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Geometric Group Theory And Its Algebraic Applications

Research Article

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Abstract: The study of finitely generated groups is the focus of the branch of mathematics known as geometric group theory. This branch of mathematics investigates the connections between the algebraic properties of such groups and the topological and geometric properties of spaces on which these groups act (that is, when the groups in question are realized as geometric symmetries or continuous transformations of some spaces). This field was first systematically studied by Walther von Dyck, a student of Felix Klein, in the early 1880s, while an early form is found in the 1856 icosian calculus of William Rowan Hamilton, where he studied the icosahedral symmetry group via the edge graph of the dodecahedral lattice. Geometric group theory evolved from combinatorial group theory, which primarily focused on the study of the properties of discrete groups through the analysis of group presentations At the moment, the field of combinatorial group theory is being mainly absorbed by the field of geometric group theory. In addition, the study of discrete groups by probabilistic, measure-theoretic, arithmetic, analytic, and other methods that fall outside of the typical armory of combinatorial group theory began to be included under the umbrella of ”geometric group theory.”

Keywords: Geometric, Group Theory, Algebraic application.

1. Introduction

There is a close relationship between places and groups. Many different communities are connected to a certain location or area. The group of symmetries, also known as the group of structure-preserving bijections, is something that we may take into consideration. There is also the basic group, in addition to the homology and cohomology groups, to mention a few more. These groupings, as Hermann Weyl has said, have the ability to provide ”a deep insight” into a certain location. The investigation of knots provides a good illustration of this phenomena. The fact that the trefoil knot cannot be untied, for example, is shown by algebraic invariants that take the form of groups, refer figure 1.

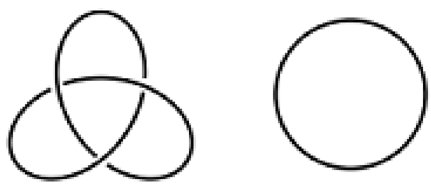

Figure 1. Groups show that these knots are distinct

The link that exists between spaces and groups is examined from a new angle by the discipline of geometric group theory. Instead of analyzing spaces by using the algebraic structure and characteristics of groups, the fundamental tenet of geometric

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group theory is to adhere to the following philosophy. Investigate the topology and geometry of the spaces that the groups operate on using your findings. That is to say, groups are the primary objects of study, and the strategies, tools, and methods that are used to analyze them are, by their very nature, dynamical, geometrical, and topological. The term "geometric group theory" is very recent in comparison to the names of other areas of mathematics1. Gromov's seminal writings [Gro87, Gro93] that introduced the concept of hyperbolic groups and initiated the study of finitely produced groups as metric spaces ignited an immense amount of research and formed lines of inquiry that are still extremely active today. Gromov's work is credited with sparking this large amount of research and establishing these lines of inquiry. Even before the development of geometric group theory, there were already geometric concepts included into group theory. These concepts can be found in the writings of authors such as Dehn, Whitehead, van Kampen, and others. In addition, the research that Thurston did on 3-manifolds demonstrated how the geometry of a manifold may affect the algebraic and computational features of the basic group of that manifold. However, it is the articles written by Gromov that represent the beginning of where these concepts first come to the forefront. This article's goal is to provide some insight into the ways in which the topology and geometry of a space have an impact on the algebraic structure of groups that operate on that space, as well as the ways in which this information may be utilized to examine group behavior. I use the approach that I learnt from my adviser Mladen Bestvina, which is to prioritize illuminating examples over broad theory. As you will see, I follow this method. In this overview of geometric group theory, as is the case with every survey of a mathematical discipline, many features and regions of the subject are not discussed at all. The last part provides a brief bibliography of further reading material on geometric group theory in the form of a list of books.

Groups and spaces: As was just discussed, geometric group theory attempts to comprehend the structure of a group via the use of group actions performed on spaces. What kinds of insights may one possibly acquire from seeing how something is carried out? Is there always going to be anything intriguing to look into? Now that we've had your attention, let's look at each of these questions.

An example $SL(2,\mathbb{Z})$: Let's take a look at an example of a group action that can be found in many different subfields of mathematics in order to demonstrate how the topology of the space that a group acts on may have an effect on the structure of the group. The group of matrices with integer entries and determinants equal to one shall be the focus of our attention here. The following members make up what is known as the special linear group:

$$SL(2,\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a,b,c,d \in \mathbb{Z} \ \ and \ \ ad-bc=1 \right\}$$

Is $SL(2,Z)$ finitely generated? That is, are there finitely many matrices $A1,\ldots,An \in SL(2,Z)$ such that any matrix $M \in SL(2,Z)$ can be expressed as a product $M = A \pm 1\ j1 \ldots A \pm 1\ jk$? (Note, each Aj may appear multiple times). The answer is "yes" and there is an algebraic approach to this problem, but let's take a geometric perspective and consider an action of $SL(2,\mathbb{Z})$ on a metric space.

The space we will consider is the Farey complex which is constructed as follows. First, we start with a graph whose vertex set is the set of rational numbers p q-always expressed in lowest terms-along with an additional point we denote 1 0. Edges join two vertices p q and r s if $ps - qr = \pm 1$. Figure 2 shows a portion of this graph, known as the Farey graph. As seen in Figure 2, the edges in the Farey graph naturally form triangles. In fact, the vertices of any such triangle always have the form p q, r s and $p + rq + s$. For instance, 1 0, 0 1 and 1 1, and also 1 0, 1 1 and 2 1. There is an action of $SL(2,\mathbb{Z})$ on the Farey graph defined by permuting the vertices using the rule:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{p}{q} = \frac{ap+bq}{cp+dq}$$

It is easy to check that two vertices $\frac{p}{q}$ and $\frac{r}{s}$ are connected by an edge only if their images $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{p}{q}$

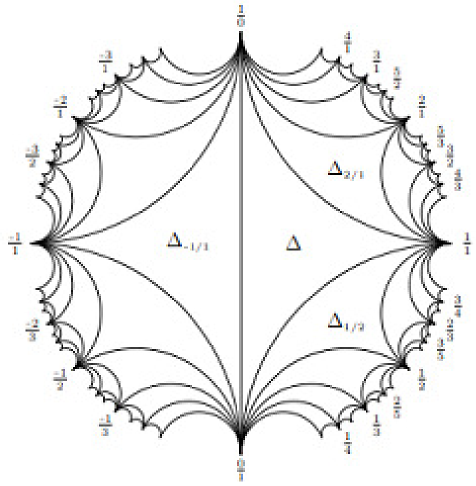


Figure 2. **The Farey graph and Farey complex**

and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{r}{s}$ are. Because of this, an action is defined on the Farey graph and, by extension, on the Farey complex. The Farey complex is the space that is created when the triangles in the Farey graph are filled in. You have very certainly seen this location and its associated activity previously, although in a different incarnation. In point of fact, the Farey complex provides a tessellation of the hyperbolic plane via the use of ideal triangles, the vertices of which, in the model of the upper half plane, are either rational or. In addition to this, the action that was described before is nothing more than the typical action of two-by-two matrices with real entries and a positive determinant, which is accomplished by fractional linear transformations of the top half plane. The conformal maps are as follows:

$$f(z) = \frac{1 - iz}{z - i} \quad and \quad g(z) = \frac{1 + iz}{z + i}$$

Now is the appropriate moment to investigate this activity. Let us designate by the symbol the triangle in the Farey complex that contains the vertices 1 0 and 0 1, as well as the triangle in the Farey complex that contains the vertices 1 0 and 1 1. Two claims are used to capture the action's most important characteristics.

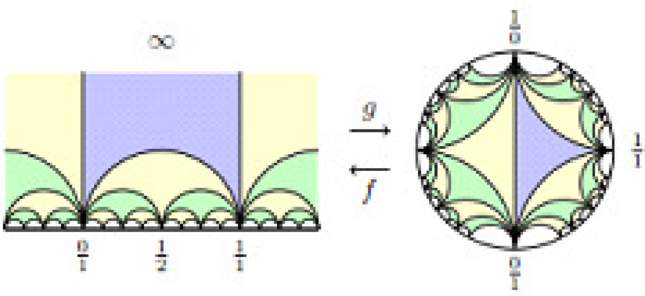


Figure 3. **The tessellation of the upper half plane by ideal triangles according to the Farey algorithm**

Claim 1: For any triangle $\Delta 0$ in the Farey complex, there is matrix $M \in SL(2, \mathbb{Z})$ such that $M\Delta = \Delta 0$.

Indeed, suppose the vertices of Δ' are $\frac{p}{q}$, $\frac{r}{s}$ and $\frac{p+r}{q+s}$, where $ps - qr = 1$. Take $M = \begin{bmatrix} p & r \\ q & s \end{bmatrix}$ and observe that $M\Delta = \Delta 0$.

A space for every group: We need to build a path-connected metric space that is capable of admitting a geometric action by the group G if we want to have a finitely generated group G. This is comparable to the requirements for proving Cayley's theorem in classical group theory, which are as follows: A permutation group and every other group have an isomorphic relationship. In the classical context, it is necessary for us to build a set that allows our group to carry out a permutation action on it. There is just one option available, the group G should be the set, and left multiplication should be the operation. The concept is analogous to what we experience right now. The metric space is constructed on top of the group, and the additional portions of the space are generated by a generating set that has a limited size. The end product is referred to as a Cayley graph. The specifics are outlined below.

Definition 1.1. *Let G be a finitely generated group and let $S \subseteq G$ be a finite generating set. The Cayley graph, denoted $\Gamma(G,S)$, is the graph whose vertex set is G and where there is an edge joining vertices $h1, h2 \in G$ if $h - 1 \quad 1 \quad h2 \in S$, i.e., $h2 = h1s$ for some generator $s \in S$.*

The group G acts on $\Gamma(G,S)$ by permuting the vertices via left multiplication. If vertices $h1, h2 \in G$ are adjacent, then so are the vertices $gh1, gh2$ as $(gh1) - 1 \quad (gh2) = h - 1 \quad 1 \quad h2$ and so the permutation action on the vertices extends to the entire graph. As S generates G, the Cayley graph $\Gamma(G,S)$ is path-connected. Figure 4 illustrates the path connecting the identity element of the group $1G$ to the element $g = s1s2 \ldots sk$, where each sj belongs to $S \cup S - 1$. The key point is that $s1 \ldots sj$ is adjacent to $s1 \ldots sj + 1$.

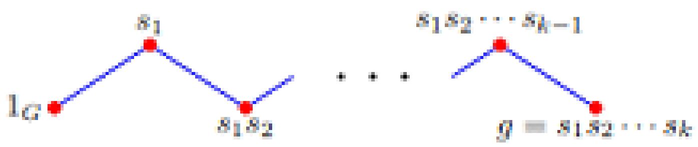


Figure 4. **A path in the Cayley graph**

Here are some examples of Cayley graphs.

1. Z and Z2: The formula $S = 1$ may be used for Z, while the formula $S = [1 \quad 0], [0 \quad 1]$ can be used for Z2. Figure 5 depicts these graphs for your viewing pleasure. There are several more potential generating sets; to see an example of one of them, create the graph Z, 2, 3. This graph may be seen in the essay that Margalit and Thomas wrote [CM17, Office Hour 7].

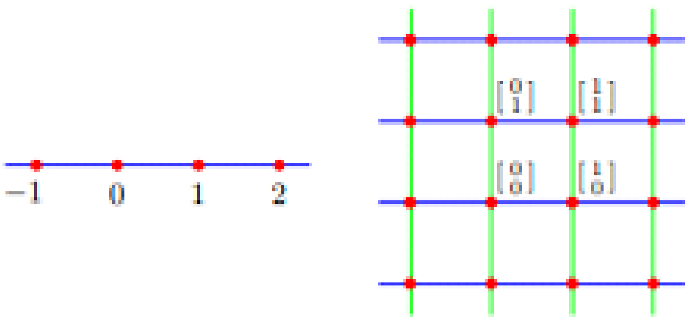


Figure 5. **Cayley graphs for Z and Z^2**

2. Sym(3): We are able to utilize the generating sets $S1 =$ "(1 2), (2 3)" or $S2 =$ "(1 2), (1 3)" for the symmetric group on three elements. These graphs may be seen in Figure 6, which also include a listing of the elements in Sym(3) written in cycle notation.

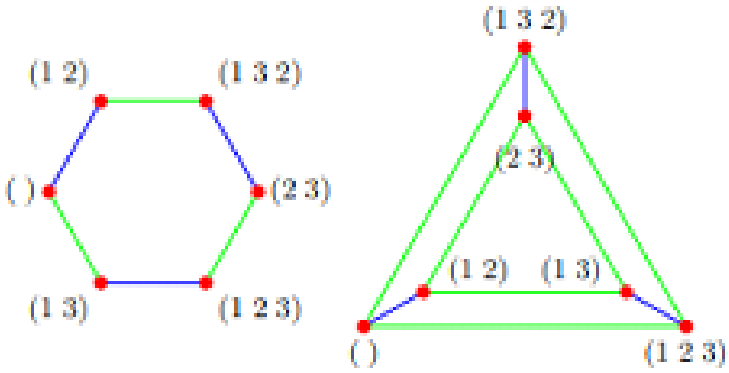


Figure 6. **Cayley graphs for Sym(3)**

3. F2: We are able to utilize a basis of the form $S = a, b$ for the free group of rank two. Remember that the components in F2 have a one-to-one correlation with the alphabetic words "a, a1, b, b1." These words are reduced in the sense that they do not include "aa1," "a 1a," "bb1," or "b 1 b." For instance, a 2 b 1a 1 b and b 2a 2 b 2 are both examples of items that may be found in F2. The concatenation step is followed by the elimination of any prohibited words in the group operation. Because the word that is reduced to represent an element is one of a kind, and because the pathways in the Cayley graphs read out a word that represents an element, as illustrated in Figure 4, there is one and only one path that does not involve backtracking that goes from 1F2 to any particular element. As a result, the Cayley graph represented by the coordinates F2, a, and b is a tree. Figure 7 is an illustration of a section of this graph.

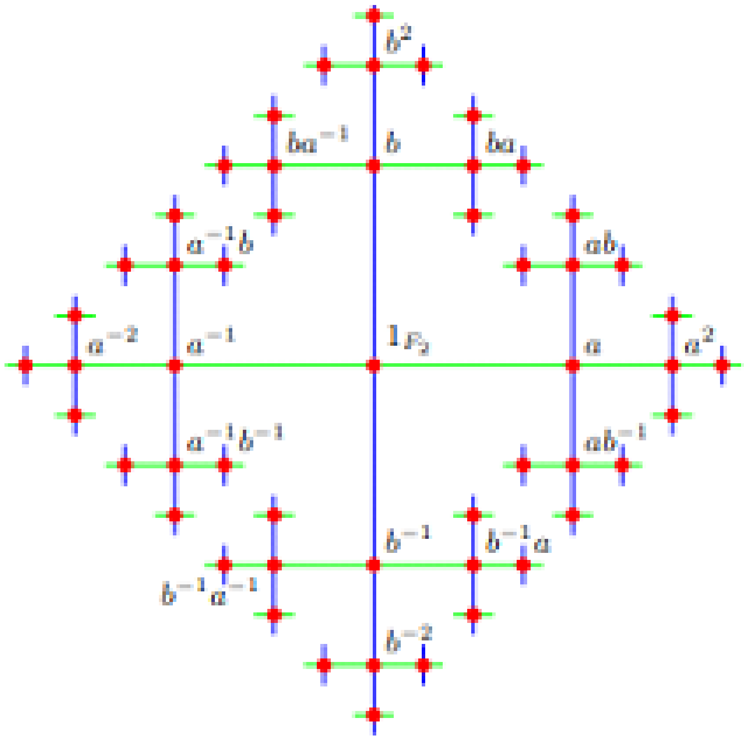


Figure 7. **A Cayley graph for F2**

There is a metric that can be applied to the vertices of (G, S), and it is defined as the minimal number of edges that must be present in an edge route in order to connect any two specified vertices. This metric may be extended to the points that lie in edges by associating (in an equivariant manner) each edge with the unit interval $[0, 1] \to R$. This will allow the metric to be applied to the points that lie in edges. On the other hand, having a metric merely on the vertices is sufficient for the majority of applications in geometric group theory. Isometries constitute the action of G on the Cayley graph denoted by (G, S) when using this metric. In order to complete the proof of Theorem 1, the last thing that needs to be shown is that the action of G on (G, S) is geometric. It won't be difficult for us to verify each of them in turn.

1. **Cocompact:** Let $K \subseteq \Gamma(G, S)$ be the union of the vertices $\{1G\} \cup S$ together with the edges incident on 1G and s for each $s \in S$. As S is finite, K is compact and clearly $S \quad g \in G \quad gK = \Gamma(G, S)$.

2. **Properly discontinuous:** Suppose that $Y \subseteq \Gamma(G, S)$ is a finite subgraph and let n denote the number of vertices in Y. If $gY \cap Y \quad 6 = \emptyset$ then $gh1 = h2$ for a pair of vertices h1, h2 in Y and hence $g = h2h - 11$. Thus the cardinality of $\{g \in G|gY \cap Y6 = \emptyset\}$ is at most n2.

Groups and spaces with negative curvature: In the prior part of this chapter, we used a path-connected space and a geometric action in order to deduce an algebraic consequence known as finite generation. The idea of a geometric action is highly limiting, despite the fact that path-connectivity is a relatively unimportant topological quality. For example, in order to satisfy the requirements of correct discontinuity, the subgroup that fixes a particular point has to be finite. What benefits may be obtained from actions performed on spaces that have stricter constraints regarding the topology and geometry of the space, but perhaps less restrictions regarding the dynamics of the action? The concept of negative curvature is a very valuable geometric attribute, as it can be shown in this example. In this lecture, we are going to examine two examples of negative curvature in geometric group theory. These examples are trees and hyperbolic spaces.

Actions on trees: Negative curvature, such as that found in the hyperbolic plane, may have a number of different effects on the geometry of a space. Some of these effects include the uniqueness of geodesics, an exponential increase in the volume of balls, and a uniform restriction on the diameter of an inscribed circle to a triangle. To be able to discuss the well-known concept of curvature that arises from differential geometry, a space needs additional structure in addition to its regular metric. In order to better understand the concept of negative curvature and how it can be stated just in terms of a distance function on any arbitrary set, let's first take a look at a straightforward illustration of a metric space that has the characteristics described earlier for the hyperbolic plane. This illustration is a tree. In order to illustrate the effectiveness of group actions on trees, let's revisit the earlier example of $SL(2, \mathbb{Z})$ and think about its finite-order elements. By "finite-order elements," we mean matrices for which some positive power is equal to the identity. This will allow us to see an example of the value of group actions on trees. It's not hard to figure out that $A3 = I = B2$, which means that $A6 = I$ and $B4 = I$, and that both A and B have a finite order. Is there room for any more? There are some that are glaringly evident. It is abundantly evident that powers of A and powers of B both have a finite order, as do their conjugates, CAkC 1 and CBkC 1, for any k that is less than Z and C that is greater than $SL(2, \mathbb{Z})$. But is that all there is? This last inquiry has a "yes" as the correct response, and we will explain why by applying the $SL(2, \mathbb{Z})$ action to the Farey tree. The Farey complex is where one may get hold of this tree. Each triangle in the Farey complex should be subdivided into three quadrilaterals that should meet pairwise along one of the legs of a tripod. The Farey tree is the name given to the structure that is created when all of these tripods are stacked together. View Figure 8 here.

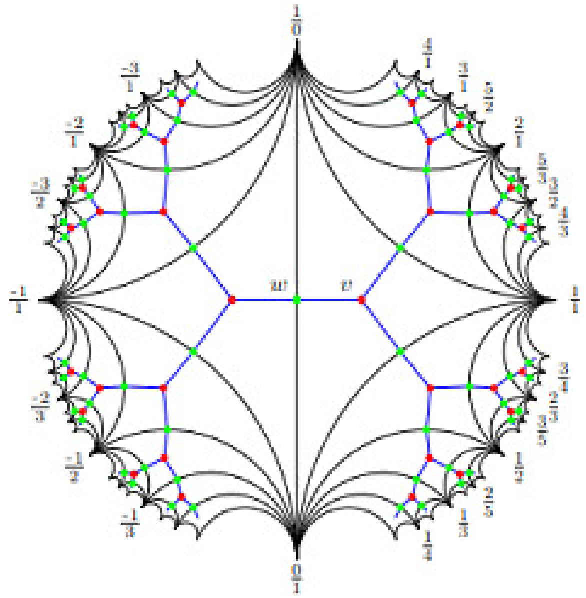


Figure 8. **The Farey tree**

In a Farey tree, there are two distinct kinds of vertices: (red) degree three vertices, which originate from the middle of a triangle, and (green) degree two vertices, which originate from the edge of a triangle. The vertex that corresponds to the center of the triangle is denoted by the letter v, while the vertex that corresponds to the edge in the Farey complex between 0 1 and 1 0 is denoted by the letter w. Figure 8 has labels for each of them.

Our investigation into the effect that $SL(2,\mathbb{Z})$ has on the Farey complex led us to the conclusion that every vertex in the Farey tree can be expressed as a translation of either v or w. This is evident both from Claim 1 and from the observation that A cyclically permutes the edges of and, as a result, each and every vertex that is near to v. Furthermore, we may deduce from Claim 2 that the stabilizer of v is the cyclic subgroup of order 6 created by A, and that the stabilizer of w is the cyclic subgroup of order 4 generated by B. This allows us to state both of these propositions as true. The following assertion is an essential aspect of an action that is performed on a tree.

Claim 3. Let's say that a member of the group G prunes a tree. In the event that g G has a finite order, then g will have a fixed point.

The key fact here is that a finite set of points $x1, \ldots, xn$ in a tree has a unique center, i.e., a point c that minimizes the quantity $\max\{d(c, xj)|j = 1, \ldots, n\}$.

The center is easy to characterize. Suppose that $x1$ and $x2$ maximize $d(xj, xj0)$ for j, $j = 0, 1, \ldots, n$. One can show that the center is the unique point c with $d(c, x1) = d(c, x2) = 12d(x1, x2)$. Now fix a point x in the tree and let c be the center of the set $O = \{x, gx, \ldots, gn - 1x\}$ where n is the order of g. Since the action is by isometries, we must have that gc is the center of the set gO. But g permutes the points in O, i.e., $gO = O$, and so $gc = c$.

Applying Claim 3 to the action of $SL(2,\mathbb{Z})$ on the Farey tree, we see if $M \in SL(2,\mathbb{Z})$ has finite order, then $Mx = x$ for some point x in this tree. If M fixes a point in the interior of an edge, then it must fix one of the incident vertices as well since these vertices have different degrees and cannot be interchanged by M. So we may assume that x is a vertex of the Farey tree. As every vertex is a translate of v or w, we have that $x = Cv$ or $x = Cw$ for some matrix $C \in SL(2,\mathbb{Z})$. In the former, we observe that $(C - 1MC)v = C - 1Mx = C - 1x = v$ and so $M = CAkC - 1$ for some $k \in Z$. Similarly, in the latter, we conclude that $M = CBkC - 1$ for some $k \in Z$. Hence every finite-order element in $SL(2,\mathbb{Z})$ is conjugate to a power of A or B. This is exactly what we desired to show.

The Farey tree is affected in a geometric way by the function $SL(2,\mathbb{Z})$. The line of reasoning that we presented demonstrates that if a group works geometrically on a tree, then there can only be a finitely large number of conjugacy classes for elements of a finite order. In point of fact, in accordance with Claim 3, and given that the action is cocompact, every finite order element may be conjugated into any one of an infinite number of stabilizer subgroups. Because the action itself is correctly discontinuous, each of these subgroups is finite, and hence, the outcome is something that should be expected. To get to the same result, we may arrive at the same conclusion by substituting the assumption of appropriate discontinuity of the action with the assumption that each point stabilizer subgroup has a finitely large number of conjugacy classes of elements of a finite order.

Theorem 1.2. *Let's say that G exhibits cocompact behavior on a tree. G is one of the point stabilizers that must have finitely many conjugacy classes of finite-order elements if every other point stabilizer must also have them.*

A popular paradigm in geometric group theory is shown by theorem 2, which may be found below. If a given property P holds for groups that act geometrically on a certain kind of metric space, then the same should be true for a group G that acts on this same type of metric space as long as certain subgroups (such as point stabilizers) have property P. This is because groups operating geometrically on a particular type of metric space should have the same property P. In other words, we should be able to promote a property P from a collection of subgroups to the full group G if we can discover the right space in which these subgroups are the point stabilizers. This should allow us to do so. This thought brings to mind a practical tactic. Suppose there is a family of groups that can be organized into the following hierarchy: G0, G1, G2, etc., and that the groups in G0 operate geometrically on a certain kind of metric space, and that the groups in Gk likewise act on this same kind of metric space, but with point stabilizers that belong to Gk1. If we are successful in validating the paradigm described above for this particular kind of metric space, then we will have an inductive technique to demonstrate that all of the groups that belong to this family have a certain quality or structure. In the next paragraph, we will discuss an instance in which the use of this tactic has shown to be very beneficial; specifically, the mapping class group of an orientable surface.
Actions on δ-hyperbolic spaces: Although actions on trees are enjoyable to work with, the class of groups that they make up is somewhat limiting. There are a lot of intriguing and natural groups where any action on a tree has a global fixed point, and one of such groups is the tree. For instance, when n is less than three, this is true for $SL(n,\mathbb{Z})$. Actions that have a global fixed point are unlikely to result in significant gains in most circumstances. Gromov's influential essay [Gro87] presented a concept of negative curvature, which unifies essential properties of the hyperbolic plane, trees, and small cancellation groups. Small cancellation groups are a thoroughly studied class of groups that were investigated in the latter half of the 20^{th} century, when geometric notions and techniques were just beginning to gain traction. Gromov's concept of a-hyperbolic space is based on the assumption that one of the helpful implications of negative curvature from the hyperbolic plane may be used as a definition for a metric space. This was Gromov's main motivation for developing this definition. Gromov provided such a description by relying merely on a metric d applied to an arbitrary set X; however, the most frequent formulation that is used-and one that is applicable to practically all of the spaces that one encounters in geometric group theory-requires a geodesic metric space, which is defined as follows. Gromov's definition is the only one that uses a metric d applied to an arbitrary set. A function $p : Y \to X$ is considered to be a geodesic in a metric space (X, d), provided that Y is a connected subset of R and that $d(p(s), p(t)) = |ts|$ holds true for any s and t that fall inside Y. A metric space (X, d) is said to be geodesic if and only if there exists a geodesic $p : [0, L] \to X$ for every pair of coordinates x and y in the space X, where $p(0) = x$ and $p(L) = y$. A geodesic metric space may be defined as a connected graph, and more specifically as the Cayley graph of a finitely formed group. Using geodesic triangles, divergence of geodesics, or closest point projections to geodesics are three of the many formulations that are equivalent for a-hyperbolic metric space. There

are many more formulations. We are going to explain the most popular formulation, which makes use of geodesic triangles and is something Gromov attributes to Rips. Any geodesic in X that goes from a to b may be represented by the notation $[a, b]$ in this statement.

Definition 1.3. *Let (X, d) be a geodesic metric space. A geodesic triangle $\Delta(a, b, c)$ is δ-thin if the δ-neighborhood of any two of the edges contains the third. That is, for all $x \in [a, c]$ there is an $x0 \in [a, b] \cup [b, c]$, where $d(x, x0) \leq \delta$. A δ-hyperbolic space is a geodesic metric space where every geodesic triangle is δ-thin.*

The key point in the definition is that the same δ works for every geodesic triangle, no matter how long the sides are. See Figure 9. Here are some examples of δ-hyperbolic spaces.

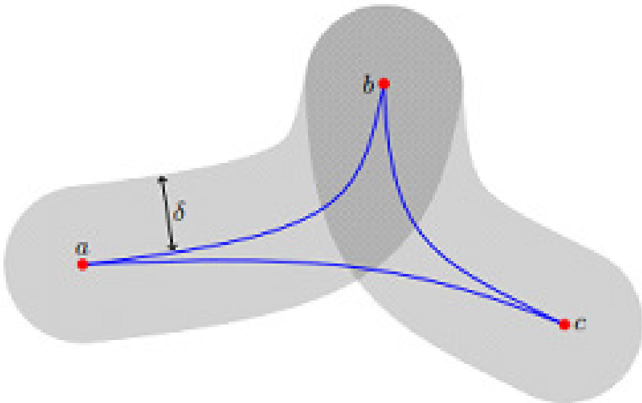


Figure 9. **A δ-thin triangle**

1. A tree has a hyperbolic degree of zero because every geodesic triangle is a tripod, and hence every side is included in the union of the other two sides. Look at figure number 10. We conceive of narrower triangles as an indication of the space being more negatively curved; this is true for the scalar curvature in Riemannian geometry; hence, in this sense, trees are negatively curved to an extreme degree.

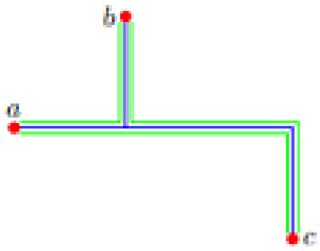


Figure 10. **One of the most common geodesic triangles seen in trees**

2. The hyperbolic plane is $\log(1 + \sqrt{2})$-hyperbolic. As every geodesic triangle is contained in an ideal triangle, we only have to compute δ for an ideal triangle, which is a fun exercise.

3. The Farey graph is 1-hyperbolic. This follows as the removal of any edge and its incident vertices disconnects the Farey graph.

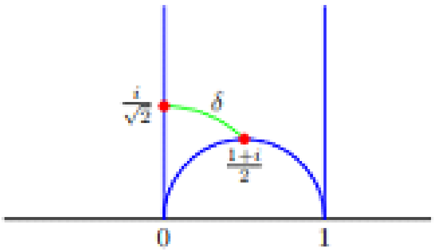


Figure 11. **The ideal triangles that exist on the hyperbolic plane have a thickness of** $\log(1+2)$

As a point of comparison, R 2 calculated using the Euclidean metric is not hyperbolic for any value of. In point of fact, the geodesic triangle with the vertices $(0, 0)$, $(n, 0)$, and $(0, n)$ can only be considered-thin if the value of is less than $n/2$. Consider the point $(n/2, n/2)$ if you want to understand this clearly. The following is a list of common questions that, when attempted to be answered using actions on hyperbolic spaces, often fall into one of the following categories:

1. Algorithmic: When do two words in a generating set represent the same element or conjugate elements?
2. Local-to-global: Are paths in the Cayley graph that are locally geodesics globally geodesics as well?
3. Rigidity: If two groups have geometrically similar Cayley graphs, are the groups algebraically similar? Can we characterize homomorphisms to and from the group?

We will discuss in turn geometric actions and other types of actions on δ-hyperbolic spaces.

Geometric actions on δ-hyperbolic spaces: A metric space is said to be appropriate if closed balls can be compressed inside it. If a group G works geometrically on a proper-hyperbolic space 2, then the group is said to be hyperbolic. The term "hyperbolic group" refers to both free groups and the basic groups of closed hyperbolic manifolds. Given that we began this section by pointing out that there are not always beneficial tree actions, it is reasonable to inquire as to the frequency with which hyperbolic groups occur. Gromov developed a model of a "random finitely presented group" in Chapter 9 of Gro93. This model contains a parameter known as the "density" that ranges from 0 to 1 and governs the number of relators in relation to the number of generators. Gromov demonstrated that a random group is both infinite and hyperbolic when d is less than half. (For those who are wondering, a random group has a maximum of two components whenever d is greater than 1/2). As a result, it is accurate to assert that hyperbolic groups are quite widespread. One definition of a hyperbolic group is identical to the statement that the group G is finitely generated and that the Cayley graph (G, S) is hyperbolic for every finite generating set that is a member of the group G. In addition, the word "some" in the above statement might be substituted by the word "every." There are several in-depth studies that are centered on hyperbolic groups, in addition to Gromov's original article, which explains how these groups meet a lengthy number of important features. See, for example, the comments that were edited by Short and published in $ABC + 91$; the chapters written by Bridson and Haefliger and published in BH99; and the references that are included within these works. from hyperbolic groups are characterized by a geometric condition (in many different ways that are equal to one another), academics have pondered from the beginning of their existence whether or not there is an algebraic characterisation. Locating algebraic roadblocks is not all that difficult. The centralizer of an infinite-order element is often one of the first to be encountered. If G is a hyperbolic group and the subgroup g of G has an infinite order, then the cyclic subgroup hgi, which was produced by g, will have a finite index in the centralizer of g, which is CG(g). To refresh your memory, the centralizer of g is the subgroup of G that consists of elements h that belong to G and has the equation $hg = gh$. The concept that underlies this fact provides a clear illustration of a common geometric argument by making use of the thin triangle condition. Assuming that $hg = gh$, we should look at the four vertices 1G, g k, hgk, and h in the Cayley graph (G, S) for a value of k that is very big. It may be deduced from the fact that $hgk = g \quad kh$ that these four locations are located on a rectangle. A geodesic [1G, gk] and its translation by h, the geodesic [h, hgk], are responsible for the formation of the horizontal sides. Use a geodesic with the coordinates [1G, h] and its translate by g k to get the vertical sides. Because of the commutativity assumption, the translate by g k yields a geodesic that goes from g k to g kh. However, the latter point is equivalent to hgk. Look at figure 12 here.

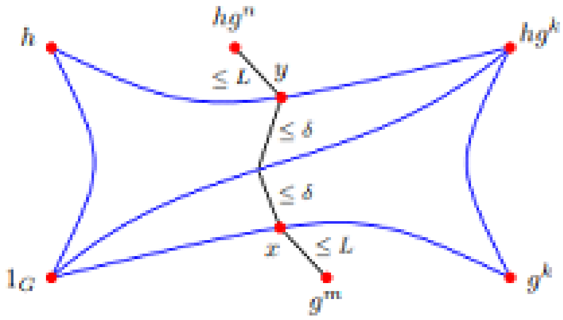
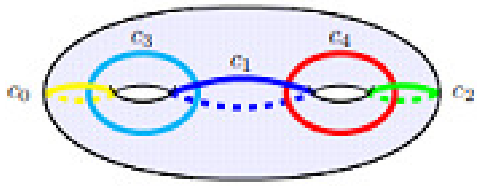


Figure 12. **A commuting rectangle in** $\Gamma(G, S)$

Other actions on δ-hyperbolic spaces: Numerous natural groups have subgroups that are isomorphic to Z 2, and as a result, these natural groups cannot be hyperbolic. Is it possible for us to continue to examine these groupings using negative curvature? Let's take an example of an important group in low-dimensional topology while relaxing the constraint of a geometric action and the necessity of a correct metric space. The group of orientation-preserving homeomorphisms of modulo isotopy is the mapping class group $MCG()$ of an orientable surface, which may or may not have a border. That is to say, two homeomorphisms of that determine the same mapping class are those that can be continuously deformed into one another in such a manner that every intermediate map along the way is likewise a homeomorphism. When has a border that is not empty, it is necessary for the homeomorphisms and the isotopies to have the same identity on each component of the boundary. In order to make this debate more manageable, let's just assume that is compact. The study of 3-manifolds, algebraic geometry, cryptography, symplectic geometry, dynamics, and configuration spaces all make use of this group. It is not difficult to locate subgroups in most mapping class groups that are isomorphic to Z 2 by making use of homeomorphisms that are supported on disjoint subsurfaces in. As a result, the $MCG()$ function is not hyperbolic in general.

The mapping class group has an effect on the graph of the curve $C()$. A simple closed curve is an embedding of the circle S 1 that does not include within itself either a disk nor an annulus (the latter only arises when has a border). The curve graph is the graph whose vertex set is the set of isotopy classes of simple closed curves. Two such curves, $[c0]$ and $[c1]$, are linked by an edge if they have representations that do not have a disjoint relationship with one another. Figure 13 displays many curves on, each of which is accompanied by the subgraph of $C()$ that corresponds to it. When a mapping class $[f]$ is applied to a vertex $[c]$ in the curve graph, the result is that the simple closed curve is transferred to the vertex's image, as shown by the equation $[f] \quad [c] = [f(c)]$. Because homeomorphisms take disjoint curves and transform them into other disjoint curves, this also applies to an operation on $C()$.

Any two non-isotopic simple closed curves will inevitably cross when the genus of is equal to 1, which is the case when is a torus S 1 S 1. Because of this, the definition given above will produce a graph that does not include any edges. In this particular instance, the definition is modified slightly such that [c0] and [c1] are considered to be united by an edge if they have representatives that overlap once. Let's take a more in-depth look at this graph of a curve. Isotopic to a curve that wraps p times around the first S 1 factor is any simple closed curve on the torus in its simplest form.



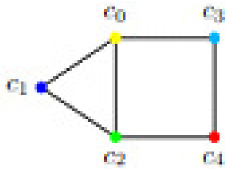


Figure 13. **A portion of the curve graph for a genus 2 surface**

and q iterations revolving around the second S 1 factor in a context in which p and q are approximately prime. Since it makes no difference either way the vector is oriented, we may safely assume that q is positive. To put it another way, isotopy classes of simple closed curves on the torus are parameterized by the set of rational numbers p q coupled with an extra element 1 0. In addition, the expression "ps qr" represents the number of times that the simple closed curves p and q and r and s cross. Does this ring a bell? Yes, you read it correctly: the Farey graph is the curve graph of the torus! In point of fact, the mapping class group of the torus is isomorphic to $SL(2, \mathbb{Z})$, meaning that both actions are equivalent to one another. The influence of $SL(2, \mathbb{Z})$ on the Farey graph is a good illustration of some of the most important characteristics of C() as well as the effect that MCG() has on C(). First, we found that the curve graph has a hyperbolic shape, similar to that of the Farey graph. This astounding truth was first shown by Masur and Minsky [MM99], however it has subsequently been disproven on several occasions. (Hennsel, Przytycki, and Webb provided the most accurate estimate of when they demonstrated that is less than 17 [HPW15]). It is hard to emphasize the significance of this finding when it comes to the investigation of the mapping class group, the geometry of 3-manifolds, and geometric group theory in general. Second, the discontinuity in the action is not executed correctly. In point of fact, there is no limit to the vertex stabilizers. However, this is not a flaw but rather a benefit! Homeomorphisms are exactly what they sound like when they fix a straightforward closed curve like c.

2. Conclusion

we hoping that the information presented here has provided you with a better understanding of how the topology and geometry of a space that a group works on may have an effect on the algebraic characteristics and structure of that space. The study of geometric groups is a burgeoning area of study. This is in part owing to the vast number of problems that the field creates about the geometry of finitely generated groups, but the subject has also witnessed a rise in attention as a consequence of its applicability to other fields of mathematics. In other words, the number of questions that are created by the field is just one reason for the growth in interest. The recent confirmation of the Virtual Haken Conjecture in hyperbolic geometry serves as a dramatic illustration of this point. This was shown by Agol [Ago13] with the use of apparatus derived from geometric group theory and developed by Scott, Sageev, Wise, and others. For a comprehensive look at this relationship, I highly recommend reading the review article [Bes14] written by Bestvina. There are several facets to the study of geometric groups, and one of them is called hyperbolicity. There are subfields of geometric group theory that make use of concepts and methods from algebra (algebraic geometry, homological algebra), analysis (L p-spaces, C and von Neumann algebras), dynamics (entropy, topological Markov chains), geometry (isoperimetric functions, Lie theory), and topology (dimension, fractals). Those interested in learning more about geometric group theory may peruse the books mentioned below, which are organized according to the year in which they were first published.

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