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On a New Type of $F-$function Fixed Point Theorem in Generalized Orthogonal $b-$Metric Spaces

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Abstract

In this paper, we propose the concept of contraction of orthogonal $F-$type functions of contractive mappings and weak exchange preservation and prove one fixed point theorems on complete orthogonal $b-$metric spaces. We also provide an example that supports our theorem.

Keywords: $O-b-$metric space; fixed point; $O-\alpha-$admissible; orthogonal generalized contractive mapping; $F-$type functions; weak exchange preservation.
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1. Introduction

Banach [4] contraction mapping principle, proved by Banach in 1922, is an important technique for solving the problem of the existence and uniqueness of fixed points in complete metric space, and plays an important role in nonlinear analysis. Based on this theorem, many scholars gave many important generalizations of this result by changing the space type or contractive conditions. In 1993, Czerwik [5] generalized the metric spaces by modifying the third condition and introduced the concept of $b-$metric space. He also studied a new class of fixed point theorems for contractive mappings in $b-$metric spaces in this paper. In the setting of $b-$metric spaces, many scholars carried out researches and got a lot of excellent results, see [1,3,14] and the literature cited therein. In 2012, Wardowski [15] gave a new type of contraction in complete metric space, named $F-$type contraction, and presented some sufficient conditions for the existence and uniqueness of fixed point of this type of mapping. Considering $F-$type contraction $b-$metric spaces, Goswami [10] provided some proofs of relevant theorems. Recently, Gordji et al. [9] introduced the concept of orthogonality, and proved the fixed point theorem in orthogonal complete metric space. For recent development on fixed point theory, we refer to [2,6–8,11–13]. In this paper, we introduced the concept of weak exchange preservation and prove one fixed point theorems on complete orthogonal $b-$metric spaces. Furthermore, we provide a specific example to demonstrate the effectiveness of the result.

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2. Preliminaries

Definition 2.1. *Suppose $s \geq 1$ and X is a nonempty set. A function $d : X \times X \to [0, +\infty)$ denotes a $b-$metric if for $x, y, z \in X$, the following conditions hold:*

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

A couple (X, d) is said to be a $b-$metric space.

Definition 2.2. *Suppose (X, d) is a $b-$metric space, $x \in X$ and $\{x_n\}$ is a sequence in X.*

(a) $\{x_n\}$ is convergent to x, if for each $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$, satisfying $d(x_n, x) < \varepsilon$ for all $n > n_\varepsilon$. We denote this as $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, where $n \to \infty$.

(b) $\{x_n\}$ is a Cauchy sequence, if for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > n_\varepsilon$.

Definition 2.3. *Let X be a nonempty set and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ holds with the constraint*

$$\exists\ x_0 \in X : \ \ (\forall\ \ x \in X,\ \ x \perp x_0)\ \ or\ \ (\forall\ \ x \in X,\ \ x_0 \perp x),$$

then $(X, \perp)$ is said to be an orthogonal set.

Definition 2.4. *Let $(X, \perp, d)$ be an orthogonal metric space. Then, X is said to be $O-$complete if every orthogonal Cauchy sequence is convergent.*

Definition 2.5. *A tripled $(X, \perp, d)$ is called an $O_b - MS$ if $(X, \perp)$ is an orthogonal set and (X, d) is a $b-$metric space.*

Definition 2.6. *Let $(X, \perp)$ be an orthogonal set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called an orthogonal sequence ($O-$sequence) if $(\forall\ \ n, x_n \perp x_{n+1})$ or $(\forall\ \ x, x_{n+1} \perp x_n)$.*

Definition 2.7. *A tripled $(X, \perp, d)$ is called an $O_b - MS$. Then, $f : X \to X$ is said to be orthogonally continuous in $x \in X$, if for each $O-$sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $x_n \to x$, we have $f(x_n) \to f(x)$. Also, f is said to be orthogonal continuous on X if f is orthogonal continuous at each $x \in X$.*

Definition 2.8. *Let X is a nonempty set, and f, g are two self-mappings on X. So f and g are called weakly compatible, if they are commutative at each coincidence point, that is, for each of $x \in C(f, g)$, we have $fx = gx \Rightarrow fgx = gfx$.*

Definition 2.9. *Let $(X, \perp, d)$ be an orthogonal set. A function $f : X \to X$ is called orthogonal-preserving if $fx \perp fy$ whenever $x \perp y$.*

Definition 2.10. *Let $(X, \perp, d)$ be an orthogonal set. $f, g : X \to X$ are called weak orthogonal exchange preserving mappings if $fx \perp gy$ and $gx \perp fy$ whenever $x \perp y$.*

Definition 2.11. *The self-mappings $f, g : X \to X$ are called α_s-orbital admissible mappings, if the following condition hold:*

$$\alpha(x, fx) \geq s^p \Rightarrow \alpha(fx, gfx) \geq s^p,$$
$$\alpha(x, gx) \geq s^p \Rightarrow \alpha(gx, fgx) \geq s^p$$

for a constant $p \geq 2$.

Definition 2.12. *Let (X, d) be a complete $b-$metric space with parameter $s \geq 1$ and let $\alpha_s : X \times X \to \mathbb{R}^+$ be a function. Then,*

(H_{s^p}) *If $\{x_n\}$ is a sequence in X such that $gx_n \to gx$ as $n \to \infty$, then there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ with $\alpha(gx_{n_k}, gx) \geq s^p$ for all $k \in \mathbb{N}$.*

(U_{s^p}) *For all $u, v \in C(f, g)$, we have the condition of $\alpha(gu, gv) \geq s^p, \alpha(gv, gu) \geq s^p$.*

(v_{s^p}) *For all $u, v, w \in X$, $\alpha_s(u, v) \geq s^p, \alpha_s(v, w) \geq s^p$, we have the condition of $\alpha_s(u, w) \geq s^p$.*

Definition 2.13. *Let Δ denote the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties:*

(F_1) *F is strictly increasing;*

(F_2) *for each sequence $\{x_n\}_{n=1}^{\infty}$ of positive numbers, we have, $\lim_{n\to\infty} x_n = 0$, $\lim_{n\to\infty} F(x_n) = -\infty$;*

(F_3) *there exists $k \in (0, 1)$ such that $\lim_{x\to 0^+} x^k F(x) = 0$;*

(F_4) *If $\forall\ n \in \mathbb{N}, \tau + F(sx_n) \leq F(x_{n-1})$, we have $\tau + F(s^n x_n) \leq F(s^{n-1}x_{n-1})$.*

Lemma 2.14. *Let (X, d) be a $b-$metric space with parameter $s \geq 1$. Assume that $\{x_n\}$ and $\{y_n\}$ are $b-$convergent to x and y respectively. Then, we have*

$$s^{-2}d(x, y) \leq \lim_{n\to\infty} \inf d(x_n, y_n) \leq \lim_{n\to\infty} \sup d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$s^{-1}d(x, z) \leq \lim_{n\to\infty} \inf d(x_n, z) \leq \lim_{n\to\infty} \sup d(x_n, z) \leq sd(x, z).$$

3. Main Results

Theorem 3.1. *Let (X, d) be an orthogonal complete $b-$metric space with parameter $s \geq 1$. $f, g : X \to X$ such that the following conditions hold:*

(1) f is orthogonal continuous;

(2) f, g are α_s-orbital admissible mapping in$X$;

(3) f, g are exchangeable;

(4) There is orthogonal elements $x_0 \in X$ with satisfying $\alpha(x_0, fx_0) \geq s^p$;

(5) f, g are weak exchange preservation mappings, g^{-1} is orthogonal preserving;

(6) If $x, y \in X$, $x \perp y$ or $y \perp x$, we have

$$\tau + F(\alpha_s(x,y)d(fx,gy)) \leq F(N(x,y)),$$

$$N(x,y) = \max\left\{d(x,y), d(x,fx), d(y,gy), \frac{1}{2s}d(x,gy), \frac{1}{2s}d(y,fx), \frac{d(x,fx)d(y,gy)\min\{d(x,fx),d(y,gy)\}}{1+d^2(x,y)}\right\}, \tag{1}$$

where $\tau > 0$, $\alpha_s : X \times X \to \mathbb{R}$, $\alpha_s(x,y) = \alpha_s(y,x) \geq s^p, p \geq 2$, properties $(H_{s^p}), (U_{s^p})$ and (V_{s^p}) are satisfied.

Then f and g possess a common fixed point in X. And f, g possess a unique common fixed point in $y^{*\perp} = \{x | x \perp y^* \ (or) \ y^* \perp x, x \in X\}$.

Proof. By the definition of orthogonality, we can find an x_0 with $x_0 \perp y$ or $x_0 \perp y$, for all $y \in X$. Define sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_n = gx_{n+1}$ for $n \in \mathbb{N}$. Since $x_1 \in X$, we obtain $x_0 \perp x_1$. Considering f, g are weak exchange preservation, let's take the first vertical relationship here, we obtain $fx_0 \perp gx_1$, $x_1 \perp x_2$, by using weak exchange, we have $gx_1 \perp fx_2$, $x_2 \perp x_3$, in turn, which implies that $\{x_n\}$ is an orthogonal sequence. Based on conditions (2), (4), we have

$$\begin{aligned} &\alpha(x_0, fx_0) \geq s^p \\ \Rightarrow\ &\alpha(fx_0, gfx_0) \geq s^p = \alpha(x_1, gx_1) \geq s^p \\ \Rightarrow\ &\alpha(gx_1, fgx_1) \geq s^p = \alpha(x_2, fx_2) \geq s^p. \end{aligned}$$

Hence $\alpha(x_{2n}, fx_{2n}) \geq s^p$, $\alpha(x_{2n+1}, gx_{2n+1}) \geq s^p$, and $\alpha(x_{2n}, x_{2n+1}) \geq s^p$, $\alpha(x_{2n+1}, x_{2n+2}) \geq s^p$. Replacing x by x_{2n} and y by x_{2n+1} in (1), we have, $\tau + F(\alpha_s(x_{2n}, x_{2n+1})d(fx_{2n}, gx_{2n+1})) \leq F(N(x_{2n}, x_{2n+1}))$. By organizing, it can be concluded that

$$\tau + F(\alpha_s(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})) \leq F(\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}).$$

If $d(x_{2n}, x_{2n+1}) < d(x_{2n+1}, x_{2n+2})$, then, we have,

$$\tau + F(\alpha_s(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})) \leq F(d(x_{2n+1}, x_{2n+2}))$$

This is a contradiction. Thus, $d(x_{2n}, x_{2n+1}) > d(x_{2n+1}, x_{2n+2})$, and the inequality becomes

$$\tau + F(\alpha_s(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})) \leq F(d(x_{2n}, x_{2n+1})).$$

Since $sd(x_{2n+1}, x_{2n+2}) \leq \alpha_s(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})$, we have

$$\tau + F(sd(x_{2n+1}, x_{2n+2})) \leq \tau + F(\alpha_s(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})) \leq F(N(x_{2n}, x_{2n+1}))$$

$$N(x_{2n}, x_{2n+1}) = \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1}), \frac{1}{2s}d(x_{2n}, gx_{2n+1}), \frac{1}{2s}d(x_{2n+1}, fx_{2n}), \frac{d(x_{2n}, fx_{2n})d(x_{2n+1}, gx_{2n+1}) \min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}}{1 + d^2(x_{2n}, x_{2n+1})}\right\}.$$

It follows that

$$\tau + F(sd(x_{2n+1}, x_{2n+2})) \leq F(d(x_{2n}, x_{2n+1})). \tag{2}$$

Similarly, it can be concluded that

$$\tau + F(sd(x_{2n+2}, x_{2n+3})) \leq F(d(x_{2n+1}, x_{2n+2})). \tag{3}$$

Then

$$\tau + F(sd(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})). \tag{4}$$

According to (F_4), we get

$$\tau + F(s^n d(x_n, x_{n+1})) \leq F(s^{n-1} d(x_{n-1}, x_n)).$$

By calculation,

$$\begin{aligned} \tau + F(s^n d(x_n, x_{n+1})) &\leq F(s^{n-1} d(x_{n-1}, x_n)), \\ \tau + F(s^{n-1} d(x_{n-1}, x_n)) &\leq F(s^{n-2} d(x_{n-2}, x_{n-1})), \\ &\vdots \\ \tau + F(sd(x_1, x_2)) &\leq F(d(x_0, x_1)). \end{aligned}$$

Obtained through organization

$$F(s^n d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau. \tag{5}$$

In (5), letting $n \to \infty$, we have $\lim_{n\to\infty} F(s^n d(x_n, x_{n+1})) = -\infty$. Thus we obtain

$$\lim_{n\to\infty} s^n d(x_n, x_{n+1}) = 0.$$

In view of (F_3), one can get that there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty}(s^n d(x_n, x_{n+1}))^k F(s^n d(x_n, x_{n+1})) = 0.$$

In (5), multiplicating $(s^n d(x_n, x_{n+1}))^k$ at both ends, we have

$$(s^n d(x_n, x_{n+1}))^k F(s^n d(x_n, x_{n+1})) - (s^n d(x_n, x_{n+1}))^k F(d(x_0, x_1)) \leq -(s^n d(x_n, x_{n+1}))^k n\tau. \tag{6}$$

In (6), letting $n \to \infty$, we have

$$\lim_{n\to\infty} n(s^n d(x_n, x_{n+1}))^k = 0.$$

According to the definition of limit, there exists $n_1 \in \mathbb{N}$, when $n \geq n_1$, we have

$$n(s^n d(x_n, x_{n+1}))^k \leq 1.$$

Then

$$s^n d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}.$$

Next we shall prove $\{x_n\}$ is Cauchy. For ease of use, letting $d_n = d(y_n, y_{n+1})$. So

$$d(x_n, x_{n+i}) \leq (sd_n + s^2 d_{n+1} + \cdots + s^{i-1} d_{n+i-2} + s^{i-1} d_{n+i-1})$$

and

$$\begin{aligned} sd_n + s^2 d_{n+1} + \cdots + s^{i-1} d_{n+i-2} + s^{i-1} d_{n+i-1} &\leq s^n d_n + s^{n+1} d_{n+1} + \cdots + s^{n+i-2} d_{n+i-2} + s^{n+i-2} d_{n+i-1} \\ &\leq s^n d_n + s^{n+1} d_{n+1} + \cdots + s^{n+i-2} d_{n+i-2} + s^{n+i-1} d_{n+i-1} \\ &= \sum_{i=n}^{n+i-1} s^i d_i \leq \sum_{i=n}^{\infty} s^i d_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Since $k \in (0,1)$, and $\frac{1}{k} > 1$, $\sum\limits_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} = 0$, then

$$\lim_{n\to\infty}(sd_n + s^2 d_{n+1} + \cdots + s^{i-1} d_{n+i-2} + s^{i-1} d_{n+i-1}) = 0, \quad \lim_{n\to\infty} d(x_n, x_{n+i}) = 0.$$

We obtain $\lim\limits_{n\to\infty} d(y_n, y_{n+i}) = 0$ from an orthogonal Cauchy sequence on orthogonal complete $b-$metric space. Therefore, the orthogonal sequence is convergent. Then, we can choose $y^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*, \quad \lim_{n\to\infty} gx_{2n+1} = x^*, \quad \lim_{n\to\infty} fx_{2n} = x^*.$$

Since $\lim_{n\to\infty} fx_{2n} = x^*$, and f is orthogonal continuous, we have

$$x^* = \lim_{n\to\infty} fx_{2n} = f \lim_{n\to\infty} fx_{2n-2} = fx^*.$$

$x_0 \perp x^*$, since f, g are weak exchange preservation, we have

$$fx_0 \perp gx^* = x_1 \perp gx^*, gx_1 \perp fgx^*.$$

f, g are exchangeable and g^{-1} is orthogonal preserving, we obtain $x_1 \perp x^*$. In turn, we have $x_n \perp x^*$. In view of the property (H_{s^p}), one can get a subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$ with $\alpha(fx_{2n_k}, x^*) \geq s^p$, for all $k \in \mathbb{N}$. Next, we will prove that $gx^* = x^*$. Replacing x by x_{2n_k} and y by x^* in (1), we have

$$\begin{aligned}\tau + F(\alpha_s(x_{2n_k}, x^*)d^2(fx_{2n_k}, gx^*)) \leq F\Bigg(\max\Bigg\{&d(x_{2n_k}, x^*), d(x_{2n_k}, fx_{2n_k}), d(x^*, gx^*), \frac{1}{2s}d(x_{2n_k}, gx^*),\\ &\frac{1}{2s}d(x^*, fx_{2n_k}), \frac{d(x_{2n_k}, fx_{2n_k})d(x^*, gx^*)\min\{d(x_{2n_k}, fx_{2n_k}), d(x^*, gx^*)\}}{1+d^2(x_{2n_k}, x^*)}\Bigg\}\Bigg) \end{aligned} \tag{7}$$

In (7), letting $n \to \infty$, and from Lemma 2.14, we obtain

$$\begin{aligned} s^p\frac{1}{s}d(x^*, gx^*) &\leq \limsup_{n\to\infty}(\alpha_s(x_{2n_k}, x^*)d(fx_{2n_k}, gx^*))\\ &\leq \max\Bigg\{\limsup_{n\to\infty} d(x_{2n_k}, x^*), \limsup_{n\to\infty} d(x_{2n_k}, fx_{2n_k}), \limsup_{n\to\infty} d(x^*, gx^*), \limsup_{n\to\infty}\frac{1}{2s}d(x_{2n_k}, gx^*),\\ &\limsup_{n\to\infty}\frac{1}{2s}d(x^*, fx_{2n_k}), \limsup_{n\to\infty}\frac{d(x_{2n_k}, fx_{2n_k})d(x^*, gx^*)\min\{d(x_{2n_k}, fx_{2n_k}), d(x^*, gx^*)\}}{1+d^2(x_{2n_k}, x^*)}\Bigg\} \end{aligned}$$

We have $d(x^*, gx^*) = 0$ and $gx^* = x^*$. Therefore, $fx^* = x^* = gx^*$. Then, f and g possess a common fixed point in X. Next, we will prove that f and g possess a unique common fixed point in $x^{*\perp} = \{x | x \perp x^* (or) x^* \perp x, x \in X\}$. First $x^{*\perp}$ is nonempty set, because $\{x_n\} \subseteq x^{*\perp}$. If there exists $t \perp x^*$ with $t \neq x^*, t$ is a common fixed point of f, g. Then $ft = gt = t \neq y^*$. Replacing x by x^* and y by t in (1),

$$\begin{aligned}\tau + F(\alpha_s(x^*, t)d(fx^*, gt)) \leq F\Bigg(\max\Bigg\{&d(x^*, t), d(x^* fx^*), d(t, gt), \frac{1}{2s}d(x^*, gt), \frac{1}{2s}d(t, fx^*),\\ &\frac{d(x^*, fx^*)d(t, gt)\min\{d(x^*, fx^*), d(t, gt)\}}{1+d^2(x^*, t)}\Bigg\}\Bigg).\end{aligned}$$

We have

$$\tau + F(\alpha_s(x^*, t)d(x^*, t)) \leq F(d(x^*, t)).$$

Since $ft = gt = t \neq x^*, \tau > 0$ and F is strictly increasing, then

$$\alpha_s(x^*, t)d(x^*, t) \leq d(x^*, t).$$

This is a contradiction. Hence $x^* = t$. Therefore, $fx^* = x^* = gx^*$, then f and g possess a common fixed point in X. And f, g possess a unique common fixed point in $x^{*\perp} = \{x | x \perp x^* (or) x^* \perp x, x \in X\}$. □

Example 3.2. *Let $X = [-1, 2]$ and $d : X \times X \to [0, +\infty)$ be a mapping defined by $d(x, y) = |x - y|^2$, for all $x, y \in X$. Define the binary relation $\perp$ on X by $x \perp y$ if $xy \leq (x + 3 \vee y + 3)$, where $x + 3 \vee y + 3 = x + 3$ or $y + 3$. Then (X, d) is an $O-$complete $b-$metric space. Define the mappings $f, g : X \to X$ by*

$$f(x) = \begin{cases} x^2, & x \in [-1, 1] \\ \frac{1}{x}, & x \in (1, 2] \end{cases}, \qquad g(x) = \begin{cases} -x, & x \in [-1, 1] \\ x, & x \in (1, 2] \end{cases},$$

$$g^{-1}(x) = \begin{cases} -x, & x \in [-1, 1] \\ x, & x \in (1, 2] \end{cases}, \qquad \alpha(x, y) = \begin{cases} 2^3 + 1, & x \in [-1, 1] \ \& \ y \in (1, 2] \\ 2^3 + 1, & x, y \in (1, 2] \\ 0, & otherwise \end{cases}$$

Clearly, g^{-1} is orthogonal preserving, f is orthogonal continuous, f, g are exchangeable. Now, let us consider the mapping F defined by $F(t) = \ln t$, $\tau = \ln(\frac{1}{2^7})$. Let $x_0 = \frac{1}{2}$. If $x \in [-1, 1]$, we have $\frac{1}{2}x \leq \frac{1}{2} + 3 \Rightarrow \frac{1}{2} \perp x$. If $x \in (1, 2]$, we have $\frac{1}{2}x \leq x + 3 \Rightarrow \frac{1}{2} \perp x$. So $\frac{1}{2}$ is orthogonal elements in X.

$$\alpha\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = \alpha\left(\frac{1}{2}, \frac{1}{4}\right) \geq 2^3,$$
$$\alpha\left(f\left(\frac{1}{2}\right), gf\left(\frac{1}{2}\right)\right) = \alpha\left(\frac{1}{4}, g\left(\frac{1}{4}\right)\right) = \alpha\left(\frac{1}{4}, -\frac{1}{4}\right) \geq 2^3,$$
$$\alpha\left(g\left(\frac{1}{4}\right), f\left(-\frac{1}{4}\right)\right) = \alpha\left(-\frac{1}{4}, \frac{1}{16}\right) \geq 2^3,$$
$$\alpha\left(f\left(-\frac{1}{4}\right), g\left(\frac{1}{16}\right)\right) = \alpha\left(\frac{1}{16}, -\frac{1}{16}\right) \geq 2^3.$$

f, g is $\alpha_{s^p}-$admissible mapping.

Case 1: Let $x \in [-1, 1], y \in [-1, 1]$. Then

$$f(x) \cdot g(y) = x^2 \cdot (-y) \leq x^2 + 3, \ \ g(x) \cdot f(y) = (-x) \cdot y^2 \leq y^2 + 3.$$

Case 2: Let $x \in (1, 2], y \in (1, 2]$. It follows that

$$f(x) \cdot g(y) = \frac{1}{x} \cdot y \leq y + 3, \ \ g(x) \cdot f(y) = x \cdot \frac{1}{y} \leq x + 3.$$

Case 3: Let $x \in [-1, 1], y \in (1, 2]$. It is easy to show

$$f(x) \cdot g(y) = x^2 \cdot y \leq y + 3, \ \ g(x) \cdot f(y) = (-x) \cdot \frac{1}{y} \leq \frac{1}{y} + 3.$$

Case 4: Let $x \in (1, 2], y \in [-1, 1]$. Obviously,

$$f(x) \cdot g(y) = \frac{1}{x} \cdot (-y) \leq \frac{1}{x} + 3, \ \ g(x) \cdot f(y) = x \cdot y^2 \leq x + 3.$$

So f, g^{-1} are weak exchange preservation. Now we consider the following cases:

Case 1: Let $f(x) \in [-1,1]$, $g(y) \in (1,2]$. Then

$$\tau + \ln(2^3 \left|x^2 - y\right|^2) \leq \ln\left(\frac{1}{2^2}\left|y - x^2\right|^2\right),$$
$$\tau \leq \ln\left(\frac{1}{2^5}\right).$$

It is clear that (1) is satisfied.

Case 2: Let $f(x) \in (1,2]$, $g(y) \in (1,2]$. We get

$$\tau + \ln\left(2^3\left|\frac{1}{x} - y\right|^2\right) \leq \ln\left(\frac{1}{4}\left|y - \frac{1}{x}\right|^2\right),$$
$$\tau \leq \ln\frac{1}{2^5}.$$

That is (1) is satisfied.

Case 3: Let $x, y \in [-1,1]$ or $x \in (1,2]$, $y \in [-1,1]$. Then

$$\alpha(x,y) = 0.$$

Hence, (1) is satisfied.

Therefore, all the conditions of Theorem 3.1 are satisfied. Hence we can conclude that f and g possess a common fixed point in X. And f, g possess a unique common fixed point in $y^{*\perp} = \{x | x \perp y^* \ \ (or) \ \ y^* \perp x, x \in X\}$, that is, $x = 1$.

4. Conclusions

In this paper, we proved the fixed point theorems of double mappings $F-$type contractions in orthogonal $b-$metric spaces by introducing the concept of weak exchange preservation. In addition, we also provided an example to explain in detail the practicality of the obtained results.

References

[1] J. Abbas, *Common fixed point of four maps in $b-$metric spaces*, Hacet. J. Math. Stat., 43(4)(2014), 613-624.

[2] A. E. Al-Mazrooei and J. Ahmad, *Fixed point approach to solve nonlinear fractional differential equations in orthogonal $F-$metric spaces*, Aims Math., 8(3)(2023), 5080-5098.

[3] H. Aydi, M. F. Bota and E. Karapinar, *A common fixed point for weak $\varphi-$contractions on $b-$metric spaces*, J. Fixed Point Theory, 13(2)(2012), 337-346.

[4] S. Banach, *Sur les operations dans les ensembles abstraits et leurs applications aux equations integrals*, Fundam. Math., 3(1992), 133–181.

[5] S. Czerwik, *Contraction mappings in $b-$metric spaces*, J. Acta Math Inform Univ Ostrav, 1(1993), 5-11.

[6] M. Gardasevic-Filipovic, K. Kukic, D. Gardasevic and Z. D. Mitrovic, *Some best proximity point results in the orthogonal $o-$complete $b-$metric$-$like spaces*, J. Contemp. Math. Anal., 58(2023), 105-115.

[7] A. J. Gnanaprakasam, G. Mani, O. Ege, A. Aloqaily and N. Mlaiki, *New fixed point results in orthogonal $b-$metric spaces with related applications*, Mathematics, 11(3)(2023), 677, 1-18.

[8] A. J. Gnanaprakasam, G. Nallaselli, A. U. Haq, G. Mani, I. A. Baloch and K. Nonlaopon, *Common fixed-points technique for the existence of a solution to fractional Integro$-$differential equations via orthogonal Branciari metric spaces*, Symmetry, 14(2022), 1859.

[9] M. E. Gordji and H. Habibi, *Fixed point theory in generalized orthogonal metric space*, J. Linear Topol. Algebra, 6(2017), 251-260.

[10] N. Goswami, N. Haokip and V. N. Mishra, *$F-$contractive type mappings in $b-$metric spaces and some related fixed point results*, Fixed Point Theory Appl., 1(2019), 13.

[11] S. K. Prakasam, A. J. Gnanaprakasam, N. Kausar, G. Mani and M. Munir, *Solution of integral equation via orthogonally modi?ed $F-$contraction mappings on $O-$complete metric$-$like space*, Int. J. Fuzzy Log. Intell. Syst., 22(2022), 287-295.

[12] S. K. Prakasam, A. J. Gnanaprakasam, G. Mani and F. Jarad, *Solving an integral equation via orthogonal generalized $\alpha - \psi-$Geraghty contractions*, Aims Mathematics, 8(3)(2023), 5899-5917.

[13] P. Senthil Kumar, G. Arul Joseph, O. Ege, M. Gunaseelan, S. Haque and N. Mlaiki, *Fixed point for an $OgF - cin$ O-complete $b-$metric-like spaces*, Aims Math., 8(2022), 1022-1039.

[14] T. Suzuki, *A new type of fixed point theorem in metric spaces*, J. Nonlinear Anal., 71(11)(2009), 5313-5317.

[15] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, J. Fixed Point Theory Appl., 2012(1)(2012), 1-6.