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Existence Solution for Nonlinear Fractional Differential Equation by the Method of Lower and Upper Solution

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Abstract

In this paper, by the use of lower and upper solutions we prove the existence and Uniqueness of Initial Value Problem containing nonlinear fractional differential equation.

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1. Introduction

The field of fractional differential equation proves the millstone in the area like science and engineering. In current days lots of work is done on existence and uniqueness of solution of nonlinear differential equations involving fractional derivative. The work on the initial value problems for Nonlinear fractional differential equation motivate us. So we choose to investigate the existence and uniqueness of positive solution of nonlinear fractional differential equation.

$$D_a^\alpha u\left(t\right) = f\left(t, u\left(t\right)\right), \qquad u\left(t_0\right) = u_0$$

Where $f \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$ and ${}^cD_a^\alpha$ is the Caputo's fractional derivative of order α, $0 < \alpha < 1$.
To our knowledge less work was done on initial value problem with the use of lower and upper solutions. The aim of this paper is to find existence and uniqueness of positive solutions of the problem. The paper is divided into following section: Section 2 is of the preliminaries and devoted to reduction of problem to the suitable equivalent equation. Section 3 is devoted for existence and uniqueness of positive solutions of nonlinear differential equation by the use of lower and upper solution method.

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2. Preliminary Results

This section is devoted for the discussion of some basic definitions common name of useful to prove our result. Now, we state the statement for initial value problem for the nonlinear fractional differential equations

$$ {}^{c}D_{a}^{\alpha}u\left(t\right) = f\left(t, u\left(t\right)\right), \quad u\left(t_0\right) = u_0 \tag{1}$$

where $f \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$ and ${}^{c}D_{a}^{\alpha}$ is the Caputo's fractional derivative of order α, $0 < \alpha < 1$. The corresponding Volterra fractional integral equation is defined as

$$ u\left(t\right) = u_0 + \int_{t_0}^{t} \frac{1}{\Gamma\left(\alpha\right)} \left(t-s\right)^{\alpha-1} f\left(s, u\left(s\right)\right) ds $$

Definition 2.1 ([1]). *A function $\delta(t)$ is called a C_m function if $\delta \in C([t_0, T], \mathbb{R})$ and $\delta(t)(t-t_0)^m \in C([t_0, T], \mathbb{R})$ with $\alpha + m = 1$.*

Next, we define the lower and upper solutions as follows:

Definition 2.2 ([1]). *A function $r \in C_m([t_0, T], \mathbb{R})$, $\alpha + m = 1$, $0 < \alpha < 1$ is said to be a lower solution of* (1) *if*

$$ {}^{c}D_{a}^{\alpha}r\left(t\right) \le f\left(t, r\left(t\right)\right), \quad r\left(t_0\right) = u\left(t_0\right) $$

It is an upper solution if the inequalities are reversed.

Now, consider the following nohomogeneous linear fractional differential equation,

$$ {}^{c}D_{a}^{\alpha}u\left(t\right) = \kappa u\left(t\right) + f(t), \;\; u\left(t_0\right) = u_0 \tag{2}$$

Where κ is a real number and $f \in C_m([t_0, T] \times \mathbb{R}, \mathbb{R})$. The equivalent Volterra fractional integral equation for $t_0 \le t \le T$ is

$$ u\left(t\right) = u_0 + \int_{t_0}^{t} \frac{\kappa}{\Gamma\left(\alpha\right)} \left(t-s\right)^{\alpha-1} u\left(s\right) ds + \int_{t_0}^{t} \frac{1}{\Gamma\left(\alpha\right)} \left(t-s\right)^{\alpha-1} f\left(s\right) ds \tag{3}$$

When we apply the method of successive approximations [46] to find the solution $u(t) = u(t, t_0, u_0)$. Explicitly for the given nonhomogeneous IVP (1), we obtain

$$ u\left(t\right) = u_0 E_{\alpha}\left(\kappa(t-t_0)^{\alpha}\right) + \int_{t_0}^{t} \frac{1}{\Gamma\left(\alpha\right)} \left(t-s\right)^{\alpha-1} E_{\alpha,\alpha}\left(\kappa(t-t_0)^{\alpha}\right) f\left(s\right) ds \quad t \in [t_0, T] \tag{4}$$

Where

$$ E_{\alpha}\left(t\right) = \sum_{i=0}^{\infty} \frac{t^i}{G\left(\alpha i + 1\right)}, \quad E_{\alpha,\alpha}\left(t\right) = \sum_{i=0}^{\infty} \frac{t^i}{G\left(\alpha i + \alpha\right)} $$

Are Mittag-Leffler functions of one parameter and two parameters, respectively. If $f(t) \equiv 0$, we get, as the solution of the corresponding homogeneous IVP

$$u(t) = u_0 E_\alpha(\kappa(t-t_0)^\alpha), \quad t \in [t_0, T] \tag{5}$$

Remark 2.3 ([4]). *Let ${}^cD_a^\alpha u\,(t) \leq Lu\,(t)\,, u\,(t_0) = u_0, u \in C_m([t_0,T], \mathbb{R}^+)$ and L is positive constant. Then we have the estimate*

$$u(t) \leq u_0 E_\alpha(L(t-t_0)^\alpha) \quad on \quad [t_0, T] \tag{6}$$

When $\alpha = 1$, that result reduces to well known Gronwall's inequality. If $u_0 = 0$, then $u(t) = 0$ identically on $[t_0, T]$.

3. Main Results

The aim of this section is to apply the technique of quasilinearization for Nonlinear Caputo's fractional differential equation (1) together with lower and upper solutions and initial time difference without using the Hölder continuity assumption on functions involved.Also,we have weaken the condition of convexity of the function $f(t,u)$ on the right hand side of the equation (1).

Theorem 3.1 (Comparison Theorem). *Let $r, s \in C_m\,([t_0,T]\,, \mathbb{R})$, $f \in C\,([t_0,T] \times \mathbb{R}, \mathbb{R})$ and*

(i) ${}^cD_a^\alpha r\,(t) \leq f\,(t, r\,(t))$ and

(ii) ${}^cD_a^\alpha s\,(t) \geq f\,(t, s\,(t))$

$t_0 < t \leq T$. Assume $f\,(t, u\,(t))$ satisfies the Lipschitz condition $f\,(t,x) - f\,(t,y) \leq L\,(x-y)$, $x \geq y$, $L > 0$. Then $r_0 < s_0$, implies that $r\,(t) \leq s\,(t)$, $t \in [t_0, T]$.

Proof. Suppose that $r(t) \leq s(t)$ for $t_0 \leq t \leq T$ is false. Then, there exists a $t_1 > t_0$ such that $r(t_1) > s(t_1)$ and $t_0 < t_1 \leq T$. Since $r(t_0) \leq s(t_0)$ with respect to the continuity of functions involved we consider following two cases:

(i) If $r(t_0) < s(t_0)$, then there exists τ_0 such that $r(\tau_0) = s(\tau_0)$, $t_0 < \tau_0 < t_1$. Thus, we get $r(t) > s(t)$ on $(\tau_0, t_1]$.

(ii) If $r(t_0) = s(t_0)$, then two cases are possible. First $r(t) > s\,(t)$ on $(\tau_0, t_1]$ or $r(t) > s(t)$ on $(\tau_0, t_1]$, where $t_0 < \tau_0 < t_1$ and $r(\tau_0) = s(\tau_0)$ as in the first case.

In above two cases, we can obtain an interval $[t_0, t_1]$ or $[\tau_0, t_1]$ on which $r(t) \geq s(t)$. Now we define

$$u(t) = r(t) - s(t) \tag{7}$$

And consider $u(t)$ is defined on $[\tau_0, t_1]$ (or on $[t_0, t_1]$). We note that $u(t) > 0$ on $(\tau_0, t_1]$ and $u(\tau_0) = 0$. By applying Caputo's fractional derivative of both sides of (7), we have

$${}^cD_a^\alpha u\,(t) = {}^cD_a^\alpha r\,(t) - {}^cD_a^\alpha s\,(t) \tag{8}$$

Using the lower and upper properties of $r(t)$ and $s(t)$, we get

$$^{c}D_{a}^{\alpha}u\left(t\right)\leq f\left(t,r\left(t\right)\right)-f\left(t,s\left(t\right)\right) \tag{9}$$

As f is Lipschitz, $L>0$ and $r(t)\geq s(t)$ on $[\tau_0,t_1]$, we have

$$^{c}D_{a}^{\alpha}u(t)\leq L[r(t)-s(t)], \tag{10}$$

By (7), we obtain

$$^{c}D_{a}^{\alpha}u\left(t\right)\leq Lu\left(t\right),u\left(\tau_0\right)=0, \tag{11}$$

In view of Remark 2.3, which implies, $u(t)=0$ for

$$\tau_0\leq t\leq t_1 \tag{12}$$

This is a contradiction. Therefore, we have $r(t)\leq s\left(t\right)$ on $[t_0,T]$. □

Corollary 3.2 ([1]). *The function $f(t,u)=\theta(t)u$, where $\theta(t)\leq L$ is admissible in Theorem 3.1 to yield $u(t)\leq 0$ on $t_0\leq t\leq T$.*

Theorem 3.3. *Assume that*

(i) $\delta\in C_m\left[[t_0,t_0+T],\mathbb{R}\right]$, $t_0,T>0$, $\gamma\in C_m\left[[t_0,t_0+T],\mathbb{R}\right]$, $\tau_0>0$, $f\in C\left[[t_0,t_0+T]\times\mathbb{R},\mathbb{R}\right]$ *and*

$$^{c}D_{a}^{\alpha}\delta(t)\leq f(t,\delta(t)),\quad t_0\leq t\leq t_0+T$$
$$^{c}D_{a}^{\alpha}\gamma(t)\geq f(t,\gamma(t)),\quad \tau_0\leq t\leq \tau_0+T$$

with $\delta\left(t_0\right)\leq x\left(s_0\right)\leq\gamma\left(\tau_0\right)$ *and* $t_0<s_0<\tau_0$, *where* $\delta\left(t\right)\leq\gamma\left(t+\mu_1\right)$, $t_0\leq t\leq t_0+T$ *and* $\mu_1=\tau_0-t_0$;

(ii) Suppose $f_x(t,x)$ exists and following relations hold $f(t,x)\geq f(t,y)+f_x(t,y)(x-y)$, whenever $x\geq y$ and $|f_x(t,x)-f_x(t,y)|\leq L|x-y|$, $L>0$;

(iii) $f\left(t,x\right)$ is nondecreasing in x for each x and $f_x\left(t,x\right)$ is nondecreasing in x for each t.

Then there exists monotone sequences $\left\{\widetilde{\delta}_n\right\}$ and $\{\widetilde{\gamma}_n\}$ which converge uniformly and monotonically to the unique solution of (?) with $x(s_0)=x_0$ on $[s_0,s_0+T]$ and the convergence is quadratic.

Proof. Suppose $\widetilde{\gamma}_0(t)=\gamma(t+\mu_1)$ and $\widetilde{\delta}_0(t)=\delta(t),t_0\leq t\leq t_0+T$, where $\mu_1=\tau_0-t_0$. Therefore, $\widetilde{\gamma}_0(t_0)=\gamma(\tau_0)\geq\delta(t_0)=\widetilde{\delta}(t_0)$. Hence,

$$^{c}D_{a}^{\alpha}\widetilde{\gamma}_0(t)={}^{c}D_{a}^{\alpha}\gamma(t+\mu_1)\geq f\left(t+\mu_1,\gamma\left(t+\mu_1\right)\right)=f(t+\mu_1,\widetilde{\gamma}_0\left(t\right))$$

As $f(t,x)$ is nondecreasing in t for each x, we have, ${}^cD_a^\alpha \widetilde{\gamma}_0(t) \geq f(t, \widetilde{\gamma}_0(t))$. In the same manner, we can get, ${}^cD_a^\alpha \widetilde{\delta}_0(t) = {}^cD_a^\alpha \delta_0(t) \leq f(t, \delta_0(t)) = f(t, \widetilde{\delta}_0(t))$. This proves that $\widetilde{\delta}_0(t)$ is a lower solution of the problem (1). Now we note that the unique solutions of the following linear fractional differential equations exist since the right hand side of the equations satisfies a Lipschitz condition.

$$ {}^cD_a^\alpha \widetilde{\delta}_{n+1}(t) = f\left(t+\mu_2, \widetilde{\delta}_n\right) + f_x\left(t+\mu_2, \widetilde{\delta}_n\right)\left(\widetilde{\delta}_{n+1} - \widetilde{\delta}_n\right), \widetilde{\delta}_{n+1}(t_0) = x_0 \tag{13} $$

$$ {}^cD_a^\alpha \widetilde{\gamma}_{n+1}(t) = f(t+\mu_2, \widetilde{\gamma}_n) + f_x(t+\mu_2, \widetilde{\gamma}_n)(\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n), \widetilde{\gamma}_{n+1}(t_0) = x_0 \tag{14} $$

where $\mu_2 = s_0 - t_0$. We will prove that

$$ \widetilde{\delta}_0 \leq \widetilde{\delta}_1 \leq \cdots \leq \widetilde{\delta}_n \leq \widetilde{\gamma}_n \leq \cdots \leq \widetilde{\gamma}_1 \leq \widetilde{\gamma}_0 \;\; on \;\; [t_0, t_0+T] \tag{15} $$

First we must show

$$ \widetilde{\delta}_0 \leq \widetilde{\delta}_1 \leq \widetilde{\gamma}_1 \leq \widetilde{\gamma}_0 \;\; on \;\; [t_0, t_0+T]. \tag{16} $$

Setting $p(t) = \widetilde{\delta}_1 - \widetilde{\delta}_0$

$$ \begin{aligned} {}^cD_a^\alpha p(t) &= {}^cD_a^\alpha \widetilde{\delta}_1 - {}^cD_a^\alpha \widetilde{\delta}_0 \\ &\geq f(t+\mu_2, \widetilde{\delta}_0) + f_x(t+\mu_2, \widetilde{\delta}_0)(\widetilde{\delta}_1 - \widetilde{\delta}_0) - f(t+\mu_2, \widetilde{\delta}_0) \\ &= f_x(t+\mu_2, \widetilde{\delta}_0)(\widetilde{\delta}_1 - \widetilde{\delta}_0) \\ {}^cD_a^\alpha p(t) &\geq f_x(t+\mu_2 \widetilde{\delta}_0)p, p(t_0) \geq 0 \end{aligned} $$

Applying Corollary 3.2, we get $\widetilde{\delta}_0 \leq \widetilde{\delta}_1$ on $[t_0, t_0+T]$. Similarly, we can prove that $\widetilde{\gamma}_1 \leq \widetilde{\gamma}_0$. We must now show that $\widetilde{\delta}_1 \leq \widetilde{\gamma}_1$ on $[t_0, t_0+T]$. For this, we set $p(t) = \widetilde{\gamma}_1(t) - \widetilde{\delta}_1(t)$, then

$$ \begin{aligned} {}^cD_a^\alpha p(t) &= {}^cD_a^\alpha \widetilde{\gamma}_1 - {}^cD_a^\alpha \widetilde{\delta}_1 \\ &= f(t+\mu_2, \widetilde{\gamma}_0) + f_x(t+\mu_2, \widetilde{\delta}_0)(\widetilde{\gamma}_1 - \widetilde{\gamma}_0) - [f(t+\mu_2, \widetilde{\delta}_0) + f_x(t+\mu_2, \widetilde{\delta}_0)(\widetilde{\delta}_1 - \widetilde{\delta}_0)] \\ &= f(t+\mu_2, \widetilde{\gamma}_0) - f(t+\mu_2, \widetilde{\delta}_0) + f_x(t+\mu_2, \widetilde{\delta}_0)\left(\widetilde{\gamma}_1 - \widetilde{\gamma}_0 - \widetilde{\delta}_1 + \widetilde{\delta}_0\right) \end{aligned} $$

By using the inequality in (ii), we can get

$$ \begin{aligned} {}^cD_a^\alpha p(t) &\geq f_x(t+\mu_2, \widetilde{\delta}_0)\left(\widetilde{\gamma}_0 - \widetilde{\delta}_0\right) + f_x(t+\mu_2, \widetilde{\delta}_0)\left(\widetilde{\gamma}_1 - \widetilde{\gamma}_0 - \widetilde{\delta}_1 + \widetilde{\delta}_0\right) \\ &\geq f_x(t+\mu_2, \widetilde{\delta}_0)\left(\widetilde{\gamma}_1 - \widetilde{\delta}_1\right) \end{aligned} $$

which implies that ${}^cD_a^\alpha p(t) \geq f_x\left(t+\mu_2, \widetilde{\delta}_0\right) p(t)$ and $p(t_0) = 0$. This is due to Corollary 3.2, which gives $p(t) \geq 0$. Hence $\widetilde{\delta}_1 \leq \widetilde{\gamma}_1$ on $[t_0, t_0+T]$. Thus (16) is proved. By the use of mathematical induction for $k > 1$, we get

$$ \widetilde{\delta}_0 \leq \widetilde{\delta}_{k-1} \leq \widetilde{\delta}_k \leq \widetilde{\gamma}_k \leq \widetilde{\gamma}_{k-1} \leq \widetilde{\gamma}_0 \;\; on \;\; [t_0, t_0+T]. \tag{17} $$

Now we have to prove that

$$\widetilde{\delta}_k \leq \widetilde{\delta}_{k+1} \leq \widetilde{\gamma}_{k+1} \leq \widetilde{\gamma}_k \ \ on \ \ [t_0, t_0 + T]. \tag{18}$$

For this, setting $p(t) = \widetilde{\delta}_{k+1} - \widetilde{\delta}_k$ and the inequality in (ii), equations (13), (14) becomes

$$\begin{aligned}
{}^cD_a^\alpha p(t) &= f(t+\mu_2, \widetilde{\delta}_k) + f_x(t+\mu_2, \widetilde{\delta}_k)(\widetilde{\delta}_{k+1} - \widetilde{\delta}_k) - [f(t+\mu_2, \widetilde{\delta}_{k-1}) + f_x(t+\mu_2, \widetilde{\delta}_{k-1})(\widetilde{\delta}_k - \widetilde{\delta}_{k-1})] \\
&\geq f_x(t+\mu_2, \widetilde{\delta}_{k-1})(\widetilde{\delta}_k - \widetilde{\delta}_{k-1}) + f_x(t+\mu_2, \widetilde{\delta}_k)(\widetilde{\delta}_{k+1} - \widetilde{\delta}_k) - f_x(t+\mu_2, \widetilde{\delta}_{k-1})(\widetilde{\beta}_k - \widetilde{\delta}_{k-1}) \\
&\geq f_x(t+\mu_2, \widetilde{\delta}_k)(\widetilde{\delta} k + 1 - \widetilde{\delta}_k).
\end{aligned}$$

Hence we obtain ${}^cD_a^\alpha p\,(t) \geq f_x \left(t+\mu_2, \widetilde{\delta}_k\right) p\,(t)$ and $p(t_0) = 0$. Again by the use of Corollary 3.2, we have $\widetilde{\delta}_k \leq \widetilde{\delta}_{k+1}$ on $[t_0, t_0+T]$. Similarly, we can show that $\widetilde{\gamma}_k \geq \widetilde{\gamma}_{k+1}$ on $[t_0, t_0+T]$. Now we have to show that $\widetilde{\delta}_{k+1} \leq \widetilde{\gamma}_{k+1}$ on $[t_0, t_0+T]$. Set $p(t) = \widetilde{\gamma}_{k+1} - \widetilde{\delta}_{k+1}$, then

$$\begin{aligned}
{}^cD_a^\alpha p\,(t) &= {}^cD_a^\alpha \widetilde{\gamma}_{k+1} - {}^cD_a^\alpha \widetilde{\delta}_{k+1} \\
&= f\,(t+\mu_2, \widetilde{\gamma}_k) + f_x(t+\mu_2, \widetilde{\delta}_k)(\widetilde{\gamma}_{k+1} - \widetilde{\gamma}_k) - [f(t+\mu_2, \widetilde{\delta}_k) + f_x\left(t+\mu_2, \widetilde{\delta}_k\right)(\widetilde{\delta}_{k+1} - \widetilde{\delta}_k)] \\
&\geq f_x(t+\mu_2, \widetilde{\delta}_k)\left(\widetilde{\gamma}_k - \widetilde{\delta}_k\right) + f_x(t+\mu_2, \widetilde{\delta}_k)\,(\widetilde{\gamma}_{k+1} - \widetilde{\gamma}_k) - f_x\left(t+\mu_2, \widetilde{\delta}_k\right)\left(\widetilde{\delta}_{k+1} - \widetilde{\delta}_k\right) \\
&= f_x(t+\mu_2, \widetilde{\delta}_k)\left(\widetilde{\gamma}_k - \widetilde{\delta}_k - \widetilde{\delta}_{k+1} + \widetilde{\delta}_k + \widetilde{\gamma}_{k+1} - \widetilde{\gamma}_k\right) \\
&\geq f_x(t+\mu_2, \widetilde{\delta}_k)\left(\widetilde{\gamma}_{k+1} - \widetilde{\delta}_{k+1}\right)
\end{aligned}$$

Hence ${}^cD_a^\alpha p(t) \geq f_x\left(t+\mu_2, \widetilde{\delta}_k\right) p\,(t)$ and $p(t_0) = 0$. It follows from the Corollary 3.2, we reach $\widetilde{\delta}_{k+1} \leq \widetilde{\gamma}_{k+1}$ on $[t_0, t_0+T]$. Thus (18) is proved. Applying standard techniques as in [3], it is easy to show that the monotone sequences $\{\widetilde{\delta}_n\}$ and $\{\widetilde{\gamma}_n\}$ converge uniformly and monotonically to the unique solution $\tilde{x}(t)$ of

$${}^cD_a^\alpha \tilde{x}\,(t) = f\,(t+\mu_2, \tilde{x}\,(t))\,, \tilde{x}\,(t_0) = x_0 \tag{19}$$

Putting $s = t + \mu_2$ and changing the variable, we have

$${}^cD_a^\alpha x = f\,(s, x)\,, \quad x\,(s_0) = x_0. \tag{20}$$

Now we shall show that the convergence is quadratic. To prove this, we consider

$$p_{n+1} = \tilde{x} - \widetilde{\delta}_{n+1}$$

Obviously, $p_{n+1}(t_0) = 0$. Hence we have

$$\begin{aligned}
{}^cD_a^\alpha p_{n+1} &= {}^cD_a^\alpha \tilde{x} - {}^cD_a^\alpha \widetilde{\delta}_{n+1} \\
&= f(t+\mu_2, \tilde{x}) - [f(t+\mu_2, \widetilde{\delta}_n) + f_x(t+\mu_2, \widetilde{\delta}_n)(\widetilde{\delta}_{n+1} - \widetilde{\delta}_n)] \\
&= [f(t+\mu_2, \tilde{x}) - f(t+\mu_2, \widetilde{\delta}_n)] - f_x(t+\mu_2, \widetilde{\delta}_n)(pn - p_{n+1})
\end{aligned}$$

$$\leq [f_x(t+\mu_2,\tilde{x}) - f_x(t+\mu_2,\widetilde{\delta}_n)]p_n + f_x(t+\mu_2,\widetilde{\delta}_n)p_{n+1}$$
$$\leq L|p_n|2 + f_x(t+\mu_2,\widetilde{\delta}_n)p_{n+1}$$
$$\leq L|p_n|_0^2 + N.p_{n+1}$$

Where $|f_x| \leq N$, $|p_n|_0 = \max|p_n(t)|$. In this inequality we get the estimate

$$p_{n+1} \ \leq \ L|p_n|_0^2 \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(N(t-s)^{\alpha})ds \leq N_0|p_n|_0^2$$

where, $E_{\alpha,\alpha}$ is Mittag-Leffler function and $N_0 = \frac{LT^{\alpha}}{\alpha}E_{\alpha,\alpha}(NT^{\alpha})$. Hence we get the desired result

$$\max_{[t_0,\ t_0+\ T]} |\tilde{x} - \widetilde{\delta}_{n+1}| \leq N_0 \max_{[t_0,t_0+T]} |\tilde{x} - \widetilde{\delta}_n|^2. \tag{21}$$

Similarly, the following quadratic convergence of $\{\widetilde{\gamma}_n\}$ is obtained by using suitable calculations

$$\max_{[t_0,t_0+T]} |\widetilde{\gamma}_{n+1} - \tilde{x}| \leq \frac{N_0 L}{2} \max_{[t_0,t_0+T]} |\tilde{x} - \widetilde{\delta}_n|^2 + \frac{3N_0 L}{2} \max_{[t_0,t_0+T]} |\widetilde{\delta}_n - \tilde{x}|^2 \tag{22}$$

The proof is complete. □

References

[1] V. Lakshmikanthan, S. Leele and J. Vasundhara Devi, *Theory and Application of Fractional Dynamical System*, Cambridge Scientific Publishers Ltd., (2009).

[2] J. Vasundhara Devi, F. A. Mc Rae and Z. Dirci, *Variational Lyapunov Method for Fractional Differential Equations*, Computers and Mathematics with Applications, 64(2012), 2981-2989.

[3] Adil Shayma and Muradand Samir Basher Hadid, *Existence and Uniqueness Theorem for Fractional Differential Equation with Integral Boundary Condition*, Journal of Fractional Calculus and Applications, 3(6)(2012), 1-9.

[4] Z. Denton and A. S. Vatsala, *Fractional Integral Inequalities and Applications*, Comput. Math. Appl., 59(2010), 1087-1094.

[5] R. P. Agarwal, M. Benchohra and S. Hamani, *A Survey on Existence for Boundary Value Problems of Nonlinear Fractional Differential Equations and Inclusions*, Acta. Appl. Math., 109(2010), 973-1033.

[6] W. M. Ahemad and R. El. Khazalib, *Fractional Order Dynamical Model of Love*, Cha. Solit, Fract., 33(2007), 1367-1375.

[7] S. Ahmad and I. Stamova, *Global Exponential Stability for Impulsive Cellular Neural Network with Time-Varying Delays*, Nonlinear Analysis, Theory, Methods & Applications, 69(3)(2008), 786-795.