Gunasekaran and S.Sridevi, On Sums of Range Quaternion Hermitian Matrices, Int. J. Modern Engineering Research, 5(2015), 44-49.

[6] K.Gunasekaran and S.Sridevi, On Product of Range Quaternion Hermitian Matrices, Int. Journal of Pure Algebra, 6(6)(2016), 318-321.

[7] K.Gunasekaran and S.Sridevi, *Generalized inverses, Group inverses and Revers order law of range quaternion Hermitian matrices*, IOSR-JM, 12(4-II)(2016), 51-55.

[8] K.Gunasekaran and S.Sridevi, *On Bicomplex Representation Methods and Application of Range Quaternion Hermitian Matrices (q-Ep)*, IJMTT, 48(4)(2017), 250-259.

[9] C.D.Meyer, *Generalized inverses of block triangular matrices*, SIAM J. Appl. Math., 19(1970), 741-750.

[10] M.H.Pearl, *On generalized inverses of matrices*, Proc. Cambridge Phil. Soc., 62(1966), 673-677.

[11] R.Penrose, *A generalizedinverse for matrices*, Proc. Cambridge Philos. Soc., 51(1955), 406-415.

[12] C.R.Rao and S.K.Mitra, *Generalized inverse of matrices and its applications*, Wiley, New York, (1971).

[13] P.Robert, *On the group inerse of a linear trandformation*, J. Math. Anal. Appl., 22(1968), 658-669.

[14] H.Schwerdtfeger, *Introduction to linear algebra and the theory of matrices*, Noordhoff, Groningen, (1950).