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Vizing's Weaker Conjecture $(\delta, \Delta) = (7, 15)$

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Abstract: Vizing conjectured that G is a simple and Δ-critical graph with m edges then $2m \geq \Delta^2$. In this paper, we prove, the conjecture for graphs with $\delta = 7$ and $\Delta = 15$.

Keywords: Critical graphs, degree sequence.

1. Introduction

Throughout this paper, $G = (V, E)$ is a graph with n vertices, m edges, maximum degree of a vertex v in G. Let n_j be the number of vertices of degree j in G. We use $\pi(G)$ to denote the valence list of G. Note $n_j = 0$, then the factor j^{nj} customary omitted $\pi(G)$. If S, T denotes the set of major and minor vertices G respectively, and $\|[S, T]\|$ denotes the sets of edges in G with one end in S and the other end in T. Also $\|[T]\|$ denotes the number of edges in T. $s(G)$ denotes the sum of degrees of minor vertices in G. Let $c(G)$ denotes the closure of G, then C denotes the Hamilton cycle of G. A well known theorem of Vizing [8] states that: if G is a simple graph with maximum degree Δ, then the edge chromatic number $X'(G)$ of G is Δ or $\Delta + 1$. A graph G is said to be of Class 1 if $X'(G) = \Delta$ and it is said to be of Class 2 if $X'(G) = \Delta + 1$. G is said to be (edge chromatic) critical if it is connected, class 2 and $X'(G - e) < X'(G)$ for every edge e. A critical graph G with maximum degree Δ is called Δ-critical.

Conjecture [7]: If G is a Δ-critical graph with n vertices m edges and maximum degree Δ then $m > \frac{1}{2}(n(\Delta - 1) + 3)$. Recognizing that conjecture is probably difficult to settle, Vizing remarks that he is enable to settle the simple problem. Is it true if G is simple and Δ-critical then $m \geq frac\Delta^2 2?$. We refer this problem as the Vizings weaker conjecture. K. Kayathri [3] proved this conjecture for graphs with $2 \leq \delta \leq 5$. M. Santhi [6] proved this conjecture for graphs with $\delta = 6$. In the following results we study the structure of 14-critical graphs with $\delta = 7$ and $2m < \Delta^2$.

2. Known Results

To prove our result, we require the following preliminary results and their consequences.

R1 [7]: Vizings Adjacency Lemma (VAL). In a Δ-critical graph G, if vw is an edge and $d(v) = k$, then w is adjacent with atleast $\Delta - k + 1$ other vertices of degree Δ.

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R2 [2]: A graph G with order $2s+1$ and maximum degree $2s-1$ is in class 2 if and only if it has size at least $2s^2 - s + 1$.

R3 [1]: A graph G with order $2s+2$ and maximum degree $2s-1$ is in class 2 if and only if $q(G) - \delta(G) \geq 2s^2 - s + 1$, where $q(G)$ denotes the size of G.

R4 [4]: If G has order $2s$ and maximum degree $2s-1$ then G is in class 2. If G has order $2s+1$ and maximum degree $2s$ then $G4$ is in class 2 if and only if the size of G is at least $2s^2+1$.

R5 [5]: There are no critical graphs with order $2s+2$ and maximum degree $2s$.

R6 [6]: Let G be Δ-critical graphs with $n = \Delta+1$ or $\Delta+2$. Then $2m \geq \Delta^2$.

R7 [6]: Let G be Δ-critical graphs with $n = \Delta+3$ and Δ odd. Then $2m \geq \Delta^2$.

R8 [6]: Let G be graph with $n_\Delta \leq \Delta+1$. If $s(G) = n_\Delta(\Delta n_\Delta + 1) + 2k$, then $\|[T]\| \leq k$ and $d(v) \leq n_\Delta + k$ for all $v \in T$.

R9 [6]: Let G be graph with $n_\Delta \leq \Delta+1$. If $s(G) = n_\Delta(\Delta n_\Delta + 1)$ then

(i) $\|[T]\| = 0$

(ii) $\|[S,T]\| = n_\Delta(\Delta n_\Delta + 1)$ and

(iii) Every vertex in S has exactly $\Delta n_\Delta + 1$ neighbours in T.

R10 [6]: Let G be Δ-critical graphs with $n_\delta \geq n - \delta + 2$ and $s(G) = n_\Delta(\Delta - n_\Delta + 1) + 2k$. Then $[T] = k$ and $d(V) \leq n_\Delta + k$ for all $v \in T$.

R11 [6]: Let G be Δ-critical graphs with $n_\Delta = \Delta - 4$ and $2m < \Delta^2$, then

(i) $5\Delta - 20 \leq s(G) < 4\Delta$.

(ii) $0 \leq \|[T]\| \leq 2$ and

(iii) $s(G)$ and Δ are of same parity.

R12 [3]: Let G be Δ-critical graphs with $n_\Delta = (\Delta - \delta + 2) + l$, where $1 \geq 0$. If $\Delta \geq (\delta - 1 - l)(\delta - 2 - l)$ then $2m \geq \Delta^2$.

If G is a Δ-critical graphs with $\delta = 7$, then by VAL $n_\Delta \geq (\Delta - \delta + 2) = \Delta - 5$.

3. Theorems

Lemma 3.1. *Let G be a 15-critical graph with $\delta = 7$ and $n_\Delta = \Delta - 3$. Then $2m \geq \Delta^2$.*

Proof. By R6 and R7 it is enough to verify the result when $n_\Delta = \Delta - 4$. By VAL $n_\Delta = \Delta - \delta + 2$. Let $n_\Delta = \Delta - \delta + 2 + l$ where $l \geq 0$. Now $n_\Delta = \Delta - 3$ and $\delta = 7$ implies that $l \geq 2$. When $l \geq 3$,

$$\begin{aligned}(\delta - 1 - l)(\delta - 2 - l) &= (6 - l)(5 - l)\\ &\geq 3 \times 2 = 6\\ &< \Delta\end{aligned}$$

and hence by R12, $2m \geq \Delta^2$. When $l = 2$, $(\delta - 1 - l)(\delta - 2 - l) = 4 \times 2 = 8 < \Delta$ and by R7, $2m \geq \Delta^2$ if $\Delta \geq 12$. Since $n_\Delta = \Delta - 3$, $n_\Delta \geq \Delta + 3$ and

$$2m \geq (\Delta - \delta)n_\Delta + \delta_n$$

$$\geq (\Delta-7)(\Delta+3)+7(\Delta+4)$$
$$\geq \Delta^2-3\Delta+49$$
$$\geq \Delta^2 \quad if \quad \Delta \leq 16$$

Hence the Result □

Lemma 3.2. *Let G be a Δ-critical graph with $\delta = 7$, $n_\Delta = \Delta - 4$, $n = \Delta + r$, $r \geq 3$. Then $2m \geq \Delta^2$ if $4\Delta \leq 28 + 7r$.*

Proof.

$$\begin{aligned} 2m \geq & \ \Delta n_\Delta + 7(n - n_\Delta) \\ \geq & \ \Delta(\Delta-4) + 7(\Delta + r - \Delta + 4) \\ \geq & \ \Delta^2 - 4\Delta + 28 + 7r \end{aligned}$$

Thus, $2m \geq \Delta^2$ if $4\Delta \leq 28 + 7r$. □

Lemma 3.3. *let G be a Δ-critical graph with $\delta = 7$, $n_\Delta = \Delta - 5$, $n = \Delta + r$ and $r \geq 3$. Then $2m \geq \Delta^2$. Then if $5\Delta \leq 35 + 7r$.*

Proof.

$$\begin{aligned} 2m &\geq \Delta n_\Delta + 7(n - n_\Delta) \\ &\geq \Delta(\Delta-5) + 7(\Delta + r - \Delta + 4) \\ &\geq \Delta^2 - 5\Delta + 35 + 7r \end{aligned}$$

Thus, $2m \geq \Delta^2$ if $4\Delta \leq 35 + 7r$. □

Lemma 3.4. *let G be a 15-critical graph with $\delta = 7$, $\Delta = 15$ and $2m \geq \Delta$. Then*

(1) $n_\Delta = \Delta - 4$ and $\Delta - 5$.

(2) $n_\Delta = \Delta + r, r = 3, 4$

(3) $0 \leq \|[T]\| \leq 2$

(4) $s(G)$ = 55 or 57 or 59.

Proof.

(1) By VAL $n_\Delta \geq \Delta - \delta + 2 \geq \Delta - 5$. Also by lemma 1, when $n_\Delta = \Delta - 4$ and $\Delta - 5$, $n_\Delta < \Delta - 3$. Hence $n_\Delta = \Delta - 4$ and $\Delta - 5$.

(2) Let $n = \Delta + r$, $r = 3$. By lemma 2, $2m \geq \Delta^2$ if $4\Delta \leq 28 + 7r$. Hence for 15, if $r \geq 5$ we have $2m \geq \Delta^2$. Also by R6 and R7 $2m \geq \Delta^2$ if $n = \Delta + 1$ and $\Delta + 2$.

(3) By R11, $0 \leq \|[T]\| \leq \frac{20-5}{2}$ and so $0 \leq \|[T]\| \leq 2$.

(4) By R11, $s(G)$ is odd. Also $6\Delta 20 \leq s(G) < 4s$ and so $55 \leq s(G) < 60$. □

Lemma 3.5. *If G is a Δ-critical graph with $\Delta = 15$, $\delta = 7$, $n = \Delta + 4$ and $n_\Delta = \Delta - 4$ then $2m \geq \Delta^2$.*

Proof. If possible let G be a Δ-critical graph with $\Delta = 15$, $\delta = 7$, $n = \Delta + 4$, $n_\Delta = \Delta - 4$ and $2m \geq \Delta^2$. Then by R11 $55 \leq s(G) < 60$ and $s(G)$ is odd and so $s(G)$ is 55, 57 or 59. Now

$$s(G) \geq \delta n_\delta + (\delta + 1(n - n_\Delta - n_\delta)\Delta$$
$$\geq (\delta + 1)(n - n_\Delta) - n_\delta$$

For $1 \leq n_7 \leq 4$, $s(G) \geq (7+1) + (19-11) - 4 \geq 60$, a contradiction. Now for $5 \leq n_7 \leq 8$, the possible degree sequences of G are as follows:

i)	7^5	8^3	15^{11}	
ii)	7^6	8	9	15^{11}
iii)	7^7	8	15^{11}	
iv)	7^7	10	15^{11}	

In all the cases, we get a contradiction in the following Lemmas (Lemma 3.5 and Lemma 3.6). Hence the Lemma. □

Lemma 3.6. *If G is a Δ-critical graph with $\Delta = 15$, $\Delta = 7$, $n = \Delta + 4$, $b_\Delta = \Delta + 4$, $s(G) = 57$, then $2m \geq \Delta^2$.*

Proof. Assume the contrary that $2m < \Delta^2$. Then the only possible degree sequence is $\pi(G) = 7^7 \;\; 8 \;\; 15^{11}$, given in Lemma 3.4. But by $\|[T]\| \leq 1$. But by VAL, $\|[T]\| = 0$. Now, $\|[S, T]\| = 57$ if $\|[T]\| = 0$. Then in G, one of the following two cases arises:

(i) $\|[T]\| = 0$, two major vertices have 6 minor neighbours and 9 major vertices have 5 minor neighbours.

(ii) $\|[T]\| = 0$, one major vertex has 7 minor neighbours and 10 major vertices have 5 minor neighbours.

Now $\pi(G) = 7^5 \;\; 8 \;\; ‘5^{11}$. Let v_1 be a vertex of degree 7. Let D be a subset of $T\{v_1\}$ with$\|D\| = 7$. Let $D' = T/D$. Then $\|D'\| = 1$ and $v_1 \in D'$. Let $D = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ and $D' = \{v_1\}$. We shall fix the degrees of u_is and v_is accordingly. Let $G_1 = G/D$. Now $\|v(G_1)\| = 13$. Then $\Delta(G_1) \leq 12$. The number of vertices of degree is in G_1. Since we have deleted 6 vertices from G and $\Delta(G) = 15$, the major vertices in G are of degree ≥ 9 in G_1. Hence

$$n'_8 + n'_9 + n'_{10} + n'_{11} + n'_{12} \geq n_\Delta(G) = 11$$
$$\|V_1\| = n'_8 + n'_9 + n'_{10} + n'_{11} + n'_{12} \geq n_\Delta(G) = 11$$

Moreover $v_1 \in G_1$ and $d_{G_1}(v_1) = 7$. Let $v_1 = \{v \in V(G_1) : d_{G_1}(v) \geq 8\}$ and $v_2 = \{v \in V(G_1) : d_{G_1}(v) < 8\}$. Then $\|V_1\| = n'_8 + n'_9 + n'_{10} + n'_{11} + n'_{12} \geq n_\Delta(G) = 11$. Let $v \in V_1$. Now for all $w \in V(G_1)$; $d_{G_1}(v) + d_{G_1}(w) \geq 8 + 6 = 14 \geq \|V(G_1)\|$. So in the closure of $c(G_1)$, every $v \in V_1$ is adjacent with every other vertex in G_1. Moreover for all $u \in V_2$, $d_{c(G_1)}(u) \geq \|V_1\| \geq n_\Delta(G) = 1$. So, for every pair of vertices u and w in V_2, $d_{c(G_1)}(u) + d_{c(G_1)}(w) \geq 11 + 11 = 22 > \|V(G_1)\|$. So, $c(G_1)$ is complete and hence G_1 is Hamiltonian. Let C be a Hamiltonian cycle of G_1. Let $G' = G/E(C)$. Since G is of class 2, G' is also of class 2. Also

$$\left.\begin{aligned} d_g(U) &= d_G(u) \;\; \textit{for } u \in D \;\; \textit{and} \\ d_{G'}(v) &= d_g(v)2 \end{aligned}\right\} \tag{1}$$

In particular $d_{G'}(V_1) = 5$ and so $\delta(G_1) = 5$. Also $\Delta(G') = \Delta(G) - 2 = 13$. Let H be a 13-critical subgraph of G'. Let h_i'' denote the number of vertices of degree is in H. Let $Sprime$, T' respectively denote the set of major and minor vertices in H. We have $\|S' \leq n_\Delta(G) = 11$. By VAL, $n_\Delta \geq \Delta S + 2$. We have

$$\begin{aligned}\delta(H) &\geq \Delta(H) - \|S'\| + 2\\ &\geq 13 - 11 + 2\\ &\geq 4\\ \delta(H) &= 5\end{aligned}$$

Now $\|S'\| = 11$. Note that $d_H(v_i) = 5$ and so v_1 has 5 major neighbourhood in Cases (i) and (ii). Then in

$$\left.\begin{array}{ll}(i) & \|[S', T']\| \leq (5 \times 4) + (4 \times 5) + (2 \times 6) = 52\\ (ii) & \|[S', T']\| \leq (5 \times 4) + (5 \times 5) + (1 \times 7) = 52\end{array}\right\} \tag{2}$$

Since $\delta(H) = 5$, it follows that $H = G'/E_1$, where $\boldsymbol{E_1} \subseteq [\boldsymbol{T}]_{\boldsymbol{G'}}$, and no edge in E1 is incident with vertices of degree 13 or 4. While removing C from G, we have removed only two edges from $[T, S]$ (two edges incident with one minor vertex in D'). So,

$$\begin{aligned}s(G') &= s(G)(1 \times 2)\\ &= 57 - 2 = 55\end{aligned}$$

Then $s(H) \geq s(G')2\|E_1|$. Now,

$$\begin{aligned}\|[S', T']\| &= s(H) - 2\|[T']\|\\ &= 55 - 2(\|E_1\| + \|[T']\|)\\ &\geq 55 - 2 \geq 53 \quad contradicting \quad (2)\\ \delta &\geq 6\end{aligned}$$

Now $n_5(H) = 0$

$$\left.\begin{array}{lll}n_5(H) = 0 \Rightarrow & H \subseteq G'/v_1 & (where\ d(v_i) = 5)\\ \Rightarrow & \|S'\| \leq 11 - 4 = 7 & (since\ v_1\ has\ atleast\ 4\ major\ neighbours\ in\ G')\\ \Rightarrow & \delta(H) \geq & (using\ VAL)\end{array}\right\} \tag{3}$$

Let u_1 be a vertex of degree 7 that has only major neighbours in G. Then $d_{G'}(u_1) = 7$. Now by (3), $\delta(H) \geq 8$

$$\begin{aligned}\delta(H) \geq 8 &\Rightarrow H \subseteq G'/u_1\\ &\Rightarrow \|S'\| \leq 11 - 7 = 4\\ &\Rightarrow 11\end{aligned}$$

We note that $n_{10} + n_{11} = 0$ in G. Hence $n_8'' + n_9'' \leq n_{11} + n_{12} + n_{13} + n_{14} + n_{15} = 11$. So, $\|V(H)\| \leq 11 n_{11}$ is a contradiction. This completes the proof. $\square$

Lemma 3.7. *If G is an Δ-critical graph with $\Delta = 15$, $\delta = 7$, $n = \Delta + 45$, $n_\Delta = \Delta - 4$ and $s(G) = 59$ then $2m \geq \Delta^2$.*

Proof. Assume the contrary that $2m < \Delta^2$. Then the possible degree sequences are

i)	7^5	8^3	15^{11}		
ii)	7^6	8	9	15^{11}	
iii)	7^7	10	15^{11}		given in Lemma 3.4.

By R8, $\|[T]\| \leq 2$. But by VAL, $\|[T]\| \leq 1$. Now

$$\|[S,T]\| = \begin{cases} 59, & if \quad \|[T]\| = 0 \\ 57, & if \quad \|[T]\| = 1 \end{cases}$$

Then in G, one of the following five cases arises:

(i) $\|[T]\| = 0$, four major vertices have 6 minor neighbours and seven major vertices have 5 minor neighbours.

(ii) $\|[T]\| = 0$, two major vertices have 6 minor neighbours and one major vertex have seven minor neighbours and 8 major vertices have 5 minor neighbours.

(iii) $\|[T]\| = 0$, two major vertices have 7 minor neighbours and nine major vertices have 5 minor neighbours.

(iv) $\|[T]\| = 1$, two major vertices have 6 minor neighbours and nine major vertices have 5 minor neighbours.

(v) $\|[T]\| = 0$, one major vertex has 7 minor neighbours and two major vertices have 5 minor neighbours and 8 major vertices have 5 minor neighbours.

Also $d(v) \leq n_\Delta + 1 = 12$ for all $v \in T$. Let

$$\|[S,T]\| = \begin{cases} 7^5 \quad 8^3 \quad 15^{11} \\ 7^6 \quad 8^3 \quad 9 \qquad 15^{11} \\ 7^7 \quad 10 \quad 15^{11} \end{cases}$$

Let v_1 be a vertex of degree 7. Let D be a subset of $T/\{v_1\}$ with $\|D\| = 6$. Let $D' = T/D$. Then $\|D'\| = 2$ and $v_1 \in D'$. Let $D = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $D' = \{v_1, v_2\}$. We shall fix the degree of u_is, v_is accordingly. Let $G_1 = /G/D$. Now $\|V(G_1)\| = 13$. Then $\Delta(G_1) \leq 12$. Since $\delta(G) = 7$ and $\|[T]\| \leq 1$, we have $\delta(G_1) = 6$. Let $V_1 = \{v \in V(G_1) : d_{G_1}(v) \geq 9\}$, $V_2 = \{v \in V(G_1) : d_{G_1})(v) < 9\}$. As in Lemma 3.6, we can check that G_1 is hamiltonian. Let c be a hamitonian cycle of G_1. Let $G' = G/E(C)$. Since G is of class 2, G' is of class 2. Let S', T' be defined as in Lemma 3.6. Then $\|S'\| \leq 11$ and $\delta(H) \geq 4 \Rightarrow \delta(H) = 5$. Now $\|S'\| = 11$. Then $V(H) = V(G')$ and $d_H(v_1) = 5$. Now we consider the Cases (i) to (v). Note that $d_H(v_1) = 5$. And so in H, V_1 has 4 major neighbourhood in Case (i) and (iii) and has at least 3 major neighbourhood in Cases (iv) and (v). Then in

$$\left.\begin{array}{rl} (i) & \|[S',T']\| \leq (4 \times 3) + (4 \times 6) + (3 \times 5) = 51 \\ (ii) & \|[S',T']\| \leq (4 \times 3) + (2 \times 6) + (1 \times 7) + (4 \times 5) = 51 \\ (iii) & \|[S',T']\| \leq (4 \times 3) + (2 \times 7) + (5 \times 5) = 51 \\ (iv) & \|[S',T']\| \leq (3 \times 3) + (2 \times 6) + (6 \times 5) = 51 \\ (v) & \|[S',T']\| \leq (4 \times 3) + (2 \times 7) + (5 \times 5) = 51 \end{array}\right\} \tag{4}$$

But in all cases,

$$\begin{aligned} s(G') &= s(G) - (2x2) \\ &= 59 - 4 \\ &= 55 \end{aligned}$$

Also $\delta(H) = 5$ and so $H = G'/E$, where $E_1 \subseteq [T]_{G'}$, and no edge in E_1 is incident with vertices of degree 13 or 5 (Then $\|E_1\| \leq 1$). Now in $\|[T']\| = \|[T]_{G'}\| \leq 1$. Then

$$\begin{aligned} s(H) &= s(G')2\|E_1\| \\ &= 55 - 2\|E_1\| \end{aligned}$$

Now in (i)-(iii), $H = G'$ and $\|[T', S']\| = 55$. In (iv) and (v) $\|[T', S']\| \geq 55 - 2 = 53$ contradicting to (1), $\delta(H) \geq 6$. Then $H \subseteq G'/v_1$ where $d(v_1) = 5$. Now

$$\begin{aligned} n_5(H) = 0 \ &\Rightarrow \|S'\| \leq 11 - 3 = && (since\ v_1\ has\ atleast\ 3 major\ neighbours\ in\ G). \\ &\Rightarrow \delta(H) \geq 7 && (using\ VAL) \end{aligned}$$

Now we have three possible degree sequences:

i)	7^6	8	9	15^{11}
ii)	7^5	8^4	5^{11}	
iii)	7^7	10	15^{11}	

Let

$$d(v_2) = \begin{cases} 8 & in\ (i)\ and\ (ii) \\ 10 & in\ (iii) \end{cases}$$

Then

$$d_{G'}(v_2) = \begin{cases} 6 & in\ (i)\ and\ (ii) \\ 8 & in\ (iii) \end{cases}$$

Also by VAL, v_2 has at most one minor neighbours in G.

$$\begin{aligned} \delta(H) \geq 7 \Rightarrow &\qquad H \subseteq G'/v_2 \\ &\Rightarrow \|S'\| \leq \begin{cases} 11 - 5 & in\ (i)\ and\ (ii) \\ 11 - 7 & in\ (iii) \end{cases} \\ &\Rightarrow \|S'\| \leq \begin{cases} 11 - 5 & in\ (i)\ and\ (ii) \\ 11 - 7 & in\ (iii) \end{cases} \\ &\Rightarrow \|S'\| \leq \begin{cases} 6 & in\ (i)\ and\ (ii) \\ 11 & in\ (iii) \end{cases} \\ &\Rightarrow \|S'\| \leq \begin{cases} 9 & in\ (i)\ and\ (ii) \\ 11 & in\ (iii) \end{cases} \\ &\Rightarrow \|\delta(H)\| \leq 11, \ a\ contradiction. \end{aligned}$$

This completes the proof. $\square$

Theorem 3.8. *If G is a 15-critical graph with $\delta = 7$ and $n_\Delta = \Delta 4$, then $2m \geq \Delta^2$.*

Proof. By Lemma 3.5 and Lemma 3.6, we get the result. □

Theorem 3.9. *If G is a 15-critical graph with $\delta = 7$ and $n_\Delta = \Delta 5$, then $2m \geq \Delta^2$.*

Proof. By R11, $55 \leq s(G) \leq 60$ and $s(G)$ is odd and so $s(G)$ is 55 or 57 or 59. Since $n_\Delta = (\Delta - 5, n_\Delta = 11)$ is odd and also $s(G)$ is odd, in the possible degree sequences the number of odd vertices is odd. It is impossible. Hence the theorem. □

Proof of the Main Theorem:

Theorem 3.10. *If G is a 15-critical graph with $\delta = 7$ then $2m \geq \Delta^2$.*

Proof. By VAL, $n_\Delta \geq \Delta 5 + 2 \geq \Delta - 5$. By R6, R7 and Lemma 3.1, it is enough to verify the result when $n_\Delta \geq \Delta + 4$ and $n_\Delta = \Delta 4$ and $\Delta - 5$. By Theorem 3.8 and 3.9, the main theorem follows. □

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