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Generalized Hyers-Ulam Stability of an Additive Functional Equation

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Abstract: In this paper is to obtain some results for the stability of new additive functional equation in matrix normed spaces with the help of fixed point method.

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1. Introduction

The problem first stated by S. M. Ulam [10] in 1940 for the case of group homomorphism, and affirmatively solved by Hyers [4]. The result of Hyers was generalized by Aoki [1] for approximate additive mappings and by Th. M. Rassias [8] for approximate linear mappings by allowing the difference Cauchy equation to be controlled by $\epsilon(\|x\|^p + \|y\|^p)$. Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Rassias is called the Hyers-Ulam-Rassias stability.In 1994, J. M. Rassias [9] solved the Ulam's problem for different mappings and for many Euler-Lagrange type quadratic mappings, by involving a product of different powers of norms.

In the same year, a generalization of Rassias's theorem was obtained by Gavruta [3], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. Since then stability problems of various functional equations and mappings have been investigated by lee [6] in matrix normed spaces. By using the fixed point method, we will find the general solution and provide the stability result for the new functional equation

$$g(2v + w) - g(2v - w) + 2g(v) = 2[g(v + w) + g(v - w)] + g(2v), \tag{1}$$

in matrix normed spaces. The above functional equation is called additive functional equation if the function $g(v) = cv$ is the solution.

All over this paper, consider $\left(X, \|.\|_n\right)$ be a matrix normed space, $\left(Y, \|.\|_n\right)$ be a matrix Banach space and n be a fixed non-negative integer.

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2. General Solution of the Functional Equation (1)

In this part, we present the general solution of the functional equation (1). All over this segment $\mathcal{C}$ and $\mathcal{D}$ be real vector spaces.

Theorem 2.1. *A mapping $g : \mathcal{C} \to \mathcal{D}$ is additive if and only if g satisfies the functional equation (1) for all $v, w \in \mathcal{C}$*

Proof. Suppose that g is additive, then consider the additive functional equation

$$g(v + w) = g(v) + g(w) \tag{2}$$

for all $v, w \in \mathcal{C}$. Letting $v = w = 0$ in (2), one gets $g(0) = 0$. Putting $w = -v$ in (2), we get $g(-v) = -g(v)$ Therefore, g is an odd mapping. Refilling $w = v$ and $w = 2v$ in (2), we obtain $g(2v) = 2g(v)$ and $g(3v) = 3g(v)$ respectively. Switching $(v, w) = (v + w, v - w)$ in (2) and then the out coming equation multiplying by 2, we get

$$4g(v) = 2[g(v + w) + g(v - w)] \tag{3}$$

Substituting $v = 2v$ in (2), we obtain

$$g(2v + w) = 2g(v) + g(w) \tag{4}$$

for all $v, w \in \mathcal{C}$. Putting $w = -w$ in (4), and then adding the out coming equation to (4), one gets

$$g(2v + w) + g(2v - w) = 2g(v) \tag{5}$$

for all $v, w \in \mathcal{C}$. Combining (3) and (7), we get (1).

Conversely, suppose that g satisfies the functional equation (1). Switching $v = w = 0$ in (1), one gets $f(0) = 0$. Instead of $v = 0$ and $w = v$ in (1), we obtain $g(-v) = -g(v)$. Thus, g is an odd mapping. Refilling $(v, w) = (v, 0)$ in (2), we obtain $g(2v) = 2g(v)$. $w = 2w$ in (1), and divide it by 2, we get

$$g(v + w) + g(v - w) = g(v + 2w) + g(v - 2w) \tag{6}$$

for all $v, w \in \mathcal{C}$. Exchanging v and w in (6), we obtain

$$g(v + w) - g(v - w) = g(2v + w) + g(2v - w) \tag{7}$$

for all $v, w \in \mathcal{C}$. Joining (1) and (7), we get

$$2g(2v - w) = g(v + w) + 3g(v - w) \tag{8}$$

for all $v, w \in \mathcal{C}$. Exchanging v and w in (8), and then combining the out coming equation to (8) we have

$$g(2v - w) - g(v - 2w) = g(v + w) \tag{9}$$

for all $v, w \in \mathcal{C}$. Let $w = v + w$ in (7), we get

$$g(2v + w) - g(w) = g(3v + w) - g(v - w) \tag{10}$$

for all $v, w \in \mathcal{C}$. Exchanging v and w in the above equation and then adding the out coming equation to (10), we get

$$g(v+2w) + g(2v+w) + g(v) + g(w) = g(3v+w) + g(v+3w) \tag{11}$$

for all $v, w \in \mathcal{C}$. Put $v = v + y$ and $y = v - y$ in (1), and then switching the out coming equation in (11), one gets

$$g(v+2w) + g(2v+w) = 3[g(v) + g(w)] \tag{12}$$

for all $v, w \in \mathcal{C}$. Combining (6) and (7) and then switching (9) and (12) in the out coming equation, one gets (2). Hence the proof. □

3. Stability Result for (1)

In this part, we will establish the stability for the functional equation (1) in matrix normed spaces by with the help of fixed point method. For a mapping $g : X \to Y$, $\mathcal{E}g : X^2 \to Y$ and $\mathcal{E}g_n : M_n(X^2) \to M_n(Y)$ defined by,

$$\mathcal{E}g(o,p) = g(2o+p) + g(2o-p) + 2g(o) - 2[g(o+p) + g(o-p)] - g(2o)$$

$$\mathcal{E}g_n([x_{rs}],[y_{rs}]) = g([2x_{rs}+y_{rs}]) + g([2x_{rs}-y_{rs}]) + 2g([x_{rs}]) - 2[g([x_{rs}+y_{rs}]) - g([x_{rs}-y_{rs}])] - g(2[x_{rs}])$$

for all $o, p \in X$ and all $x = [x_{rs}], y = [y_{rs}] \in M_n(X)$.

Theorem 3.1. *Let $q = \pm 1$ be fixed and let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists a $\zeta < 1$ with*

$$\varphi(o,p) \le 2^q \zeta \varphi\left(\frac{o}{2^q}, \frac{p}{2^q}\right) \tag{13}$$

for all $o, p \in X$. Let $g : X \to Y$ be a mapping satisfying

$$\|\mathcal{E}g_n([x_{rs}],[y_{rs}])\| \le \sum_{i,j=1}^{n} \varphi(x_{rs}, y_{rs}) \forall\ x = [x_{rs}], y = [y_{rs}] \in M_n(X). \tag{14}$$

Then there exists a unique additive mapping $\mathbb{A} : X \to Y$ such that

$$\|g_n([x_{rs}]) - \mathbb{A}_n([x_{rs}])\|_n \le \sum_{r,s=1}^{n} \frac{\zeta^{\frac{1-q}{2}}}{2(1-\zeta)} \varphi(x_{rs}, 0) \forall\ x = [x_{rs}] \in M_n(X). \tag{15}$$

Proof. Set $n = 1$. Then (14) is equivalent to

$$\|\mathcal{E}g(o,p)\| \le \varphi(o,p) \qquad \forall\ a, b \in X. \tag{16}$$

Setting $o = 0$ and $p = o$in (16), we get

$$\|g(2o) - g(o)\| \le \frac{1}{2}\varphi(o,0) \qquad \forall\ o \in X. \tag{17}$$

$$\left\| g(o) - \frac{1}{2^q} g(2^q o) \right\| \le \frac{\zeta^{\left(\frac{1-q}{2}\right)}}{2} \varphi(o,0) \qquad \forall\ o \in X. \tag{18}$$

Set $\mathcal{N} = \{f : X \to Y\}$ and offer the generalized metric ρ on $\mathcal{N}$:

$$\rho(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(o) - h(o)\| \le \varphi(o,0), \forall o \in X\right\},$$

and $\mathcal{S}:\mathcal{N}\to\mathcal{N}$ be the mapping defined by, $\mathcal{S}g(o)=\frac{1}{2^q}g(2^q o)$ for all $g\in\mathcal{N}$ and $o\in X$. Consequently

$$\|\mathcal{S}g(o)-\mathcal{S}h(o)\|=\left\|\frac{1}{2^q}g(2^q o)-\frac{1}{2^q}h(2^q o)\right\|\le\zeta\varphi(o,0)\quad \text{for all } o\in X.$$

This means that $\mathcal{S}$ is a contractive mapping with $L=\zeta<1$. It follows from (18) that $\rho(g,\mathcal{S}g)\le\frac{\zeta^{\left(\frac{1-q}{2}\right)}}{|2|}$. So

$$\rho(g,\mathbb{A})\le\frac{\zeta^{\frac{1-q}{2}}}{2(1-\zeta)} \tag{19}$$

From (13) and (14), we obtain

$$\|\mathcal{E}\mathbb{A}(o,p)\|=\lim_{k\to\infty}\frac{1}{2^{qk}}\left\|\mathcal{E}f(2^{qk}o,2^{qk}p)\right\|\le\lim_{k\to\infty}\frac{1}{2^{qk}}\varphi(2^{qk}o,2^{qk}p)\le\lim_{k\to\infty}\zeta^k\varphi(o,p)=0$$

By using Lemma 2.1 in [6] and (19), one gets (15). Therefore $\mathbb{A}:X\to Y$ is a unique additive mapping. □

Corollary 3.2. *Let $q=\pm1$ be fixed and let ι,Υ be non-negative real numbers with $\iota\neq1$. Let $g:X\to Y$ be a mapping such that*

$$\|\mathcal{E}g_n([x_{rs}],[y_{rs}])\|_n\le\sum_{r,s=1}^{n}\Upsilon(\|x_{rs}\|^\iota+\|y_{rs}\|^\iota)\forall\ x=[x_{rs}],y=[y_{rs}]\in M_n(X). \tag{20}$$

Then there exists a unique additive mapping $\mathbb{A}:X\to Y$ such that

$$\|g_n([x_{rs}])-\mathbb{A}_n([x_{rs}])\|_n\le\sum_{r,s=1}^{n}\frac{\Upsilon}{2-2^\iota}\|x_{rs}\|^\iota\qquad\forall\ x=[x_{rs}]\in M_n(X).$$

Proof. The proof is related to the proof of Theorem 3.1 by taking $\varphi(o,p)=\Upsilon(\|a\|^\iota+\|b\|^\iota)$ for all $o,p\in X$. Then we can take $\zeta=2^{q(\iota-1)}$. □

4. J. M. Rassias Stability for (1)

The lessening corollary gives the John. M. Rassias stability for the additive functional equation (1). This stability containing the mixed product of sum of powers of norms.

Corollary 4.1. *Let $q=\pm1$ be fixed and let ι,Υ be non-negative real numbers with $\iota=v+w\neq1$. Let $f:X\to Y$ be a mapping such that*

$$\|\mathcal{E}g_n([x_{rs}],[y_{rs}])\|_n\le\sum_{r,s=1}^{n}\Upsilon(\|x_{rs}\|^v\,.\,\|y_{rs}\|^w+\|x_{rs}\|^{v+w}+\|y_{rs}\|^{v+w}) \tag{21}$$

$\forall\ x=[x_{rs}],y=[y_{rs}]\in M_n(X)$. *Then there exists a unique additive mapping $\mathbb{A}:X\to Y$ such that*

$$\|g_n([x_{rs}])-\mathbb{A}_n([x_{rs}])\|_n\le\sum_{r,s=1}^{n}\frac{\Upsilon}{2-2^\iota}\|x_{rs}\|^\iota\qquad\forall\ x=[x_{rs}]\in M_n(X).$$

Proof. The proof arise from Theorem 3.1. □

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