Int. J. Math. And Appl., 6(1)(2018), 7–13. (Special Issue)
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Stability of Quindecic Functional Equation

R. Murali[1,*] and V. Vithya[1]

1 PG and Research Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, Vellore, Tamil Nadu, India.

Abstract: In this paper, we find the general solution of a quindecic functional equation and prove the stability of quindecic functional equation in matrix normed spaces by using the fixed point method.

MSC: 39B52, 39B82, 46L07, 47H10, 47L25.

Keywords: Generalized Hyers-Ulam stability, J. M. Rassias stability, fixed point, quindecic functional equation, matrix normed spaces.

Accepted on: 13th April 2018

1. Introduction

In 1940, S. M. Ulam [19] investigated the stability problems for various functional equations. He raised a question relating to the stability of homomorphism. In the following year, D. H. Hyers [5] was able to give a partial solution to Ulam's question. The result of Hyers was generalized by Aoki [1] for additive mappings. In 1978, Th. M. Rassias [12] succeeded in extending the result of Hyers theorem by weakening the condition for the Cauchy difference. In 1982, J. M. Rassias [14] solved the Ulam problem for different mappings and for many Euler-Lagrange type quadratic mappings, by involving a product of different powers of norms. In 1994, a generalization of the Rassias theorem was obtained by Gavruta [4] by replacing the unbounded Cauchy difference by a general control function. A further generalization of the stability for a large class of mapping was obtained by Isac and Th. M. Rassias [6]. They also presented some applications in non-linear analysis, especially in fixed point theory. This terminology may also be applied to the cases of other functional equations [2, 3, 11, 13, 16, 18, 21]. Also, the generalized Hyers-Ulam stability of functional equations and inequalities in matrix normed spaces has been studied by number of authors [7–9, 20]. Very recently, K. Ravi et. al., [17] discussed the general solution of quattuordecic functional equation

$$\begin{aligned} f(x+7y) &- 14f(x+6y) + 91f(x+5y) - 364f(x+4y) + 1001f(x+3y) - 2002f(x+2y) \\ &- 3003f(x+y) - 3432f(x) + 3003f(x-y) - 2002f(x-2y) + 1001f(x-3y) \\ &- 364f(x-4y) + 91f(x-5y) - 14f(x-6y) + f(x-7y) = 14!f(y) \end{aligned}$$

where, $14! = 87178291200f$ and its stability in quasi β - normed spaces. In this paper, we offer the following new functional equation

$$f(x+8y) - 15f(x+7y) + 105f(x+6y) - 455f(x+5y) + 1365f(x+4y) - 3003f(x+3y)$$

* *E-mail: shcrmurali@yahoo.co.in*

$$+5005f(x+2y)-6435f(x+y)+6435f(x)-5005f(x-y)+3003f(x-2y)-1365f(x-3y)$$
$$+455f(x-4y)-105f(x-5y)+15f(x-6y)-f(x-7y)=15!f(y) \quad (1)$$

where $15! = 1307674368000$, is said to be quindecic functional equation if the function $f(x) = cx^{15}$ is its solution. In this paper, we determine the general solution of the functional equation (1) and we also present the Generalized Hyers-Ulam stability and J. M. Rassias stbility of the functional equation (1) in matrix normed spaces by using fixed point method. Throughout this paper, let us consider $(X, \|.\|_n)$ be a matrix normed space, $(Y, \|.\|_n)$ be a matrix Banach space and let n be a fixed non-negative integer.

2. General Solution of Quindecic Functional Equation (1)

In this segment, we derive the general solution of quindecic functional equation (1). Throughout this part, let us consider $\mathcal{D}$ and $\mathcal{E}$ be real vector spaces.

Theorem 2.1. *If $f : \mathcal{D} \to \mathcal{E}$ be a mapping satisfying (1) for all $x, y \in \mathcal{D}$, then $f(2x) = 2^{15}f(x)$ for all $x, y \in \mathcal{D}$.*

Proof. Set $x = y = 0$ in (1), one gets $f(0) = 0$. Refilling $x = 0$, $y = x$ and $x = x$, $y = -x$ in (1) and joining the two out coming equations, we get $f(-x) = -f(x)$. Hence, f is an odd mapping. Refilling $x = 0$, $y = 2x$ and $x = 8x$, $y = x$ in (1) and eliminating the two out coming equations, one gets

$$15f(15x)-119f(14x)+455f(13x)-1275f(12x)+3003f(11x)-5355f(10x)$$
$$+6435f(9x)-5525f(8x)+5005f(7x)-4641f(6x)+1365f(5x)$$
$$+1547f(4x)+105f(3x)-1307674369000f(2x)+15!f(x)=0 \quad (2)$$

for all $x \in \mathcal{D}$. Refilling (x, y) by $(7x, x)$ in (1), and increasing the out coming equation by 15, and then eliminating the out coming equation from (2), we get

$$106f(14x)-1120f(13x)-43680f(5x)+22022f(4x)-6720f(3x)+5550f(12x)$$
$$-17472f(11x)+39690f(10x)-68640f(9x)+91000f(8x)-91520f(7x)$$
$$+70434f(6x)-1307674367000f(2x)+2092278989000f(x)=0 \quad (3)$$

$\forall\ x \in \mathcal{D}$. Refilling (x, y) by $(6x, x)$ in (1), and increasing the out coming equation by 106, and then eliminating the out coming equation from (3), we have

$$470f(13x)-5580f(12x)+30758f(11x)-105000f(10x)+249678f(9x)-439530f(8x)$$
$$+590590f(7x)-611676f(6x)+486850f(5x)-296296f(4x)$$
$$+137970f(3x)-1307674416000f(2x)+159536272900000f(x)=0 \quad (4)$$

for all $x \in \mathcal{D}$. Replenishing (x, y) by $(5x, x)$ in (1), and increasing the out coming equation by 470, and then the out coming equation from (4), we have

$$1470f(12x)+108850f(10x)-391872f(9x)+971880f(8x)-1761760f(7x)$$

$$+2412774f(6x)-2537600f(5x)+2056054f(4x)-1273440f(3x)$$
$$-18592f(11x)-1307673775000f(2x)+774143225900000f(x)=0 \quad (5)$$

$\forall\ x\in\mathcal{D}$. Replenishing (x,y) by $(4x,x)$ in (1), and increasing the out coming equation by 1470, and then eliminating the out coming equation from (5), we arrive at

$$3458f(11x)+276978f(9x)-1034670(8x)+2652650f(7x)-4944576f(6x)$$
$$+6921850f(5x)-7403396f(4x)+6082440f(3x)$$
$$-455000f(10x)-1307678167000f(2x)+2696424548000000f(x)=0 \quad (6)$$

for all $x\in\mathcal{D}$. Replenishing (x,y) by $(3x,x)$ in (1), and increasing the out coming equation by 3458, and then eliminating the out coming equation from (6), we arrive at

$$6370f(10x)-86112f(9x)+538720f(8x)-2067520f(7x)+5439798f(6x)-10385440f(5x)$$
$$+14845376f(4x)-16117920f(3x)-1307661223000f(2x)+7218362502000000f(x)=0 \quad (7)$$

$\forall\ x\in\mathcal{D}$. Replenishing (x,y) by $(2x,x)$ in (1), and increasing the out coming equation by 6370, and then eliminating the out coming equation from (7), we have

$$9438f(9x)-130130f(8x)+830830f(7x)+24204180f(3x)-3255252f(6x)+87373000f(5x)$$
$$-16940924f(4x)+15548248250000000f(x)-1307699315000f(2x)=0 \quad (8)$$

for all $x\in\mathcal{D}$. Replenishing (x,y) by (x,x) in (1), and increasing the out coming equation by 9438, and then eliminating the out coming equation from (8), we have

$$11440f(8x)-160160f(7x)+1029600f(6x)-4004000f(5x)+10410400f(4x)$$
$$-18738720f(3x)-1307651465000f(2x)+27890078910000000f(x)=0 \quad (9)$$

$\forall\ x\in\mathcal{D}$. Replenishing (x,y) by $(0,x)$ in (1), and increasing the out coming equation by 11440 , and then eliminating the out coming equation from (9) , we have $f(2x)=2^{15}f(x)$ for all $x\in\mathcal{D}$. Thus $f:\mathcal{D}\to\mathcal{E}$ is a quindecic mapping. □

3. Generalized Hyers-Ulam Stability of Quindecic Functional Equation (1)

In this segment, we will investigate the Generalized Hyers-Ulam stability for the functional equation (1) in matrix normed spaces by using the fixed point method. For a mapping $f:X\to Y$, define $\mathcal{H}f:X^2\to Y$ and $\mathcal{H}f_n:M_n(X^2)\to M_n(Y)$ by,

$$\mathcal{H}f(c,d)=f(c+8d)-15f(c+7d)+105f(c+6d)-455f(c+5d)+1365f(c+4d)-3003f(c+3d)$$
$$+5005f(c+2d)-6435f(c+d)+6435f(c)-5005f(c-d)+3003f(c-2d)$$
$$-1365f(c-3d)+455f(c-4d)-105f(c-5d)+15f(c-6d)-f(c-7d)-15!f(y)$$
$$\mathcal{H}f_n(x_{kl},y_{kl})=f_n([x_{kl}+8y_{kl}])-15f_n([x_{kl}+7y_{kl}])+105f_n([x_{kl}+6y_{kl}])-455f_n([x_{kl}+5y_{kl}])$$

$$+ 1365f_n([x_{kl} + 4y_{kl}]) - 3003f_n([x_{kl} + 3y_{kl}]) + 5005f_n([x_{kl} + 2y_{kl}])$$
$$- 6435f_n([x_{kl} + y_{kl}]) + 6435f_n([x_{kl}]) - 5005f_n([x_{kl} - y_{kl}])$$
$$+ 3003f_n([x_{kl} - 2y_{kl}]) - 1365f_n([x_{kl} - 3y_{kl}]) + 455f_n([x_{kl} - 4y_{kl}])$$
$$- 105f_n([x_{kl} - 5y_{kl}]) + 15f_n([x_{kl} - 6y_{kl}]) - f_n([x_{kl} - 7y_{kl}]) - 15!f_n([y_{kl}])$$

for all $c, d \in X$ and all $x = [x_{kl}], y = [y_{kl}] \in M_n(X)$.

Theorem 3.1. *Let $r = \pm 1$ be fixed and $\varsigma : X^2 \to [0, \infty)$ be a function such that there exists a $\zeta < 1$ with*

$$\varsigma(c, d) \leq 2^{15r}\zeta\varsigma(\frac{c}{2^r}, \frac{d}{2^r})\forall\ c, d \in X. \tag{10}$$

Let $f : X \to Y$ be a mapping satisfying

$$\|\mathcal{H}f_n([x_{kl}], [y_{kl}])\| \leq \sum_{k,l=1}^{n} \varsigma(x_{kl}, y_{kl})\forall\ x = [x_{kl}], y = [y_{kl}] \in M_n(X). \tag{11}$$

Then there exists a unique quindecic mapping $\mathbb{Q} : X \to Y$ such that

$$\|f_n([x_{kl}]) - \mathbb{Q}_n([x_{kl}])\|_n \leq \sum_{k,l=1}^{n} \frac{\zeta^{\frac{1-r}{2}}}{2^{19}(1-\zeta)}\varsigma^*(x_{kl})\forall\ x = [x_{kl}] \in M_n(X), \tag{12}$$

where

$$\varsigma^*(x_{kl}) = \frac{1}{15!}[\varsigma(0, 2x_{kl}) + \varsigma(8x_{kl}, x_{kl}) + 15\varsigma(7x_{kl}, x_{kl}) + 106\varsigma(6x_{kl}, x_{kl})$$
$$+ 470\varsigma(5x_{kl}, x_{kl}) + 1470\varsigma(4x_{kl}, x_{kl}) + 3458\varsigma(3x_{kl}, x_{kl})$$
$$+ 6370\varsigma(2x_{kl}, x_{kl}) + 9438\varsigma(x_{kl}, x_{kl}) + 11440\varsigma(0, x_{kl})]$$

Proof. Switching $n = 1$ in (11), we obtain

$$\|\mathcal{H}f(c, d)\| \leq \varsigma(c, d) \tag{13}$$

By utilizing Theorem 2.1, one gets

$$\left\|-f(2c) + 2^{15}f(c)\right\| \leq \frac{1}{15!}\left[\varsigma(0, 2c) + \varsigma(8c, c) + 15\varsigma(7c, c) + 106\varsigma(6c, c) + 470\varsigma(5c, c)\right.$$
$$\left. + 1470\varsigma(4c, c) + 3458\varsigma(3c, c) + 6370\varsigma(2c, c) + 9438\varsigma(c, c) + 11440\varsigma(0, c)\right] \tag{14}$$

Consequently,

$$\left\|f(2c) - 2^{15}f(c)\right\| \leq \varsigma^*(c)\forall\ c \in X. \tag{15}$$

Therefore

$$\left\|f(c) - \frac{1}{2^{15r}}f(2^r c)\right\| \leq \frac{\zeta^{\left(\frac{1-r}{2}\right)}}{2^{15}}\varsigma^*(c) \qquad \forall\ c \in X. \tag{16}$$

Set $\mathcal{T} = \{f : X \to Y\}$ and offer the generalized metric ρ on $\mathcal{T}$ as follows:

$$\rho(f, g) = \inf\left\{\mu \in \mathbb{R}_+ : \|f(c) - g(c)\| \leq \mu\varsigma^*(c), \forall c \in X\right\},$$

It is easy to check that $(\mathcal{T}, \rho)$ is a complete generalized metric (see also [10]). Define the mapping $\mathcal{S} : \mathcal{T} \to \mathcal{T}$ by

$$\mathcal{S}f(c) = \frac{1}{2^{15r}} f(2^r c) \qquad \forall\ f \in \mathcal{T} \text{ and } c \in X.$$

Set $f, g \in \mathcal{T}$ and τ be an arbitrary constant with $\rho(f,g) = \tau$. Then $\|f(c) - g(c)\| \leq \tau\varsigma^*(c)$ for all $c \in X$. Utilizing (10), we find that

$$\|\mathcal{S}f(c) - \mathcal{S}g(c)\| = \left\| \frac{1}{2^{15r}} f(2^r c) - \frac{1}{2^{15r}} g(2^r c) \right\| \leq \zeta\tau\varsigma^*(c)$$

for all $c \in X$. Hence it holds that $\rho(\mathcal{S}f, \mathcal{S}g) \leq \zeta\tau$, that is, $\rho(\mathcal{S}f, \mathcal{S}g) \leq \zeta\rho(f,g)$ for all $f, g \in \mathcal{T}$. By (16), we have $\rho(f, \mathcal{S}f) \leq \frac{\zeta^{(\frac{1-r}{2})}}{2^{15}}$. According to the Theorem 2.2 in [3], there exists a mapping $\mathbb{Q} : X \to Y$ which satisfying:

(1). $\mathbb{Q}$ is a unique fixed point of $\mathcal{S}$, which is satisfied $\mathbb{Q}(2^r c) = 2^{15r}\mathbb{Q}(c)\ \forall\ c \in X$.

(2). $\rho(\mathcal{S}^m f, \mathbb{Q}) \to 0$ as $m \to \infty$. This implies that $\lim_{m\to\infty} \frac{1}{2^{15mr}} f(2^{mr}c) = \mathbb{Q}(c)\ \forall\ c \in X$.

(3). $\rho(f, \mathbb{Q}) \leq \frac{1}{1-\zeta}\rho(f, \mathcal{S}f)$, which implies

$$\|f(c) - \mathbb{Q}(c)\| \leq \frac{\zeta^{\frac{1-r}{2}}}{2^{15}(1-\zeta)}\varsigma^*(c) \qquad \forall\ c \in X. \tag{17}$$

It follows from (10) and (11) that

$$\|\mathcal{H}\mathbb{Q}(c,d)\| = \lim_{m\to\infty} \frac{1}{2^{15mr}} \|\mathcal{H}f(2^{mr}c, 2^{mr}d)\| \leq \lim_{m\to\infty} \frac{1}{2^{15mr}}\varsigma(2^{mr}c, 2^{mr}d) \leq \lim_{m\to\infty} \frac{2^{mr}\zeta^r}{2^{15mr}}\varsigma(c,d) = 0$$

for all $c, d \in X$. Therefore, the mapping $\mathbb{Q} : X \to Y$ is quindecic mapping. By Lemma 2.1 in [8] and (17), we can get (12) Thus $\mathbb{Q} : X \to Y$ is a unique quindecic mapping satisfying (12). ☐

Corollary 3.2. *Let $r = \pm 1$ be fixed and let s, ω be non-negative real numbers with $s \neq 15$. Let $f : X \to Y$ be a mapping such that*

$$\|\mathcal{H}f_n([x_{kl}], [y_{kl}])\|_n \leq \sum_{k,l=1}^{n} \omega(\|x_{kl}\|^s + \|y_{kl}\|^s) \forall\ x = [x_{kl}], y = [y_{kl}] \in M_n(X). \tag{18}$$

Then there exists a unique quindecic mapping $\mathbb{Q} : X \to Y$ such that

$$\|f_n([x_{kl}]) - \mathbb{Q}_n([x_{kl}])\|_n \leq \sum_{k,l=1}^{n} \frac{\omega_0}{|2^{15} - 2^s|} \|x_{kl}\|^s \qquad \forall\ x = [x_{kl}] \in M_n(X),$$

where $\omega_0 = \frac{\omega}{15!}[42206 + 6371(2^s) + 3458(3^s) + 1470(4^s) + 470(5^s) + 106(6^s) + 15(7^s) + 8^s]$.

Proof. The proof is related to the proof of Theorem 3.1 by taking $\varsigma(c,d) = \omega(\|c\|^s + \|d\|^s)$ for all $a, b \in X$. Then we can choose $\zeta = 2^{r(s-15)}$, and we can obtain the required result. ☐

Corollary 3.3. *Let $r = \pm 1$ be fixed and let s, ω be non-negative real numbers with $s = a + b \neq 15$. Let $f : X \to Y$ be a mapping such that*

$$\|\mathcal{H}f_n([x_{kl}], [y_{kl}])\|_n \leq \sum_{k,l=1}^{n} \omega(\|x_{kl}\|^a . \|y_{kl}\|^b) \forall\ x = [x_{kl}], y = [y_{kl}] \in M_n(X). \tag{19}$$

Then there exists a unique quindecic mapping $\mathbb{Q} : X \to Y$ such that

$$\|f_n([x_{kl}]) - \mathbb{Q}_n([x_{kl}])\|_n \leq \sum_{k,l=1}^{n} \frac{\omega_0}{|2^{15} - 2^s|} \|x_{kl}\|^s \qquad \forall\ x = [x_{kl}] \in M_n(X),$$

where $\omega_0 = \frac{\omega}{15!}[9438 + 6370(2^a) + 3458(3^a) + 1470(4^a) + 470(5^a) + 106(6^a) + 15(7^a) + 8^a]$

Proof. The proof is related to the proof of Theorem 3.1. □

Corollary 3.4. *Let $r = \pm 1$ be fixed and let s, ω be non-negative real numbers with $s = a + b \neq 15$. Let $f : X \to Y$ be a mapping such that*

$$\|\mathcal{H}f_n([x_{kl}], [y_{kl}])\|_n \leq \sum_{k,l=1}^{n} \omega(\|x_{kl}\|^a \cdot \|y_{kl}\|^b + \|x_{kl}\|^{a+b} + \|y_{kl}\|^{a+b}) \forall\ x = [x_{kl}], y = [y_{kl}] \in M_n(X). \quad (20)$$

Then there exists a unique quindecic mapping $\mathbb{Q} : X \to Y$ such that

$$\|f_n([x_{kl}]) - \mathbb{Q}_n([x_{kl}])\|_n \leq \sum_{k,l=1}^{n} \frac{\omega_0}{|2^{15} - 2^s|} \|x_{kl}\|^s \qquad \forall\ x = [x_{kl}] \in M_n(X),$$

where

$$\omega_0 = \frac{\omega}{15!}\left[51644 + 6371(2^s) + 6370(2^a)3458(3^s + 3^a) + 1470(4^s + 4^a) + 470(5^s + 5^a) + 106(6^s + 6^a) + 15(7^s + 7^a) + 8^s + 8^a\right].$$

Proof. The proof is related to the proof of Theorem 3.1. □

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