Int. J. Math. And Appl., 6(1)(2018), 15–20. (Special Issue)
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Ulam Stability for Particular Difference Equation of Second Order

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Abstract: We study the Hyers-Ulam stability of the homogeneous and non-homogeneous linear difference equation of second order with initial conditions by using Z-Transforms method.

MSC: 26D10, 44A10, 39A10, 34A30.

Keywords: Hyers-Ulam stability, homogeneous and non-homogeneous, linear difference equation, Z-Transforms method.
 Accepted on: 13th April 2018

1. Introduction

In 1940, S.M. Ulam [15] posed a question concerning the stability of functional equations. This problem was solved by D.H. Hyers [6] in 1941 for Cauchy additive functional equation in Banach spaces. After that many researchers have been extensively established the stability results to the various functional equations in different directions (see [3, 4, 7, 12]). Now a days, the theory of difference equations and their applications have been receiving exhaustive attention of many researchers (see [1, 5, 16]). Some authors have proved the Hyers-Ulam stability of linear and non-linear reccurences, linear reccurence with constant co-efficients, linear reccurence of higher order in [2, 10, 13, 14, 16]. Recently, Jung et.al., [11] proved the Hyers-Ulam stability and non-stability for the difference equation of first order. Motivation of the above researchers, the authors are interested in proving the Hyers-Ulam stability of the homogeneous linear difference equation

$$m\,(w+2) - r\,(1+s)\;\; m\,(w+1) + rs\; m\,(w) = 0 \tag{1}$$

and the non-homogeneous linear difference equation

$$m\,(w+2) - r\,(1+s)\;\; m\,(w+1) + rs\; m\,(w) = l\,(w) \tag{2}$$

with initial conditions

$$m(0) = 0 \text{ and } m(1) = 1 \tag{3}$$

by using Z-Transforms method for all $w > 0$, where $r, s \neq 0$ be a constant.

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2. Preliminaries

Now, we give the definitions of Hyer-Ulam stability of the homogeneous and non-homogeneous linear difference equations (1) and (2) with (3).

Definition 2.1. *Let the homogeneous linear difference equation (1) has the Hyers-Ulam stability, if for every $\epsilon > 0$ there exists a positive constant $\boldsymbol{C}$ such that $m(w)$ satisfies the inequality*

$$|m(w+2) - r(1+s)\ m(w+1) + rs\ m(w)| \leq \epsilon,$$

with (3) then there exists a function $n(w)$ satisfying (1) with $n(0) = 0$ and $n(1) = 1$ such that $|m(w) - n(w)| \leq \boldsymbol{C}\,\epsilon$.

Definition 2.2. *We say that the non-homogeneous linear difference equation (2) has the Hyers-Ulam stability, if for every $\epsilon > 0$ there exists a constant $\boldsymbol{C}$ such that $m(w)$ be a function satisfies*

$$|m(w+2) - r(1+s)\ m(w+1) + rs\ m(w) - l(w)| \leq \epsilon,$$

with (3) then there exists a function $n(w)$ satisfying (2) with $n(0) = 0$ and $n(1) = 1$ such that $|m(w) - n(w)| \leq \boldsymbol{C}\,\epsilon$.

3. Hyers-Ulam Stability (1) and (2)

In this section, we prove the Hyers-Ulam stability of the homogeneous linear difference equations (1) by applying Z-Transforms method.

Theorem 3.1. *For every $\epsilon > 0$, there exists a positive constant $\boldsymbol{C}$ such that a function $m : (0, \infty) \to \mathbb{F}$ satisfies the inequality*

$$|m(w+2) - r(1+s)\ m(w+1) + rs\ m(w)| \leq \epsilon, \tag{4}$$

for each value of w with initial conditions $m(0) = 0$ and $m(1) = 1$, then there exists a solution function $n : (0, \infty) \to \mathbb{F}$ of the difference equation (1) with $n(0) = 0$ and $n(1) = 1$ such that $|m(w) - n(w)| \leq \boldsymbol{C}\,\epsilon$, for all $w > 0$.

Proof. Consider a function $g : (0, \infty) \to \mathbb{F}$ such that $g(w) = m(w+2) - r(1+s)\ m(w+1) + rs\ m(w)$, for all $w > 0$. By (4), we have $|g(w)| \leq \epsilon$. Now, taking Z-Transform to $g(w)$, then we have

$$Z[g(w)] = G(z) = \left(z^2 - r(1+s)\ z + rs\right)\ M(z) - z\ \ (z - r(1+s))\ \ m(0) - z\ m(1) \tag{5}$$

Using the initial conditions (3) in (5), we obtain

$$G(z) = \left(z^2 - r(1+s)\ z + rs\right)\ M(z) - z. \tag{6}$$

Since we have $z^2 - r(1+s)\ z + rs = (z-p)\ \ (z-q)$, where

$$p = \frac{r(1+s) + \sqrt{r^2\left(1+s^2+2s\right) - 4rs}}{2}$$

and

$$q = \frac{r(1+s) - \sqrt{r^2\left(1+s^2+2s\right) - 4rs}}{2}.$$

Then (6) becomes

$$M(z) - \frac{z}{(z-p)\ (z-q)} = \frac{G(z)}{(z-p)\ (z-q)}. \tag{7}$$

Now, we define a function $n(w)$ such that $n(w) = \dfrac{p^w - q^w}{p-q}\ m(1)$, then applying Z-Transforms to $n(w)$, we get

$$Z\left[n(w)\right] = N(z) = \frac{z\ m(1)}{(z-p)\ (z-q)}.$$

Hence

$$(z-p)\ (z-q)\ N(z) - z\ (z - r(1+s))\ m(0) - z\ m(1) = 0. \tag{8}$$

Since we have $n(0) = m(0) = 0$ and $n(1) = m(1) = 1$. Now,

$$\begin{aligned} Z\left[n\left(w+2\right) - r(1+s)\ n\left(w+1\right) + rs\ n\left(w\right)\right] &= Z\left[n\left(w+2\right)\right] - r(1+s)\ Z\left[n\left(w+1\right)\right] + rs\ Z\left[n\left(w\right)\right] \\ &= (z-p)\ (z-q)\ N(z) - z\ (z - r(1+s))\ n(0) - z\ n(1) \\ &= 0. \end{aligned}$$

Since Z is one-to-one operator, it holds that $n\left(w+2\right) - r(1+s)\ n\left(w+1\right) + rs\ n\left(w\right) = 0$.

Therefore $n(w)$ is a solution of (1). Then we have

$$Z\left[m(w)\right] - Z\left[n(w)\right] = M(z) - N(z) = \frac{G(z)}{(z-p)\ (z-q)} = Z\left[h(w) * g(w)\right],$$

where $h(w) = \dfrac{1}{z}\left\{\dfrac{p^w - q^w}{p-q}\right\}$. Since Z-Transforms is linear and one-to-one, we have $m(w) - n(w) = (h(w) * g(w))$. Now taking modulus on both sides, we get

$$|m(w) - n(w)| = |h(w) * g(w)| = \left|\sum_{w=-\infty}^{\infty} h(w-k)\ g(w)\right| \le \sum_{w=-\infty}^{\infty} |h(w-k)|\,|g(w)| \le \mathbf{C}\ \epsilon,$$

for each $w > 0$, where $\mathbf{C} = \sum_{w=-\infty}^{\infty} |h(w-k)| = \sum_{w=-\infty}^{\infty} \left|\dfrac{1}{z}\left\{\dfrac{a^{w-k} - b^{w-k}}{a-b}\right\}\right|$ exists for each value of w. Then by the virtue of Definition 2.1, the linear difference equation (1) has the Hyers-Ulam stability. □

By the similar argument of the Theorem 3.1 we can prove the Hyers-Ulam-Rassias stability of (1) with (3). The following corollary shows that the Hyers-Ulam-Rassias stability of the homogeneous linear difference equation (1) with initial condition (3).

Corollary 3.2. *For every $\epsilon > 0$ and $\theta : (0,\infty) \to (0,\infty)$ be a function, there exists a positive constant $\boldsymbol{C}$ such that a difference function $m : (0,\infty) \to \mathbb{F}$ satisfies the inequality*

$$\left|m\left(w+2\right) - r\left(1+s\right)\ m\left(w+1\right) + rs\ m\left(w\right)\right| \le \epsilon\ \theta(w),$$

for each value of w with initial condition (3), then there exists a solution function $n : (0,\infty) \to \mathbb{F}$ of the difference equation (1) with $n(0) = 0$ and $n(1) = 1$ such that $|m(w) - n(w)| \le \boldsymbol{C}(\epsilon)\ \theta(w)$, for all $w > 0$.

Now, we prove the Hyers-Ulam stability of the non-homogeneous linear difference equation (2) with initial conditions (3).

Theorem 3.3. *For every* $\epsilon > 0$, *there exists a positive constant* $\boldsymbol{C}$ *such that a function* $m : (0, \infty) \to \mathbb{F}$ *satisfies the inequality*

$$|m\,(w+2) - r\,(1+s)\;\; m\,(w+1) + rs\;m\,(w) - l(w)| \leq \epsilon, \tag{9}$$

for each value of w *with initial conditions (3), then there exists a solution function* $n : (0, \infty) \to \mathbb{F}$ *of the difference equation (2) with* $n(0) = 0$ *and* $n(1) = 1$ *such that* $|m(w) - n(w)| \leq \boldsymbol{C}\,\epsilon$, *for all* $w > 0$ *with* $l(0) = 0$*and* $l(1) = 1$.

Proof. Let us define a function $g : (0, \infty) \to \mathbb{F}$ such that $g(w) = m\,(w+2) - r\,(1+s)\;\; m\,(w+1) + rs\;m\,(w) - l(w)$, for all $w > 0$. By (9), we have $|g(w)| \leq \epsilon$. Now, taking Z-Transforms to $g(w)$, then

$$Z\,[g(w)] = G(w) = \left(z^2 - r(1+s)\;z + rs\right)\;\; M(z) - z\;\;(z - r(1+s))\;\; m(0) - z\;m(1) - L(z). \tag{10}$$

From (10), we have a function $m_0 : (0, \infty) \to \mathbb{F}$ is a solution of (2) if and only if

$$\left(z^2 - r(1+s)\;z + rs\right)\;\; M(z) - z\;\;(z - r(1+s))\;\; m_0(0) - z\;m_0(1) = L(z). \tag{11}$$

Using the initial conditions (3) in (11), we have

$$G(z) = \left(z^2 - r(1+s)\;z + rs\right)\;\; M(z) - z - L(z). \tag{12}$$

Since we have $z^2 - r(1+s)\;z + rs = (z-p)\;\;(z-q)$, where

$$p = \frac{r(1+s) + \sqrt{r^2\,(1+s^2+2s) - 4rs}}{2}$$

and

$$q = \frac{r(1+s) - \sqrt{r^2\,(1+s^2+2s) - 4rs}}{2}.$$

Then (12) becomes

$$M(z) - \frac{z}{(z-p)\;\;(z-q)} = \frac{G(z) + L(z)}{(z-p)\;\;(z-q)}. \tag{13}$$

Now, we define a function $n(w)$ such that $n(w) = \dfrac{p^w - q^w}{p-q}\;m(1) + [h(w) * l(w)]$, then applying Z-Transform to n(w), we get

$$Z\,[n(w)] = N(z) = \frac{z\;m(1)}{(z-p)\;\;(z-q)} + \frac{L(z)}{(z-p)\;\;(z-q)}.$$

Hence

$$(z-p)\;\;(z-q)\;\;N(z) - z\;\;(z - r(1+s))\;\;m(0) - z\;m(1) = L(z). \tag{14}$$

Since we have $n(0) = m(0) = 0$ and $n(1) = m(1) = 1$. Now,

$$\begin{aligned} Z\,[n\,(w+2) - r(1+s)\;n\,(w+1) + rs\;n\,(w)] &= Z\,[n\,(w+2)] - r(1+s)\;Z\,[\;n\,(w+1)] + rs\;Z\,[\;n\,(w)] \\ &= \left(z^2 - r(1+s)\;z + rs\right)\;\;N(z) - z\;\;(z - r(1+s))\;\;n(0) - z\;n(1) \\ &= H(z) = Z\,[h(w)] \end{aligned}$$

Since Z is one-to-one operator, it holds that $n\,(w+2) - r(1+s)\;n\,(w+1) + rs\;n\,(w) = l(w)$. Therefore, $n(w)$ is a solution of (2). Then we have

$$Z\,[m(w)] - Z\,[n(w)] = M(z) - N(z) = \frac{G(z)}{(z-p)\;(z-q)} = Z\,[h(w) * g(w)]\,,$$

where $h(w) = \frac{1}{z}\left\{\frac{p^w - q^w}{p-q}\right\}$. Since Z-Transforms is linear and one-to-one, we have $m(w) - n(w) = (h(w) * g(w))$. Now taking modulus on both sides, we get

$$|m(w) - n(w)| = |h(w) * g(w)| = \left|\sum_{w=-\infty}^{\infty} h(w-k)\; g(w)\right| \leq \sum_{w=-\infty}^{\infty} |h(w-k)|\,|g(w)| \leq \mathbf{C}\;\epsilon,$$

for each $w > 0$, where $\mathbf{C} = \sum\limits_{w=-\infty}^{\infty} |h(w-k)| = \sum\limits_{w=-\infty}^{\infty} \left|\frac{1}{z}\left\{\frac{p^{w-k} - q^{w-k}}{p-q}\right\}\right|$ exists for each value of w. Then by the virtue of Definition 2.2, the linear difference equation (2) has the Hyers-Ulam stability. □

By the similar argument of the Theorem 3.3 we can prove the Hyers-Ulam-Rassias stability of (2) with (3). The following corollary shows that the Hyers-Ulam-Rassias stability of the non-homogeneous linear difference equation (2) with initial conditions (3).

Corollary 3.4. *For every $\epsilon > 0$ and $\theta : (0,\infty) \to (0,\infty)$ be a function there exists a positive constant $\boldsymbol{C}$ such that a difference function $m : (0,\infty) \to \mathbb{F}$ satisfies the inequality*

$$|m\,(w+2) - r\,(1+s)\;\;m\,(w+1) + rs\;m\,(w) - l(w)| \leq \epsilon\;\theta(w),$$

for each value of w with initial conditions (3), then there exists a solution function $n : (0,\infty) \to \mathbb{F}$ of the difference equation (2) with $n(0) = 0$ and $n(1) = 1$ such that $|m(w) - n(w)| \leq \boldsymbol{C}(\epsilon)\;\theta(w)$, for all $w > 0$ with $q(0) = 0$ and $q(1) = 1$.

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