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Finite Fourier Series Decomposition of Functions (Signals) of Three Variables

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Abstract: In this paper, we apply the properties of discrete orthonormal system of functions and derive certain results on trigonometric functions by using inverse of generalized difference operator. Also we develop a method to find out finite Fourier series decomposition of real valued functions of three variables

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1. Introduction

The development of the Fourier series and the beginnings of the field of Harmonic Analysis can be traced back to France at the beginning of the 19^{th} century. In the year 1804, Jean Baptiste Joseph Fourier published a paper dealing with a solution to a specific form of the heat equation. In order to derive this solution he utilized an infinite series expansion with trigonometric terms. While some work on trigonometric expansions had been done by earlier mathematicians, Fourier legitimized their use. By deriving a general solution to the heat equation, at the time an open and difficult problem, Fourier could be said to have started the field of Harmonic Analysis [1, 8]. A Fourier series is an expansion of a periodic function $f(x)$ in terms of an infinite sum of sines and cosines. Fourier series make use of the orthonormal relationships of the sine and cosine functions. If such a function forms a complete orthogonal system over $[-\pi,\pi]$, the Fourier series of the function $f(x)$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

where

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx\quad and\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

In this paper, we develop and analyse a new type of Finite Fourier Series Decomposition (FFSD) of two variable functions (signals) by defining discrete orthonormal family of functions by using inverse generalized difference operator $\underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}}$. This FFSD becomes Fourier Series as ℓ_1,ℓ_2 tends to zero. The main findings are verified and the diagrams are generated using MATLAB and they are given.

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2. Preliminaries

The primitive N^{th} roots of unity ($z^N = 1$ but $z^r \neq 1$; $0 < r < N$)

$$z_n = e^{i(2\pi/N)n}, \ n = 1, 2, 3, ..., N-1, \tag{1}$$

where n and N are co-prime, satisfies the geometric series expressed as

$$\sum_{k=0}^{N-1} z_n^k = \Delta^{-1} z_n^k \Big|_{k=0}^{N} = \frac{z_n^N - 1}{z_n - 1} = \begin{cases} 1 & if \quad N = 1 \\ 0 & if \quad N > 1. \end{cases} \tag{2}$$

From (1) and (2), the complex discrete-time sequence $e_r(k)$ is defined as

$$e_n(k) = (z_n)^k = e^{i(2\pi/N)nk}; \ n, \ k = 0, 1, 2, ..., N-1. \tag{3}$$

For the positive integers n, r and N, the $e_n(k)$ defined in (3) satisfies the identity

$$\sum_{k=0}^{N-1} e_n(k) = \Delta^{-1} e_n(k)\Big|_{k=0}^{N} = \Delta^{-1} e^{i(2\pi n/N)k}\Big|_{k=0}^{N} = \begin{cases} N & if \quad n = rN \\ 0 & if \quad n \neq rN. \end{cases} \tag{4}$$

This mathematical property is utilized with the factorization into two orthogonal exponential functions, $\{e_n(k)\}$ satisfying

$$\Delta^{-1} e_n(k) e_m^*(k)\Big|_{k=0}^{N} = \Delta^{-1} e^{i(\frac{2\pi(n-m)k}{N})}\Big|_{k=0}^{N} = \begin{cases} N & if \quad n - m = rN \\ 0 & if \quad n - m \neq rN, \end{cases} \tag{5}$$

where m, n and r are integers, and the notation $(*)$ represents the complex conjugate. The equation (5) induces us to define a generalized discrete orthonormal system of two variables and a finite Fourier series by replacing Δ by $\underset{\ell_1,\ell_2,\ell_3}{\Delta}$ and $e_n(k)$ by $u_n(k_1, k_2, k_3, k_3)$.

3. Basic Results

In this section, we provide some basic definitions and results to obtain the Fourier coefficients using generalized difference equation. Here $u(k_1, k_2, k_3, k_3)$ and $v(k_1, k_2, k_3)$ are real valued functions of two variables.

Definition 3.1. *Let $u(k_1, k_2, k_3), k_1, k_2, k_3 \in [0, \infty)$, be a real or complex valued function and $\ell_1, \ell_2, \ell_3 > 0$ be fixed. Then the generalized difference operator $\underset{\ell_1,\ell_2,\ell_3}{\Delta}$ on $u(k_1, k_2, k_3)$ is defined as*

$$\underset{\ell_1,\ell_2,\ell_3}{\Delta} u(k_1, k_2, k_3) = u(k_1 + \ell_1, k_2 + \ell_2, k_3 + \ell_3) - u(k_1, k_2, k_3) \tag{6}$$

and its inverse is defined as if there is a function $v(k_1, k_2, k_3)$ such that $\underset{\ell_1,\ell_2,\ell_3}{\Delta} v(k_1, k_2, k_3) = u(k_1, k_2, k_3)$, then

$$v(k_1, k_2, k_3) = \underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}} u(k_1, k_2, k_3) + c_j, \tag{7}$$

where c_j is constant.

Lemma 3.2. *If* $\underset{\ell_1,\ell_2,\ell_3}{\Delta} v(k_1,k_2,k_3) = u(k_1,k_2,k_3)$ *and* $\ell_1,\ell_2 > 0$ *where* m *is any positive integer, then we have*

$$v(k_1,k_2,k_3) - v(k_1 - m\ell_1, k_2 - m\ell_2, k_3 - m\ell_3) = \sum_{r=1}^{m} u(k_1 - r\ell_1, k_2 - r\ell_2, k_3 - r\ell_3) \tag{8}$$

Proof. Since $\underset{\ell_1,\ell_2,\ell_3}{\Delta} v(k_1,k_2,k_3) = u(k_1,k_2,k_3)$, from the Definition 3.1, we have

$$v(k_1+\ell_1, k_2+\ell_2, k_3+\ell_3) - v(k_1,k_2,k_3) = u(k_1,k_2,k_3) \tag{9}$$

Replacing k_1 by $k_1 - \ell_1$ and k_2 by $k_2 - \ell_2$, we get

$$v(k_1,k_2,k_3) = u(k_1-\ell_1, k_2-\ell_2, k_3-\ell_3) + v(k_1-\ell_1, k_2-\ell_2, k_3-\ell_3) \tag{10}$$

Again replacing k_1 by $k_1 - \ell_1$ and k_2 by $k_2 - \ell_2$ in (10), we get

$$v(k_1-\ell_1, k_2-\ell_2, k_3-\ell_3) = u(k_1-2\ell_1, k_2-2\ell_2, k_3-2\ell_3) + v(k_1-2\ell_1, k_2-2\ell_2, k_3-2\ell_3)$$

and (10) becomes

$$v(k_1,k_2,k_3) = u(k_1-\ell_1, k_2-\ell_2, k_3-\ell_3) + u(k_1-2\ell_1, k_2-2\ell_2, k_3-2\ell_3) + v(k_1-2\ell_1, k_2-2\ell_2, k_3-2\ell_3)$$

Proceeding like this upto m steps, we get (8). □

Lemma 3.3 ([4])**.** *Let* $u(k_1,k_2,k_3)$ *and* $v(k_1,k_2,k_3)$ *are the two functions, then we have*

$$\underset{\ell_1,\ell_2}{\overset{-1}{\Delta}}(u(k_1,k_2,k_3)v(k_1,k_2,k_3)) = u(k_1,k_2,k_3)\underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}} v(k_1,k_2,k_3) - \underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}}(\underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}} v(k_1+\ell_1,k_2+\ell_2,k_3+\ell_3)\underset{\ell_1,\ell_2,\ell_3}{\Delta} u(k_1,k_2,k_3)). \tag{11}$$

Lemma 3.4. *Let* s_r^m *and* S_r^m *are the Stirling numbers of first and second kinds,* $(k_1+k_2+k_3+k_3)^{(0,0)}_{\ell_1,\ell_2} = 1$, $(k_1+k_2+k_3)^{(1,1)}_{\ell_1,\ell_2} = k_1+k_2+k_3$ *and the polynomial factorial as*

$$(k_1+k_2+k_3)^{(m,m)}_{\ell_1,\ell_2} = (k_1+k_2+k_3)(k_1+k_2+k_3-(\ell_1+\ell_2+\ell_3))\dots(k_1+k_2+k_3-(m-1)(\ell_1+\ell_2+\ell_3)).$$

Then

$$(k_1+k_2+k_3)^{(m,m)}_{\ell_1,\ell_2} = \sum_{r=1}^{m} s_r^m(\ell_1+\ell_2+\ell_3)^{m-r}(k_1+k_2+k_3)^r,\ (k_1+k_2+k_3)^m = \sum_{r=1}^{m} S_r^m(\ell_1+\ell_2+\ell_3)^{m-r}(k_1+k_2+k_3)^{(r,r)}_{\ell_1,\ell_2}, \tag{12}$$

$$\underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}}(k_1+k_2+k_3)^{(m,m)}_{\ell_1,\ell_2} = \frac{(k_1+k_2+k_3)^{(m+1,m+1)}_{\ell_1,\ell_2}}{(\ell_1+\ell_2+\ell_3)(m+1)},\ \underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}}(k_1+k_2+k_3)^m = \sum_{r=1}^{m}\frac{S_r^m(\ell_1+\ell_2+\ell_3)^{m-r}(k_1+k_2+k_3)^{(r,r)}_{\ell_1,\ell_2}}{(r+1)(\ell_1+\ell_2+\ell_3)}. \tag{13}$$

Lemma 3.5. *Let* p *be real,* $\ell_1,\ \ell_2 > 0$, $k_1 \in (\ell_1,\infty), k_2 \in (\ell_2,\infty)$ *and* $p\ell_1, p\ell_2 \neq m2\pi$*. Then, we have*

$$\underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}} \cos p(k_1+k_2+k_3) = \frac{\cos p(k_1-\ell_1+k_2-\ell_2) - \cos p(k_1+k_2+k_3)}{2(1-\cos p(\ell_1+\ell_2+\ell_3))} \tag{14}$$

$$\underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}} \sin p(k_1+k_2+k_3) = \frac{\sin p(k_1-\ell_1+k_2-\ell_2) - \sin p(k_1+k_2+k_3)}{2(1-\sin p(\ell_1+\ell_2+\ell_3))} \tag{15}$$

Proof. From Defintion 3.1, $\underset{\ell_1,\ell_2,\ell_3}{\Delta} \cos p(k_1+k_2+k_3) = \cos p(k_1+\ell_1+k_2+\ell_2) - \cos p(k_1+k_2+k_3)$

$$R.P\left(\underset{\ell_1,\ell_2,\ell_3}{\Delta} e^{ip(k_1+k_2+k_3)}\right) = R.P\ \ (e^{ip(k_1+k_2+k_3)})Re(e^{ip(\ell_1+\ell_2+\ell_3)}-1)$$

Applying $\underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}}$ both sides, we obtain

$$R.P\left(\underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}} e^{ip(k_1+k_2+k_3)}\right) = R.P\ \left(\frac{e^{ip(k_1+k_2+k_3)}}{e^{ip(\ell_1+\ell_2+\ell_3)}-1}\right)$$

By taking complex conjugate and equating the real parts, we get (14). Similarly, we get the proof of (15) by equating the imaginary part. □

4. Computation of Finite Fourier Series Decomposition

In this section, we compute the Fourier series and obtain the Fourier coefficients using generalized difference equation and the orthonormal property of trigonometric functions .

Theorem 4.1. *Let $u(k_1,k_2,k_3)$ be bounded function on $[a,a+2\pi]$ and $\ell_1+\ell_2+\ell_3=\frac{2\pi}{N}$. Then we have FFSD as*

$$u(k_1,k_2,k_3)=\frac{a_{0,0}}{2}+\sum_{n=1}^{N-1}(a_{n,n}cos\ n(k_1+k_2+k_3)+b_{n,n}sin\ n(k_1+k_2+k_3))+\frac{a_{N,N}}{2}cos\ N(k_1+k_2+k_3), \tag{16}$$

where the coefficients are obtained by

$$\begin{aligned}
a_{0,0} &= \frac{\ell_1+\ell_2+\ell_3}{2\pi}\overset{-1}{\underset{\ell_1,\ell_2,\ell_3}{\Delta}} u(k_1,k_2,k_3)\Big|_a^{a+2\pi}\\
a_{n,n} &= \frac{\ell_1+\ell_2+\ell_3}{2\pi}\overset{-1}{\underset{\ell_1,\ell_2,\ell_3}{\Delta}} u(k_1,k_2,k_3)\cos\ n(k_1+k_2+k_3)\Big|_a^{a+2\pi}\\
b_{n,n} &= \frac{\ell_1+\ell_2+\ell_3}{2\pi}\overset{-1}{\underset{\ell_1,\ell_2,\ell_3}{\Delta}} u(k_1,k_2,k_3)\sin\ n(k_1+k_2+k_3)\Big|_a^{a+2\pi}
\end{aligned}$$

Proof. To prove orthogonality, consider

$$(\ell_1+\ell_2+\ell_3)\overset{-1}{\underset{\ell_1,\ell_2}{\Delta}}\frac{\cos n(k_1+k_2)}{\sqrt{2\pi}}\frac{\cos m(k_1+k_2)}{\sqrt{2\pi}}\Big|_0^{2\pi}=\frac{(\ell_1+\ell_2+\ell_3)}{2\pi}\overset{-1}{\underset{\ell_1,\ell_2}{\Delta}}\left(\cos(nk_1+mk_2)+\cos(mk_1-mk_2)\Big|_0^{2\pi}\right)=0.$$

and

$$\begin{aligned}
\overset{-1}{\underset{\ell_1,\ell_2,\ell_3}{\Delta}}\cos^2 n(k_1+k_2)\Big|_0^{2\pi}\Big|_0^{2\pi} &= \overset{-1}{\underset{\ell_1,\ell_2,\ell_3}{\Delta}}\left(\frac{1+\cos 2n(k_1+k_2+k_3)}{2}\right)\Big|_0^{2\pi}\\
&= \overset{-1}{\underset{\ell_1,\ell_2,\ell_3}{\Delta}}\left(\frac{1}{2}\right)\Big|_0^{2\pi}\Big|_0^{2\pi}+\frac{1}{2}\overset{-1}{\underset{\ell_1,\ell_2,\ell_3}{\Delta}}\cos^2 n(k_1+k_2)\Big|_0^{2\pi}\Big|_0^{2\pi}\\
&= \frac{2\pi}{\ell_1+\ell_2+\ell_3},
\end{aligned}$$

which is the required term for the FFSD coefficients $a_{n,n}$ and $b_{n,n}$. □

5. Main Results and Decomposition of Functions

In this section, we obtain FFSD for the functions like polynomial, polynomial factorial and trigonometric. Also we decompose the real valued functions of two variables into sum of sine and cosines using the generalized difference operator defined in (6).

Theorem 5.1. *Let $k_1,k_2,k_3\in(-\infty,\infty)$ and $\ell_1,\ell_2>0$. If $n(\ell_1+\ell_2+\ell_3)\neq 2m\pi$, then*

$$\overset{-1}{\underset{\ell_1,\ell_2,\ell_3}{\Delta}}(k_1+k_2+k_3)^{(m,m)}_{\ell_1,\ell_2}\cos n(k_1+k_2+k_3)$$

$$=\sum_{t=0}^{m}\sum_{r=0}^{t+1}\binom{t+1}{r}\frac{(m)_1^{(t)}(k_1+k_2+k_3)^{(m-t,m-t)}_{\ell_1,\ell_2}\cos n((k_1+k_2+k_3)-(r-1)(\ell_1+\ell_2+\ell_3))}{(-1)^{(r-1)}(\ell_1+\ell_2+\ell_3)^{-t}(2(\cos n(\ell_1+\ell_2+\ell_3)-1))^{(t+1)}} \tag{17}$$

$$\overset{-1}{\underset{\ell_1,\ell_2,\ell_3}{\Delta}}(k_1+k_2+k_3)^{(m,m)}_{\ell_1,\ell_2}\sin n(k_1+k_2+k_3)$$

$$=\sum_{t=0}^{m}\sum_{r=0}^{t+1}\binom{t+1}{r}\frac{(m)_1^{(t)}(k_1+k_2+k_3)^{(m-t,m-t)}_{\ell_1,\ell_2}\sin n((k_1+k_2+k_3)-(r-1)(\ell_1+\ell_2+\ell_3))}{(-1)^{(r-1)}(\ell_1+\ell_2+\ell_3)^{-t}(2(\sin n(\ell_1+\ell_2+\ell_3)-1))^{(t+1)}} \tag{18}$$

Proof. Taking $u(k_1,k_2,k_3)=(k_1+k_2+k_3)^{(1,1)}_{\ell_1,\ell_2}$ and $v(k_1,k_2,k_3)=\cos n(k_1+k_2+k_3)$ in (11) and using (14) we get

$$\underset{\ell_1,\ell_2}{\overset{-1}{\Delta}}(k_1+k_2+k_3)^{(1,1)}_{\ell_1,\ell_2}\cos n(k_1+k_2+k_3)=(k_1+k_2+k_3)^{(1,1)}_{\ell_1,\ell_2}\frac{\cos n(k_1+k_2+k_3)-\cos n(k_1+\ell_1+k_2+\ell_2)}{2(1-\cos n(\ell_1+\ell_2+\ell_3))}$$
$$-\underset{\ell_1,\ell_2}{\overset{-1}{\Delta}}\left(\underset{\ell_1,\ell_2}{\overset{-1}{\Delta}}\cos n(k_1+2\ell_1+k_2+2\ell_2)\underset{\ell_1,\ell_2,\ell_3}{\Delta}(k_1+k_2+k_3)^{(1,1)}_{\ell_1,\ell_2}\right)$$

Applying (14) for the above expansion we get ,

$$\underset{\ell_1,\ell_2}{\overset{-1}{\Delta}}(k_1+k_2+k_3)^{(1,1)}_{\ell_1,\ell_2}\cos n(k_1+k_2+k_3)=(k_1+k_2+k_3)^{(1,1)}_{\ell_1,\ell_2}\frac{\cos n(k_1+k_2+k_3)-\cos n(k_1+\ell_1+k_2+\ell_2)}{2(1-\cos n(\ell_1+\ell_2+\ell_3))}$$

$$-\frac{(\ell_1+\ell_2+\ell_3)\,(\cos n(k_1+k_2+k_3)-2\cos n(k_1+k_2+k_3+(\ell_1+\ell_2+\ell_3)+\cos n(k_1+k_2+k_3+2(\ell_1+\ell_2+\ell_3))}{(2(1-\cos n(\ell_1+\ell_2+\ell_3)))^2} \tag{19}$$

Taking $u(k_1,k_2,k_3)=(k_1+k_2+k_3)^{(2,2)}_{\ell_1,\ell_2}$ and $v(k_1,k_2,k_3)=\cos n(k_1+k_2+k_3)$ in (11), using (14) and (19), we get

$$\begin{aligned}\underset{\ell_1,\ell_2}{\overset{-1}{\Delta}}(k_1+k_2+k_3)^{(2,2)}_{\ell_1,\ell_2}\cos n(k_1+k_2+k_3)&=(k_1+k_2+k_3)^{(2,2)}_{\ell_1,\ell_2}\frac{\cos n(k_1+k_2+k_3-(\ell_1+\ell_2+\ell_3))-\cos n(k_1+k_2+k_3)}{2(1-\cos n(\ell_1+\ell_2+\ell_3))}\\&-\frac{2(\ell_1+\ell_2+\ell_3)(k_1+k_2+k_3)^{(1,1)}_{\ell_1,\ell_2}}{(2(1-\cos n(\ell_1+\ell_2+\ell_3)))^2}\\&(\cos n(k_1+k_2+k_3-(\ell_1+\ell_2+\ell_3))-2\cos n(k_1+k_2+k_3)\\&+\cos n(k_1+k_2+k_3+(\ell_1+\ell_2+\ell_3))\\&+\frac{2(\ell_1+\ell_2+\ell_3)^2}{(2(1-\cos n(\ell_1+\ell_2+\ell_3)))^3}(\cos n(k_1+k_2+k_3-(\ell_1+\ell_2+\ell_3))\\&-3\cos n(k_1+k_2+k_3)+3\cos n(k_1+k_2+k_3+(\ell_1+\ell_2+\ell_3))\\&-\cos n(k_1+k_2+k_3+2(\ell_1+\ell_2+\ell_3)),\end{aligned} \tag{20}$$

and hence RHS of (20) can be expressed as

$$\sum_{t=0}^{2}\sum_{r=0}^{t+1}\binom{t+1}{r}\frac{(2)^{(t)}_1(\ell_1+\ell_2+\ell_3)^t(k_1+k_2+k_3)^{(2-t,2-t)}_{\ell_1,\ell_2}\cos n((k_1+k_2+k_3)-(r-1)(\ell_1+\ell_2+\ell_3))}{(-1)^{(r-1)}(2(\cos n(\ell_1+\ell_2+\ell_3)-1))^{(t+1)}}$$

By continuing the above process upto m steps, we get (17). Now, (18) follows by replacing $\cos n(k_1+k_2+k_3)$ by $\sin n(k_1+k_2+k_3)$ in (17). □

Corollary 5.2. *When $I=[0,2\pi]$, $\ell_1+\ell_2+\ell_3=\frac{2\pi}{N}$, $k_1,k_2,k_3\in\{r(\ell_1+\ell_2+\ell_3)\}_0^{2N-1}$, the finite Fourier coefficients $a_{n,n}$ and $b_{n,n}$ for the polynomial factorial $(k_1+k_2+k_3)^{(m,m)}_{\ell_1,\ell_2}$ are given by,*

$$a_{0,0}=\frac{\ell_1+\ell_2+\ell_3}{2\pi}\underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}}(k_1+k_2+k_3)^{(m,m)}_{\ell_1,\ell_2}\Big|_0^{2\pi}=\frac{(4\pi)^{(m+1,m+1)}_{\ell_1,\ell_2}(\ell_1+\ell_2+\ell_3)}{2\pi3(\ell_1+\ell_2+\ell_3)} \tag{21}$$

$$\begin{aligned}a_{n,n}&=\frac{\ell_1+\ell_2+\ell_3}{2\pi}\underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}}(k_1+k_2+k_3)^{(m,m)}_{\ell_1,\ell_2}\cos n(k_1+k_2+k_3)\Big|_0^{2\pi}\\&=\sum_{t=0}^{m-1}\sum_{r=0}^{t+1}\binom{t+1}{r}\frac{(m)^t_1(\ell_1+\ell_2+\ell_3)^t(4\pi)^{(m-t,m-t)}_{\ell_1,\ell_2}\cos n(r-1)(\ell_1+\ell_2+\ell_3)}{N(-1)^{(r-1)}(2(\cos n(\ell_1+\ell_2+\ell_3)-1))^{t+1}}\end{aligned} \tag{22}$$

$$\begin{aligned}b_{n,n}&=\frac{\ell_1+\ell_2+\ell_3}{2\pi}\underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}}(k_1+k_2+k_3)^{(m,m)}_{\ell_1,\ell_2}\sin n(k_1+k_2+k_3)\Big|_0^{2\pi}\\&=\sum_{t=0}^{m-1}\sum_{r=0}^{t+1}\binom{t+1}{r}\frac{(m)^t_1(\ell_1+\ell_2+\ell_3)^t(4\pi)^{(m-t,m-t)}_{\ell_1,\ell_2}\sin n(r-1)(\ell_1+\ell_2+\ell_3)}{N(-1)^{(r-1)}(2(\sin n(\ell_1+\ell_2+\ell_3)-1))^{t+1}}\end{aligned} \tag{23}$$

Proof. The proof follows by applying the limit 0 to 2π in (17)and (18) and then multiplying by $\frac{\ell_1+\ell_2+\ell_3}{2\pi}$. □

Theorem 5.3. *Let* $k_1, k_2, k_3 \in (-\infty, \infty), \ell_1 + \ell_2 + \ell_3 > 0$ *. If* $n(\ell_1 + \ell_2 + \ell_3) \neq m2\pi$*, then*

$$\underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}} (k_1 + k_2 + k_3)^p \cos n(k_1 + k_2 + k_3)$$

$$= \sum_{m=1}^{p} \sum_{t=0}^{m} \sum_{r=0}^{t+1} \binom{t+1}{r} \frac{S_m^p (m)_1^{(t)} (k_1 + k_2 + k_3)_{\ell_1,\ell_2}^{(m-t),(m-t)} \cos n((k_1 + k_2 + k_3) - (r-1)(\ell_1 + \ell_2 + \ell_3))}{(-1)^{r-1}(\ell_1 + \ell_2 + \ell_3)^{m-t-p}\big(2(\cos n(\ell_1 + \ell_2 + \ell_3) - 1)\big)^{t+1}} \tag{24}$$

$$\underset{\ell_1,\ell_2,\ell_3}{\overset{-1}{\Delta}} (k_1 + k_2 + k_3)^p \sin n(k_1 + k_2 + k_3)$$

$$= \sum_{m=1}^{p} \sum_{t=0}^{m} \sum_{r=0}^{t+1} \binom{t+1}{r} \frac{S_m^p (m)_1^{(t)} (k_1 + k_2 + k_3)_{\ell_1,\ell_2}^{(m-t),(m-t)} \sin n((k_1 + k_2 + k_3) - (r-1)(\ell_1 + \ell_2 + \ell_3))}{(-1)^{r-1}(\ell_1 + \ell_2 + \ell_3)^{m-t-p}\big(2(\sin n(\ell_1 + \ell_2 + \ell_3) - 1)\big)^{t+1}}. \tag{25}$$

Proof. The proof follows by second term of (12) and applying (17). □

Corollary 5.4. *When* $I = [0,\ 2\pi]$*,* $\ell_1 + \ell_2 + \ell_3 = \frac{\pi}{N}$*, the finite Fourier coefficients* $a_{n,n}$ *and* $b_{n,n}$ *for* $n = 0, 1, 2, \cdots, N$ *for polynomial* $(k_1 + k_2 + k_3)^p$ *are given by*

$$a_{n,n} = \frac{\ell_1 + \ell_2 + \ell_3}{2\pi} \Delta_{\ell_1,\ell_2}^{-1} (k_1 + k_2 + k_3)^p \cos n(k_1 + k_2 + k_3)\Big|_0^{2\pi}$$

$$= \sum_{m=1}^{p-1} \sum_{t=0}^{m} \sum_{r=0}^{t+1} \binom{t+1}{r} \frac{S_m^p (m)_1^{(t)} (4\pi)_{\ell_1,\ell_2}^{(m-t),(m-t)} \cos n(r-1)(\ell_1 + \ell_2 + \ell_3}{(-1)^{r-1} N (\ell_1 + \ell_2 + \ell_3)^{m-t-p}\big(2(\cos n(\ell_1 + \ell_2 + \ell_3) - 1)\big)^{t+1}} \tag{26}$$

$$b_{n,n} = \frac{\ell_1 + \ell_2 + \ell_3}{2\pi} \Delta_{\ell_1,\ell_2}^{-1} (k_1 + k_2 + k_3)^p \sin n(k_1 + k_2 + k_3)\Big|_0^{2\pi}$$

$$= \sum_{m=1}^{p-1} \sum_{t=0}^{m} \sum_{r=0}^{t+1} \binom{t+1}{r} \frac{S_m^p (m)_1^{(t)} (4\pi)_{\ell}^{(m-t)} \sin n(r-1)(\ell_1 + \ell_2 + \ell_3)}{(-1)^{r-1} N (\ell_1 + \ell_2 + \ell_3)^{m-t-p}\big(2(\cos n(\ell_1 + \ell_2 + \ell_3) - 1)\big)^{t+1}}. \tag{27}$$

Proof. The proof follows by applying the limits 0 to 2π in (24) and (25), and then multiplying it by $(\ell_1 + \ell_2 + \ell_3)/2\pi$. □

6. Conclusion

Here, we have provided FFSD expression(decomposition) for the functions using summation solution form of inverse of generalized difference operator and orthonormal conditions. The nature of Fourier series are discussed and illustrated the numerical example. The Fourier series decomposition of input functions, considering as signals are generated by MATLAB.

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