



Total Resolving Number of Power Graphs

Research Article

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Abstract: Let $G = (V, E)$ be a simple connected graph. An ordered subset W of V is said to be a resolving set of G if every vertex is uniquely determined by its vector of distances to the vertices in W . The minimum cardinality of a resolving set is called the *resolving number* of G and is denoted by $r(G)$. As an extension, the total resolving number was introduced in [5] as the minimum cardinality taken over all resolving sets in which $\langle W \rangle$ has no isolates and it is denoted by $tr(G)$. In this paper, we obtain the bounds on the total resolving number of power graphs. Also, we characterize the extremal graphs.

MSC: Primary 05C12, Secondary 05C35.

Keywords: Resolving number, total resolving number, power graph.

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1. Introduction

Let $G = (V, E)$ be a finite, simple, connected and undirected graph. The *degree* of a vertex v in a graph G is the number of edges incident to v and it is denoted by $d(v)$. The maximum degree in a graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest $u - v$ path in G . The maximum value of distance between vertices of G is called its *diameter*. P_n denote the *path* on n vertices. C_n denote the *cycle* on n vertices. K_n denote the *complete graph* on n vertices. A graph is *acyclic* if it has no cycles. A *tree* is a connected acyclic graph. A *spider* is a tree with one vertex of degree at least 3 and all others with degree at most 2. A complete bipartite graph is denoted by $K_{s,t}$. A *star* is denoted by $K_{1,n-1}$. A tree obtained by joining the centres of two stars $K_{1,s}$ and $K_{1,t}$ by an edge is called a *bistar* and it is denoted by $B_{s,t}$. A (k, l) -*kite* is a graph obtained by identifying any vertex of a cycle C_k with an end vertex of a path P_l . If $W = \{w_1, w_2, \dots, w_k\} \subseteq V(G)$ is an ordered set, then the ordered k -tuple $(d(v, w_1), d(v, w_2), \dots, d(v, w_k))$ is called the representation of v with respect to W and it is denoted by $r(v|W)$. Since the representation for each $w_i \in W$ contains exactly one 0 in the i^{th} position, all the vertices of W have distinct representations. W is called a *resolving set* for G if all the vertices of $V \setminus W$ also have distinct representations. The minimum cardinality of a resolving set is called the *resolving number* of G and it is denoted by $r(G)$. In [5] we introduced and studied total resolving number. If W is a resolving set and the induced subgraph $\langle W \rangle$ has no isolates, then W is called a *total resolving set* of G . The minimum cardinality taken over all total resolving sets of G is called the *total resolving number* of G and is denoted by $tr(G)$. In this paper, we obtain the bounds on the total resolving number of power graphs. Also, we characterize the extremal graphs.

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2. Total Resolving Number of Graphs

The following results are used in next section.

Observation 2.1 ([5]). *Let $\{w_1, w_2\} \subset V(G)$ be a total resolving set in G . Then the degrees of w_1 and w_2 are at most 3.*

Theorem 2.2 ([5]). *For $n \geq 3$, $tr(P_n) = 2$ and $tr(C_n) = 2$.*

Observation 2.3 ([5]). *For any graph G of order $n \geq 3$, $2 \leq tr(G) \leq n - 1$.*

Theorem 2.4 ([5]). *Let G be a graph of order $n \geq 3$. Then $tr(G) = n - 1$ if and only if $G \cong K_n$ or $K_{1,n-1}$.*

3. Power Graphs

In this section, we determine the total resolving number of square of cycles, bistar, spider and power of paths. Also, we obtain the bounds of the total resolving number of power graphs and characterize the extremal graphs.

Definition 3.1. *For any integer $k \geq 2$, the power G^k of a graph G is a graph whose vertex set is $V(G)$ and two distinct vertices of G^k are adjacent if their distance in G is at most k .*

Theorem 3.2. *For $n \geq 3$, $tr(C_n^2) = \begin{cases} 4 & \text{if } n \equiv 1 \pmod{4} \\ 3 & \text{otherwise.} \end{cases}$*

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Since C_n^2 is a 4-regular graph, $tr(C_n^2) \geq 3$. Let d be the diameter of C_n^2 . Let x, y be two distinct vertices of $V(C_n^2) \setminus W$. We consider the following two cases.

Case 1: $n \not\equiv 1 \pmod{4}$.

Let $W = \{v_1, v_2, v_3\}$. Then we consider the following two subcases.

Sub case 1.1: $n \equiv 0 \pmod{4}$ or $n \equiv 2 \pmod{4}$.

If either $d(x, v_1) \neq d(y, v_1)$ or $d(x, v_2) \neq d(y, v_2)$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, v_1) = d(y, v_1)$ and $d(x, v_2) = d(y, v_2)$. If $n \equiv 0 \pmod{4}$, then $x = v_{\frac{n}{2}}$ and $y = v_{\frac{n}{2}+1}$. But $d(x, v_3) = d(y, v_3) - 1$. It follows that $r(x|W) \neq r(y|W)$.

If $n \equiv 2 \pmod{4}$, then $n \geq 8$ and $x = v_{\frac{n}{2}}$ and $y = v_{\frac{n}{2}+3}$. But $d(x, v_3) = d(y, v_3) - 2$. It follows that $r(x|W) \neq r(y|W)$.

Sub case 1.2: $n \equiv 3 \pmod{4}$.

Let x lie on $v_1-v_{\lceil \frac{n}{2} \rceil+2}$ path of C_n . If either $d(x, v_1) \neq d(y, v_1)$ or $d(x, v_2) \neq d(y, v_2)$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, v_1) = d(y, v_1)$ and $d(x, v_2) = d(y, v_2)$. Then x lies on $v_4-v_{\lfloor \frac{n}{2} \rfloor}$ path in the graph C_n and y lies on $v_n-v_{\lfloor \frac{n}{2} \rfloor}$ path in the graph C_n . But $d(x, v_3) = d(y, v_3) - 2$. It follows that $r(x|W) \neq r(y|W)$.

Therefore W is a resolving set of C_n^2 and hence $tr(C_n^2) \leq 3$. Thus $tr(C_n^2) = 3$.

Case 2: $n \equiv 1 \pmod{4}$.

Then $n = 4k + 1$, $k \geq 1$. If $k = 1$, then $C_n^2 \cong K_5$. But $tr(K_5) = 4$. So we consider $k \geq 2$. In this case, we claim that $tr(C_n^2) = 4$. Suppose $tr(C_n^2) \leq 3$. If $\langle W \rangle = P_3$, then without loss of generality, let $W = \{v_1, v_2, v_4\}$ or $\{v_1, v_3, v_5\}$. If $W = \{v_1, v_2, v_4\}$, then $r(v_{\lceil \frac{n}{2} \rceil}|W) = r(v_{\lceil \frac{n}{2} \rceil+1}|W) = (k, k, k - 1)$, which is a contradiction to $tr(C_n^2) = 3$. If $W = \{v_1, v_3, v_5\}$, then $r(v_{\lceil \frac{n}{2} \rceil}|W) = r(v_{\lceil \frac{n}{2} \rceil+2}|W) = (k, k, k - 1)$. If $\langle W \rangle = K_3$, then without loss of generality, let $W = \{v_1, v_2, v_3\}$. Then $r(v_{\lceil \frac{n}{2} \rceil+1}|W) = r(v_{\lceil \frac{n}{2} \rceil+2}|W) = (k, k, k)$, which is a contradiction. Thus $tr(C_n^2) \geq 4$. Let $W = \{v_1, v_2, v_3, v_4\}$. If $d(x, v_i) \neq d(y, v_i)$ for some $i (i = 1, 2, 3)$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, v_1) = d(y, v_1)$, $d(x, v_2) = d(y, v_2)$ and $d(x, v_3) = d(y, v_3)$. Then $x = v_{\lceil \frac{n}{2} \rceil+1}$ and $y = v_{\lfloor \frac{n}{2} \rfloor+2}$. But $d(x, v_4) = d(y, v_4) - 1$. It follows that $r(x|W) \neq r(y|W)$. Thus W is a resolving set of C_n^2 and hence $tr(C_n^2) \leq 4$. Thus $tr(C_n^2) = 4$. □

Observation 3.3 ([6]). *If a connected graph G contains a set S of vertices of G of cardinality $p \geq 2$ such that $d(u, x) = d(v, x) \forall u, v \in S$ and $x \in V(G) \setminus \{u, v\}$, then every resolving set must contain at least $p - 1$ vertices of S .*

Theorem 3.4. *For $s, t \geq 2$, $tr(B_{s,t}^2) = s + t - 1$.*

Proof. Let $V(B_{s,t}) = \{u_0, u_1, \dots, u_s\} \cup \{v_0, v_1, \dots, v_t\}$ and $E(B_{s,t}) = \{u_0u_i / 1 \leq i \leq s\} \cup \{v_0v_j / 1 \leq j \leq t\} \cup \{u_0v_0\}$. Let W be a total resolving set of $B_{s,t}^2$. By Observation 3.3, every W contain at least one vertex from $\{u_0, v_0\}$, $s - 1$ vertices from $\{u_1, u_2, \dots, u_s\}$ and $t - 1$ vertices from $\{v_1, v_2, \dots, v_t\}$. Therefore, $tr(B_{s,t}^2) \geq s + t - 1$. Let $W = \{u_i / 0 \leq i \leq s - 1\} \cup \{v_j / 1 \leq j \leq t - 1\}$. Then all the coordinates of the representation of v_0 are 1, only the 1^{st} s coordinates of the representation of u_s is 1 and only the last $r - 1$ coordinates of the representation of v_t is 1. Since $\langle W \rangle$ has no isolates, $tr(B_{s,t}^2) \leq s + t - 1$ and hence $tr(B_{s,t}^2) = s + t - 1$. □

Theorem 3.5. *Let T be a spider. Then $tr(T^2) = \Delta(T)$.*

Proof. Let $V(T) = \{v, v_{i1}, v_{i2}, \dots, v_{ir_i} / 1 \leq i \leq t\}$, where $d(v) = t \geq 3$ in T and $E(T) = \{vv_{i1}, v_{i1}v_{i2}, v_{i2}v_{i3}, \dots, v_{i(r_i-1)}v_{ir_i} / 1 \leq i \leq t\}$, where $|V(T)| = r_1 + r_2 + \dots + r_t + 1$. Then $V(T^2) = V(T)$ and $E(T^2) = E(T) \cup \{vv_{i2}, v_{i1}v_{i3}, v_{i2}v_{i4}, v_{i3}v_{i5}, v_{i4}v_{i6}, \dots, v_{i(r_i-2)}v_{ir_i} / 1 \leq i \leq t\}$. Let W be a minimum total resolving set of T^2 . Then we claim that W contains at least one vertex from the set $\{v_{i1}, v_{i2}, \dots, v_{it_i}\}$ for all $1 \leq i \leq t$ with one exception. Suppose no vertex of $\{v_{11}, v_{12}, \dots, v_{1t_1}\}$ and $\{v_{21}, v_{22}, \dots, v_{2t_2}\}$ belongs to W . Then $r(v_{i1}|W) = r(v_{2j}|W)$ for $i = j$, which is a contradiction. Since W is a minimum total resolving set, $t - 1$ vertices from the set $\{v_{11}, v_{21}, \dots, v_{t1}\}$ belong to W . Without loss of generality, let $v_{11}, v_{21}, \dots, v_{(t-1)1}$ belongs to W . But each coordinate of the representation of v_{t1} and v_1 is 1. It follows that $r(v_{t1}|W) = r(v|W)$. Therefore v or v_{t1} belongs to W . Thus $tr(T^2) \geq t$. Let $W = \{v, v_{11}, v_{21}, \dots, v_{(t-1)1}\}$. We claim that W is a resolving set of T^2 . Let x, y be two distinct vertices of $V(T^2) \setminus W$. We consider the following two cases.

Case 1 : x lies on $v_{i1}-v_{ir_i}$ path of T for some $1 \leq i \leq t - 1$.

Then $r(x|W) \neq r(y|W)$ for all $x, y \in V(T^2) \setminus W$ with respect to $\{v, v_{i1}\}$, $1 \leq i \leq t - 1$.

Case 2 : x lies on $v_{t1}-v_{tr_t}$ path of T .

For $1 \leq i \leq t - 1$, if x lies on $v_{i1}-v_{ir_i}$ path of T , then by case 1, $r(x|W) \neq r(y|W)$ for all $x, y \in V \setminus W$. So we may assume that y lies on $v_{t1}-v_{tr_t}$ path of T . If $d(x, v) \neq d(y, v)$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, v) = d(y, v)$. If x lies on $y-v$ path of T , then $d(y, v_{11}) = d(x, v_{11}) + 1$ and if y lies on $x-v$ path of T , then $d(x, v_{11}) = d(y, v_{11}) + 1$. So $r(x|W) \neq r(y|W)$ for all $x, y \in V(T^2) \setminus W$. Therefore each vertex of $V(T^2) \setminus W$ have distinct representations. Thus $tr(T^2) \leq t$ and hence $tr(T^2) = t = \Delta(T)$. □

Observation 3.6. *Let G be a graph of order $n \geq 3$ and diameter d . Then $2 \leq tr(G^k) \leq n - 1$, $2 \leq k \leq d$.*

Proof. The proof follows from Observation 2.3. □

Theorem 3.7. *Let G be a graph of order $n \geq 4$. Then $tr(G^k) = 2$ if and only if $G \cong P_n$.*

Proof. Assume that $tr(G^k) = 2$. Let $W = \{w_1, w_2\}$ be a total resolving set of G^2 . Then by Observation 2.1, $d(w_1) \leq 3$ and $d(w_2) \leq 3$ and hence $k = 2$. First, we claim that $\delta(G) = 1$. Suppose $\delta(G) \geq 2$. If $n = 4$, then $tr(G^2) = 3$. If $n \geq 5$, then $\delta(G) \geq 4$. By Observation 2.1, $tr(G^2) \geq 3$, which is a contradiction. Thus $\delta(G) = 1$. Now, we claim that $\Delta(G) = 2$. Suppose $\Delta(G) \geq 3$. Suppose $G \cong (3, l)$ -kite. If $l = 1$ or 2 , then $tr(G^2) = 3$. Let $l \geq 3$. Let $v_1v_2v_3v_1$ be the cycle of $(3, l)$ -kite, u be the pendant and v be its neighbor. Let $d(v_1) = 3$. Then by Observation 2.1, one vertex of W is u and another one is v . But $d(v_2, u) = d(v_3, u)$ and $d(v_2, v) = d(v_3, v)$. It follows that $r(v_2|W) = r(v_3|W)$, which is a contradiction. Suppose $G \cong (k, l)$ -kite, $k \geq 4$. If $l = 1$ or 2 , then we can easily verify that $tr(G^2) \neq 2$. Let $l \geq 3$. Let $v_1v_2v_3 \dots v_kv_1$ be the

cycle C_k of (k, l) -kite and $v_k v_{k+1} v_{k+2} \dots v_n$ be the path of (k, l) -kite. Then $d_{G^2}(v_n) = 2$, $d_{G^2}(v_{n-1}) = 3$ and $d(v_i) \geq 4$, $1 \leq i \leq n - 2$. So $W = \{v_n, v_{n-1}\}$. But $d_{G^2}(v_1, v_n) = d_{G^2}(v_{k-1}, v_n)$ and $d_{G^2}(v_1, v_{n-1}) = d_{G^2}(v_{k-1}, v_{n-1})$. It follows that $r(v_1|W) = r(v_{k-1}|W)$, which is a contradiction. If $G \not\cong (k, l)$ -kite, then we use the similar argument we get $tr(G^2) \geq 3$. Thus $\Delta(G) = 2$. Since $\delta(G) = 1$, $G \cong P_n$.

The converse can be easily verified. □

Theorem 3.8. *Let G be a graph of order $n \geq 3$ and diameter d . Then $tr(G^k) = n - 1$, $2 \leq k \leq d$ if and only if $diam(G) = k$.*

Proof. Assume that $tr(G^k) = n - 1$. Then we claim that $diam(G) = k$. Suppose $diam(G) \geq k + 1$. Then $diam(G^k) \geq 2$. Since $\delta(G^k) \geq 2$ and $diam(G^k) \geq 2$, by Theorem 2.4, $tr(G^k) \leq n - 2$, which is a contradiction. Thus $diam(G) = k$. Conversely, let $diam(G) = k$. Then $G^k \cong K_n$. By Theorem 2.4, $tr(G^k) = n - 1$. □

Theorem 3.9. *For $n \geq 3$ and $n > k$, $tr(P_n^k) = k$.*

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$. Let v_1 and v_n be the end vertices of P_n and v_2, v_3, \dots, v_{n-1} be the internal vertices of P_n . Let W be a total resolving set of P_n^k . Let $W = \{v_1, v_2, \dots, v_k\}$. Then we can easily verify that each vertex of $V(P_n^k) \setminus W$ have distinct representations. Therefore, $tr(P_n^k) \leq k$. Next, we prove that $tr(P_n^k) \geq k$. Suppose that $tr(P_n^k) \leq k - 1$. If $n = k + 1$, then $P_n^k \cong K_{k+1}$. We know that $tr(K_{k+1}) = k$, which is a contradiction to W . So, we assume that $n \geq k + 2$. Let $W = \{w_1, w_2, \dots, w_{k-1}\}$ and $U = \{v_1, v_2, \dots, v_k, v_{k+1}\}$. Let $W \subset U$. Since $|W| = k - 1$, let $u, v \in U$ but not in W and $\langle U \rangle$ is K_{k+1} , $r(u|W) = r(v|W) = (1, 1, \dots, 1)$, which is a contradiction. Let $W \not\subset U$. Then at least one vertex of W must be in $V(P_n^k) \setminus U$. If exactly one vertex of W does not in U , then without loss of generality, let $w_{k-1} \notin U$. Therefore there exist exactly three vertices of U not in W , say y_1, y_2, y_3 . Then there exists a vertex, say v_i in $U \setminus \{v_1\}$ such that $d(v_i, v_{i+\tau k}) = r$, where $v_{i+\tau k} = w_{k-1}$, $r \geq 1$, $i \neq 1$. Then $d(y_1, w_i) = d(y_2, w_i) = d(y_3, w_i)$ for all $1 \leq i \leq k - 2$. Let $x = v_i$.

If y_1, y_2 and y_3 do not lie on $x - w_{k-1}$ path of P_n , then $d(y_1, w_{k-1}) = d(y_2, w_{k-1}) = d(y_3, w_{k-1}) = r + 1$ in P_n^k . Thus $r(y_1|W) = r(y_2|W) = r(y_3|W)$, which is a contradiction. If y_1 and y_2 do not lie on $x - w_{k-1}$ path and y_3 lies on $x - w_{k-1}$ path of P_n , then $d(y_1, w_{k-1}) = d(y_2, w_{k-1}) = r + 1$ in P_n^k . Thus $r(y_1|W) = r(y_2|W)$ which is a contradiction. If y_1 does not lie on $x - w_{k-1}$ path and y_2 and y_3 lie on $x - w_{k-1}$ path of P_n , then $d(y_2, w_{k-1}) = d(y_3, w_{k-1}) = r$ in P_n^k . Thus $r(y_2|W) = r(y_3|W)$, which is a contradiction. If y_1, y_2 and y_3 lies on $x - w_{k-1}$ path of P_n , then $d(y_1, w_i) = d(y_2, w_i) = d(y_3, w_i) = 1$, $1 \leq i \leq k - 2$. Thus $r(y_1|W) = r(y_2|W) = r(y_3|W)$, which is a contradiction. Therefore at least two vertices of U have the same representations, which is a contradiction. Similarly, if more than one vertex of W do not in U , then we can prove that $tr(P_n^k) \geq k$. Thus $tr(P_n^k) \geq k$ and hence $tr(P_n^k) = k$. □

Open Problem 3.10. *If G is a connected graph of order $n \geq 3$ and d is the diameter of G , then characterize G for which $tr(G^k) = k$, $2 \leq k \leq d$.*

Acknowledgement

The research work of the second author is supported by the University Grants Commission, New Delhi through Basic Science Research Fellowship (vide Sanction No.F.7-201/2007(BSR)).

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