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L-Fuzzy Almost Subgroups

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Abstract: Algebraic structures like fuzzy sub groups of L-Fuzzy sets have been well studied in the case where the lattice L is distributive. By slightly relaxing the relevant axioms the concepts can be generalised to the case where the lattice is not distributive. The aim of this current paper is to define the concept of L-Fuzzy Almost Subgroup (LFASG) and investigate some of its properties.

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1. Introduction

Zadeh [7] introduced the concept of fuzzy sets in 1965. The theory of L-fuzzy sets was initiated by J.A.Goguen [1]. A.Rosenfeld [2] has applied the cocept of fuzzy sets to the theory of groups. In [6] Wang-Jin Liu studied fuzzy subgroups and fuzzy ideals in 1982. In [3] P. Sivaramakrishna Das investigated Fuzzy Groups and Level Subgroups. In our previous papers [4] and [5], the concept of L-Fuzzy Almost Ideal (LFAI) was introduced and concept of primality in LFAI was studied. L-Fuzzy Almost Ideal is a generalisation of L-fuzzy ideal and L-Fuzzy Almost Ideal are defined even when the lattice L is not necessarily distributive. The aim of this current paper is to define the LFASG which is a generalisation of L-fuzzy subgroup and investigate some of its properties.

2. Preliminaries

Let X be a nonempty subset, $(L, \leq, \vee, \wedge)$ be a complete distributive lattice, which has least and greatest elements, say 0 and 1 respectively. Relevant definitions are recalled in this section.

Definition 2.1. *Let X be a nonempty set. A mapping $\mu : X \to [0, 1]$ is called a fuzzy subset of X.*

Definition 2.2. *Let X be a nonempty set. A mapping $\mu : X \to L$ is called a L-fuzzy subset of X.*

Definition 2.3. *let μ be any fuzzy subset of a set X, $t \in [0, 1]$. Then the set $\mu_t = \{x \in X/\mu(x) \geq t\}$ is called a level set of μ. More generally if μ is L-fuzzy set defined by $\mu : X \to L$ then the set $\mu_t = \{x \in X/\mu(x) \geq t\}$ is called a level set of μ.*

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Notation 2.4. *Consider $\mu : X \to L$. If L is totally ordered then for all $x, y \in R$, $\mu(x)$ and $\mu(y)$ are comparable. That is either $\mu(x) > \mu(y)$ or $\mu(x) = \mu(y)$ or $\mu(x) < \mu(y)$. But if L is not totally ordered then are four possibilities $\mu(x) > \mu(y)$ or $\mu(x) = \mu(y)$ or $\mu(x) > \mu(y)$ or $\mu(x)$ and $\mu(y)$ are not comparable. We use the notation $\mu(x) \not< \mu(y)$ to mean, $\mu(x) > \mu(y)$ or $\mu(x) = \mu(y)$ or $\mu(x)$ and $\mu(y)$ are not comparable.*

Definition 2.5. *AL-fuzzy subset μ of a group G is called a L-fuzzy subgroup of G if, for all x, y in G, the following conditions are satisfied:*

(1). $\mu(xy) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$.

(2). $\mu(x^{-1}) \geq \mu(x)$ for all $x, y \in G$.

Definition 2.6 ([4])**.** *If A is a L-fuzzy set of X defined by $\mu : X \to L$ then the Almost level set A_t is defined as $\approx A_t = \{x \in X | \mu(x) \not< t\}$.*

Proposition 2.7 ([4])**.** *If $t = s$, $\approx A_t \supseteq \approx A_s$.*

Definition 2.8. *Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Then the composition of the function f and g is denoted by $g \circ f$ is a function from X to Z and is defined by $(g \circ f)(x) = g[f(x)]$.*

Note: Generally composition of functions is not commutative.

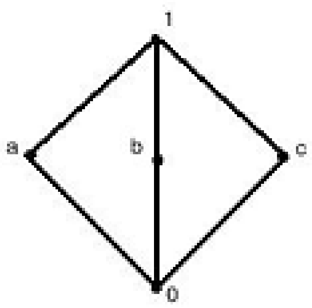
3. L-Fuzzy Almost Subgroups

Definition 3.1. *Let G be a group with the identity element e. Let L be a lattice $(L, \leq, \bigvee, \bigwedge)$ not necessarily distributive with least and greatest element 0 and 1 respectively. Then L-fuzzy subset H defined by $\mu : G \to L$ with $\mu(e) = 1$ is said to be L-fuzzy almost subgroup (LFASG) if,*

(1). $\mu(xy) \not< \mu(x) \wedge \mu(y)$ for all $x, y \in G$.

(2). $\mu(x^{-1}) \not< \mu(x)$ for all $x, y \in G$.

Example 3.2. *Let $G = \{1, 2, 3, 4, 5, 6\}$ be the group under the operation multiplication modulo 7. L is defined by the following Hasse diagram.*



$$\text{If } \mu(x) = \begin{cases} 1; & \text{if } x = 1 \\ a; & \text{if } x = 2, 4 \\ b; & \text{if } x = 5 \\ c; & \text{if } x = 3, 6 \end{cases} \quad \text{then it can be verified that } \mu \text{ defines an LFASG.}$$

x \ y	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

Table 1. **Multiplication table of the group**

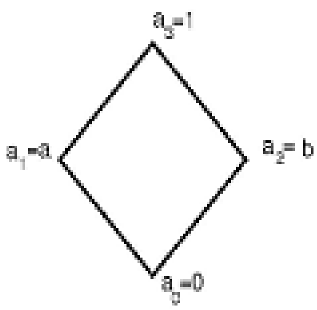
x \ y		1	2	3	4	5	6
1	$\mu(x)\wedge\mu(y)$	$1\wedge 1=1$	$1\wedge a=a$	$1\wedge c=c$	$1\wedge a=a$	$1\wedge b=b$	$1\wedge c=c$
	$\mu(xy)$	1	a	c	a	b	c
2	$\mu(x)\wedge\mu(y)$	$a\wedge 1=a$	$a\wedge a=a$	$a\wedge c=0$	$a\wedge a=a$	$a\wedge b=0$	$a\wedge c=0$
	$\mu(xy)$	a	a	c	1	c	b
3	$\mu(x)\wedge\mu(y)$	$c\wedge 1=c$	$c\wedge a=0$	$c\wedge c=c$	$c\wedge a=0$	$c\wedge b=0$	$c\wedge c=c$
	$\mu(xy)$	c	c	a	b	1	a
4	$\mu(x)\wedge\mu(y)$	$a\wedge 1=a$	$a\wedge a=a$	$a\wedge c=0$	$a\wedge a=a$	$a\wedge b=0$	$a\wedge c=0$
	$\mu(xy)$	a	1	b	a	c	c
5	$\mu(x)\wedge\mu(y)$	$b\wedge 1=b$	$b\wedge a=0$	$b\wedge c=0$	$b\wedge a=0$	$b\wedge b=b$	$b\wedge c=0$
	$\mu(xy)$	b	c	1	c	a	a
6	$\mu(x)\wedge\mu(y)$	$c\wedge 1=c$	$c\wedge a=0$	$c\wedge c=c$	$c\wedge a=0$	$c\wedge b=0$	$c\wedge c=c$
	$\mu(xy)$	c	b	a	c	a	1

Table 2. **Axiom (i) verification**

x	$\mu(x)$	$\mu(x^{-1})$
1	1	1
2	a	a
3	c	b
4	a	a
5	b	c
6	c	c

Table 3. **Axiom (ii) verification**

Example 3.3. *Let $G=\{Z,+\}$ be the group. L is defined by the following Hasse diagram.*



$$\text{If } \mu(x)=\begin{cases} a; & \text{if } x\in 2Z-6Z \\ b; & \text{if } x\in 3Z-6Z \\ 1; & \text{if } x\in 6Z \\ 0; & \text{if } x\in Z-2Z-3Z \end{cases}$$ *then it can be verified that μ defines an LFASG.*

Note: The level sets of LFSAG need not be subgroups. However we can say

Theorem 3.4. *Let L-fuzzy subset H defined by $\mu : G \to L$ be a LFASG. Then $H_1 = \{x \,|\mu (x) = 1\}$ is a subgroup of G.*

Proof.

(i). If $x, y \in H_1$ then to prove $(xy) \in H_1$. If $x, y \in H_1$ then $\mu(x) = 1$ and $\mu(y) = 1$. Since $\mu(xy) \nless \mu(x) \wedge \mu(y) = 1 \wedge 1$, $\mu(xy)$ must be 1. Hence $(xy) \in H_1$.

(ii). If $x \in H_1$ then to show $x^{-1} \in H_1$. If $x \in H_1$ then $\mu(x) = 1$. Since $\mu(x^{-1}) \nless \mu(x) = 1$, $\mu(x^{-1})$ must be 1. Hence $x^{-1} \in H_1$. Hence H_1 is a subgroup of G. □

4. Almost Level Sets and LFASG

Theorem 4.1. *Let G be a group and $(L, \vee, \wedge, =)$ be a lattice not necessarily distributive with least and greatest element 0 and 1 respectively. Let H be defined by $\mu : G \to L$. If for all $t \in Im(\mu)$, H_t is a subgroup of G then H is a L-fuzzy almost subgroup.*

Proof. Let G be a group and L be a lattice not necessarily distributive. Let H be defined by $\mu : G \to L$. If for all $t \in Im(\mu)$, $\approx H_t$ is a subgroup of G. Suppose for all $t \in Im(\mu)$, $\approx H_t$ is a subgroup. Let $x, y \in G$. Let $\mu(x) = t$ and $\mu(y) = s$. So $x \in\approx H_t$ and $y \in\approx H_t$.

Case 1: If t and s are comparable

Without loss of generality say $t = s$. Then by Proposition 4.2 $\approx H_t \supseteq\approx H_t$. So $y \in\approx H_t$. Since x and y belong to $\approx H_t$ and $\approx H_t$ is a subgroup of G, $xy \in\approx H_t$ and $x^{-1} \in\approx H_t$. Therefore $\mu(xy) \nless t$. But $\mu(x) \wedge \mu(y) = t \wedge s = t$. So

$$\mu(xy) \nless \mu(x) \wedge \mu(y) \tag{1}$$

Since $\approx H_t$ is a subgroup of G, $x^{-1} \in\approx H_t$. So

$$\mu(x^{-1}) \nless t = \mu(x) \tag{2}$$

Case 2: If t and s are not comparable

Since t and s are not comparable, $\mu(x) = t$ means $\mu(x) \nless s$. So $x \in H_s$. We already know that $x \in H_t$. So $x \in (H_s) \cap (H_t)$. Similarly $y \in (H_s) \cap (H_t) (H_s) \cap (H_t)$ is a subgroup of G, since it is the intersection of two subgroup. Therefore $xy \in (H_s) \cap (H_t)$. It means $\mu(xy) \nless s$ and $\mu(xy) \nless t$. So

$$\mu(xy) \nless \mu(x) \wedge \mu(y) \tag{3}$$

Since H_t is a subgroup of G, $x^{-1} \in H_t$. So

$$\mu(x^{-1}) \nless t = \mu(x) \tag{4}$$

In both cases the two conditions for L-fuzzy almost subgroup are satisfied. Hence H is a L-fuzzy almost subgroup. Converse is also true. □

5. Homomorphisms and LFASG

Theorem 5.1. *Let $: G_1 \to G_2$ be a group homomorphism. Let $H_2 : G_1 \to L$ be a L -fuzzy almost subgroup of G_2. Then $H_2 \circ f : G_1 \to L$ is an L-fuzzy almost subgroup of G.*

Proof. Let $x, y \in G_1$. Then

(i). $H_2 \circ f(xy) = H_2(f(xy))$

$= H_2(f(x) * f(y))$

$\not< H_2(f(x)) \wedge H_2(f(y))$

(ii). $H_2 \circ f(x^{-1}) = H_2(f(x^{-1}))$

$= H_2(f(x)^{-1})$

$\not< H_2(f(x))$

□

Definition 5.2. *Let* $: G_1 \to L$ *be a L-fuzzy subset of* G_1. *Also* $f : G_1 \to G_2$ *be a group homomorphism. Then* μ *is said to be f-invariant if whenever* $f(x) = f(y)$ *we have* $\mu(x) = \mu(y)$.

Theorem 5.3. *Let* $\mu : G_1 \to G_2$ *be a group epimorphism. Let* $\mu : G_1 \to L$ *define a L-fuzzy almost subgroup of* G_1. *If* μ_1 *is f-invariant then let* $\mu_2 : G_2 \to L$ *be defined by* $\mu_2(g) = \mu_1(x)$ *where* $x \in f^{-1}(g)$. *Then* μ_2 *is L-fuzzy almost subgroup of* G_2.

Proof. First we check that μ_2 is well defined. Suppose $x, y \in f^{-1}(g)$. So $f(x) = f(y) = g$. Since μ_1 is f-invariant we have $\mu_1(x) = \mu_1(y)$. Hence μ_2 is well defined. The verification of necessary axioms is straightforward. □

Theorem 5.4. *Let* $\mu : G \to L_1$ *be a L -fuzzy almost subgroup of* G *and* $f : L_1 \to L_2$ *be a lattice homomorphism. Then* $f \circ \mu : G \to L_2$ *is a L-fuzzy almost subgroup of* G.

Proof. Let $x, y \in G_1$. Then

(i). $f \circ \mu(xy) = f(\mu(xy))$

$\not< f(\mu(x) \wedge \mu(y))$

$\not< f(\mu(x)) \wedge f(\mu(y))$

(ii). $f \circ \mu(x^{-1}) = f(\mu(x^{-1}))$

$= f(\mu(x)^{-1})$

$\not< f(\mu(x))$

□

6. Conclusion

In this paper we have developed the concept of L-fuzzy almost subgroup. We also investigate the effect of homomorphism on L-fuzzy almost subgroup. We have proved some theorems connecting almost level set and L-fuzzy almost subgroup.

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