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Stability of Bi-Cubic Functional Equation in Multi-Banach Spaces

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Abstract: In this Paper, We obtain the generalized Hyers-Ulam-Rassias Stability for bi-cubic functional equation in Multi-Banach Spaces.

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1. Introduction

The first stability problem of functional equation was raised by S.M. Ulam [10] about seventy seven years ago. Since then, this question has attracted the attention of many researchers. Note that the affirmtive solution to this question was given in the next year by D.H. Hyers [4] in 1941. Thereafter, the stability problem of several functional equations has been extensively investigated by a number of authors, and there are many interesting results concerning this problem ([3, 5, 6, 8]). In 2016 M.Almahalebi, A.Chahbi and S. Kabbaj ([1]), investigated the generalized Hyers-Ulam stability of bi-cubic functional equation in 2-Banach spaces.

Definition 1.1 ([2]). *A Multi-norm on* $\left\{\mathcal{A}^k : k \in \mathbb{N}\right\}$ *is a sequence* $(\|.\|) = \left(\|.\|_k : k \in \mathbb{N}\right)$ *such that* $\|.\|_k$ *is a norm on* $\mathcal{A}^k$ *for each* $k \in \mathbb{N}, \|x\|_1 = \|x\|$ *for each* $x \in \mathcal{A}$, *and the following axioms are satisfied for each* $k \in \mathbb{N}$ *with* $k \geq 2$:

(1). $\left\|\left(x_{\sigma(1)}, ..., x_{\sigma(k)}\right)\right\|_k = \|(x_1...x_k)\|_k$, *for* $\sigma \in \Psi_k, x_1, ..., x_k \in \mathcal{A}$;

(2). $\|(\alpha_1 x_1, ..., \alpha_k x_k)\|_k \leq (\max_{i\in\mathbb{N}_k} |\alpha_i|) \|(x_1...x_k)\|_k$ *for* $\alpha_1...\alpha_k \in \mathbb{C}, x_1, ..., x_k \in \mathcal{A}$;

(3). $\|(x_1, ..., x_{k-1}, 0)\|_k = \|(x_1, ..., x_{k-1})\|_{k-1}$, *for* $x_1, ..., x_{k-1} \in \mathcal{A}$;

(4). $\|(x_1, ..., x_{k-1}, x_{k-1})\|_k = \|(x_1, ..., x_{k-1})\|_{k-1}$ *for* $x_1, ..., x_{k-1} \in \mathcal{A}$.

In this case, we say that $\left(\left(\mathcal{A}^k, \|.\|_k\right) : k \in \mathbb{N}\right)$ *is a multi - normed space. Suppose that* $\left(\left(\mathcal{A}^k, \|.\|_k\right) : k \in \mathbb{N}\right)$ *is a multi - normed space, and take* $k \in \mathbb{N}$. *We need the following two properties of multi - norms. They can be found in [2].*

(a). $\|(x, ..., x)\|_k = \|x\|$, *for* $x \in \mathcal{A}$,

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(b). $\max_{i\in\mathbb{N}_k} \|x_i\| \leq \|(x_1, ..., x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i\in\mathbb{N}_k} \|x_i\|, \forall x_1, ..., x_k \in \mathcal{A}.$

It follows from (b) that if $(\mathcal{A}, \|.\|)$ is a Banach space, then $(\mathcal{A}^k, \|.\|_k)$ is a Banach space for each $k \in \mathbb{N}$. In this case, $\left((\mathcal{A}^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi - Banach space.

In this paper, we estabilish the stability of the Bi-Cubic functional equation in multi-Banach spaces

$$\begin{aligned}\mathcal{D}g(p, q, r, s) &= g(2p + q, 2r + s) + g(2p - q, 2r + s) + g(2p + q, 2r - s) + g(2p - q, 2r - s)\\ &\quad - 4g(p + q, r + s) - 4g(p + q, r - s) - 24g(p + q, r) - 4g(p - q, r + s)\\ &\quad - 4g(p - q, r - s) - 24g(p - q, r) - 24g(p, r + s) - 24g(p, r - s) - 144g(p, r).\end{aligned}$$

Throughout this paper, let $\mathcal{X}$ be an linear space and $\left((\mathcal{Y}^k, \|.\|_k) : k \in \mathbb{N}\right)$ be a multi-Banach space.

2. Main Results

Theorem 2.1. *Let $g : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exists a function $\phi : X^{4k} \to [0, \infty)$ satisfying*

$$\mathcal{D}g(p_1, q_1, r_1, s_1, ..., p_k, q_k, r_k, s_k) \leq \phi(p_1, q_1, r_1, s_1, ..., p_k, q_k, r_k, s_k) \tag{1}$$

$$\phi(p_1, q_1, r_1, s_1, ..., p_k, q_k, r_k, s_k) \leq 64L\phi\left(\frac{p_1}{2}, \frac{q_1}{2}, \frac{r_1}{2}, \frac{s_1}{2}, ..., \frac{p_k}{2}, \frac{q_k}{2}, \frac{r_k}{2}, \frac{s_k}{2}\right) \tag{2}$$

for all $p_i, q_i, r_i, s_i \in \mathcal{X}$ where $i = 1, .., k$, and for some $0 < \mathcal{L} < 1$. Then there exists a unique sextic mapping $\mathcal{S} : \mathcal{X}^2 \to \mathcal{Y}$ satisfying the inequality

$$\|(g(p_1, r_1) - \mathcal{S}(p_1, r_1), ..., g(p_k, r_k) - \mathcal{S}(p_k, r_k))\| \leq \frac{1}{256(1 - \mathcal{L})}\phi(p_1, 0, r_1, 0, ..., p_k, 0, r_k, 0) \tag{3}$$

for all $p_i, q_i, r_i, s_i \in \mathcal{X}$ where $i = 1, .., k$.

Proof. Let us consider the set $\mathcal{M} = \{\beta : \mathcal{X}^2 \to \mathcal{Y}\}$ and introduce a generalized metric on $\mathcal{M}$ as follows:

$$\begin{aligned}d(\beta, h) = inf\,\{&\alpha \in [0, \infty) : \|\beta(p_1, r_1) - h(p_1, r_1), ..., \beta(p_k, r_k) - h(p_k, r_k)\|_k\\ &\leq \alpha\phi(p_1, 0, r_1, 0, ..., p_k, 0, r_k, 0)\}\end{aligned}$$

for all $p_i, q_i, r_i, s_i \in \mathcal{X}$ where $i = 1, .., k$. It is easy to show that $(\mathcal{M}, d)$ is complete [7]. Now, we consider the linear mapping $\Im : \mathcal{M} \to \mathcal{M}$ such that

$$\Im\beta(p, r) = \frac{1}{64}\beta(2p, 2r)$$

for all $p, r \in \mathcal{X}$. Given $\beta, h \in \mathcal{M}$, let $\alpha \in [0, \infty)$ be an arbitary constant with $d(\beta, h) \leq \alpha$, that is

$$\|\beta(p_1, r_1) - h(p_1, r_1), ..., \beta(p_k, r_k) - h(p_k, r_k)\| \leq \alpha\phi(p_1, 0, r_1, 0, .., p_k, 0, r_k, 0)$$

so we have

$$\begin{aligned}\|\Im\beta(p_1, r_1) - \Im h(p_1, r_1), ..., \Im\beta(p_k, r_k) - \Im h(p_k, r_k)\| &= \frac{1}{64}\|\beta(2p_1, 2r_1) - h(2p_1, 2r_1), ..., \beta(2p_k, 2r_k) - h(2p_k, 2r_k)\|\\ &\leq \alpha\mathcal{L}\phi(p_1, 0, r_1, 0, .., p_k, 0, r_k, 0)\end{aligned}$$

for all $p_i, r_i \in \mathcal{X}$ where $i = 1, .., k$. Hence, we see that $d(\Im\beta, \Im h) \leq \mathcal{L}d(\beta, h)$, for $\beta, h \in \mathcal{M}$. So $\Im$ is a strictly contractive operator. Putting $q_i = 0$ and $s_i = 0$ in (1), we have

$$\left\| \frac{1}{64}g(2p_1, 2r_1) - g(p_1, r_1), ..., \frac{1}{64}g(2p_k, 2r_k) - g(p_k, r_k) \right\| \leq \frac{1}{256}\phi\left(p_1, 0, r_1, 0, ..., p_k, 0, r_k, 0\right) \tag{4}$$

for all $p_i, r_i \in \mathcal{X}$ where $i = 1, .., k$. Thus we get $d(g, \Im g) \leq \frac{1}{256}$. By Theorem 2.2 in [9], there exists a unique mapping $\mathcal{S} : \mathcal{X}^2 \to \mathcal{Y}$ satisfying the following:

(1). $\mathcal{S}$ is fixed point of $\Im$, that is $\mathcal{S}(2p, 2r) = 64\mathcal{S}(p, r)$ for all $p, r \in \mathcal{X}$. The $\mathcal{S}$ is a unique fixed point of $\Im$ in the set $\mathcal{B} = \{\beta \in \mathcal{M} : d(g, \beta) < \infty\}$. This implies that $\mathcal{S}$ is a unique mapping such that there exists $\alpha \in (0, \infty)$ such that

$$\|g(p_1, r_1) - \mathcal{S}(p_1, r_1), ..., g(p_k, r_k) - \mathcal{S}(p_k, r_k)\| \leq \alpha\phi\left(p_1, 0, r_1, 0, ..., p_k, 0, r_k, 0\right)$$

for all $p_i, r_i \in \mathcal{X}$ where $i = 1, .., k$.

(2). $d(\Im^n, \mathcal{S}) \to 0$ as $n \to \infty$, which implies the equality

$$\lim_{n\to\infty} \Im^n g(p, r) = \lim_{n\to\infty} \frac{g(2^n(p), 2^n(q))}{2^{6n}} = \mathcal{S}(p). \tag{5}$$

(3). $d(g, \mathcal{S}) \leq \frac{1}{1 - \mathcal{L}}d(g, \Im g) \leq \frac{1}{256(1 - \mathcal{L})}\phi\left(p_1, 0, r_1, 0, ..., p_k, 0, r_k, 0\right)$,

which implies the inequality (3) holds. It follows from (1),(2) and (5), that

$$\begin{aligned} \|\mathcal{D}\mathcal{S}\left(p_1, q_1, r_1, s_1\right), ..., \mathcal{D}\mathcal{S}\left(p_k, q_k, r_k, s_k\right)\| &= \lim_{n\to\infty} \frac{1}{2^{6n}} \|(\mathcal{D}g(2^n(p_1), 2^n(q_1), 2^n(r_1), 2^n(s_1)), ..., \mathcal{D}g(2^n(p_k)2^n(q_k)2^n(r_k)2^n(s_k)))\| \\ &\leq \lim_{n\to\infty} \frac{1}{2^{6n}}\phi\left(\mathcal{D}g(2^n(p_1), 2^n(q_1), 2^n(r_1), 2^n(s_1)), ..., \mathcal{D}g(2^n(p_k)2^n(q_k)2^n(r_k)2^n(s_k))\right) \\ &= 0. \end{aligned}$$

Hence $\mathcal{S} : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is a sextic mapping, as desired. □

Corollary 2.2. *Let $\mathcal{X}$ be a linear space, and let $(\mathcal{Y}^n, \|.\|_n)$ be a multi-Banach space. Let $\theta > 0, \varpi < 6$ and $\phi : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $g(0) = 0$*

$$\|(\mathcal{D}g(p_1, q_1, r_1, s_1), ..., \mathcal{D}g(p_k, q_k, r_k, s_k))\| \leq \theta\left(\|p_1\|^\varpi + \|q_1\|^\varpi + \|r_1\|^\varpi + \|s_1\|^\varpi, .., \|p_k\|^\varpi + \|q_k\|^\varpi + \|r_k\|^\varpi + \|s_k\|^\varpi\right)$$

for all $p_i, q_i, r_i, s_i \in \mathcal{X}$ where $i = 1, .., k$. Then there exists a unique sextic mapping $\mathcal{S} : \mathcal{X}^2 \to \mathcal{Y}$ such that

$$\|g(p_1, r_1) - \mathcal{S}(p_1, r_1), ..., g(p_k, r_k) - \mathcal{S}(p_k, r_k)\| \leq \frac{\theta}{256 - 2^{\varpi+2}}\left(\|p_1\|^\varpi + \|r_1\|^\varpi, ..., \|p_k\|^\varpi + \|r_k\|^\varpi\right)$$

for all $p_i, r_i \in \mathcal{X}$ where $i = 1, .., k$.

Proof. We get the desired result from Theorem 2.1 by taking

$$\phi\left(p_1, q_1, r_1, s_1, ..., p_k, q_k, r_k, s_k\right) = \theta\left(\|p_1\|^\varpi + \|q_1\|^\varpi + \|r_1\|^\varpi + \|s_1\|^\varpi, .., \|p_k\|^\varpi + \|q_k\|^\varpi + \|r_k\|^\varpi + \|s_k\|^\varpi\right)$$

for all $p_i, q_i, r_i, s_i \in \mathcal{X}$ where $i = 1, .., k.$, and choosing $\mathcal{L} = 2^{\varpi-6}$. □

Corollary 2.3. *Let $\mathcal{X}$ be a linear space, and let $(\mathcal{Y}^n, \|.\|_n)$ be a multi-Banach space. Let $\theta > 0, \varpi < 3$ and $\phi : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $g(0) = 0$*

$$\|(\mathcal{D}g(p_1, q_1, r_1, s_1), ..., \mathcal{D}g(p_k, q_k, r_k, s_k))\| \leq \theta \left(\|p_1\|^{\varpi} . \|r_1\|^{\varpi} + \|q_1\|^{\varpi} . \|s_1\|^{\varpi}, ..., \|p_k\|^{\varpi} . \|r_k\|^{\varpi} + \|q_k\|^{\varpi} . \|s_k\|^{\varpi}\right)$$

for all $p_i, q_i, r_i, s_i \in \mathcal{X}$ where $i = 1, .., k$. Then there exists a unique sextic mapping $\mathcal{S} : \mathcal{X}^2 \to \mathcal{Y}$ such that

$$\|g(p_1, r_1) - \mathcal{S}(p_1, r_1), ..., g(p_k, r_k) - \mathcal{S}(p_k, r_k)\| \leq \frac{\theta}{256 - 2^{2\varpi+2}} \left(\|p_1\|^{\varpi} . \|r_1\|^{\varpi}, ..., \|p_k\|^{\varpi} . \|r_k\|^{\varpi}\right)$$

for all $p_i, r_i \in \mathcal{X}$ where $i = 1, .., k$.

Proof. We get the desired result from Theorem 2.1 by taking

$$\phi\left(p_1, q_1, r_1, s_1, ..., p_k, q_k, r_k, s_k\right) = \theta\left(\|p_1\|^{\varpi} . \|r_1\|^{\varpi} + \|q_1\|^{\varpi} . \|s_1\|^{\varpi}, .., \|p_k\|^{\varpi} . \|r_k\|^{\varpi} + \|q_k\|^{\varpi} . \|d_k\|^{p}\right)$$

for all $p_i, q_i, r_i, s_i \in \mathcal{X}$ where $i = 1, .., k.$, and choosing $\mathcal{L} = 2^{2\varpi-6}$. □

Lemma 2.4. *Let $\mathcal{X}$ be an linear space and let $\left((\mathcal{Y}^k, \|.\|_k) : k \in \mathbb{N}\right)$ be a multi-Banach space. Suppose that δ is a non-negative real number and $\phi : \mathcal{X} \to \mathcal{Y}$ be a function fulfills*

$$\sup_{k \in \mathbb{N}} \|((\mathcal{D}(p_1, q_1, r_1, s_1), ..., \mathcal{D}(p_k, q_k, r_k, s_k)))\|_k \leq \delta \tag{6}$$

$p_i, q_i, r_i, s_i \in \mathcal{X}$ where $i = 1, .., k$. Then there exists a unique sextic mapping $\mathcal{S} : \mathcal{X} \to \mathcal{Y}$ such that

$$\sup_{k \in \mathbb{N}} \|(g(p_1, r_1) - \mathcal{S}(p_1, r_1), ..., g(p_k, r_k) - \mathcal{S}(p_k, r_k))\|_k \leq \frac{\delta}{252}. \tag{7}$$

$p_i, r_i \in \mathcal{X}$ where $i = 1, .., k$.

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