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On Signless Laplacian Energy and its Variations

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Abstract: The signless Laplacian energy of a simple connected graph G of order n and size m is defined in [1] as $LE^{+}(G) = \sum_{i=1}^{n} \mid \lambda_i - \frac{2m}{n} \mid$, where $\lambda_i, i = 1, 2, \ldots, n$ are the eigen values of the signless Laplacian matrix $L^{+}(G)$. In this paper, a new concept called signless Laplacian maximum eccentricity energy is introduced and value of the same is determined for some families of graphs. Also, signless Laplacian energy and signless Laplacian maximum eccentricity energy are compared for some classes of graphs. Further, signless Laplacian energy of direct product of graphs is obtained.

MSC: 05C50, 05C76.

Keywords: Graph energy, signless Laplacian energy, signless Laplacian maximum eccentricity energy.

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1. Introduction

Let G be a simple, connected and undirected graph of order n and size m with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix $A(G) = [a_{ij}]$ of the graph G is a square matrix of order n where a_{ij} is equal to one if v_i and v_j, $1 \leq i, j \leq$ n, are adjacent and is equal to zero otherwise. The eigen values $\beta_1, \beta_2, \ldots \beta_n$ of $A(G)$ are called the eigen values of the graph G. The energy of the graph G is defined by I. Gutman [14] in 1978 as the sum of the absolute values of its eigen values:

$$E(G) = \sum_{i=1}^{n} \mid \beta_i \mid$$

According to Gutman, Klobucar, and Majstorović [15], chemists studying total π-electron energy were aware of graph theoretic connection but considered only those graphs which had pure chemical origins and they approximated total π-electron energy of the special class of molecules called conjugated hydrocarbons. The research in graph energy has grown exponentially over the last few decades and several variations of graph energy have been conceived. Among them, we consider signless Laplacian energy for our study. In 2006, I. Gutman and B. Zhou [4] introduced the concept of Laplacian energy which depends on the Laplacian matrix. For a graph $G(n, m)$, let $D(G) = [b_{ij}]$ be the diagonal matrix of order n whose elements are given by

$$b_{ij} = \begin{cases} d_i, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

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where, d_i is the degree of the vertex $v_i \in V(G)$, the matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix whose eigen values are denoted by $\mu_1, \mu_2, \ldots \mu_n$, then, I. Gutman [1] defined Laplacian energy $LE(G)$ as

$$LE(G) = \sum_{i=1}^{n} \mid \mu_i - \frac{2m}{n} \mid$$

D. Cvetković [8] first introduced signless Laplacian spectra in 2007, however, signless Laplacian energy was defined by I. Gutman [1] in 2009 in which additive variation of the Laplacian matrix was considered. If $\gamma_1, \gamma_2, \ldots, \gamma_n$ denote the eigen values of the signless Laplacian matrix $L^+(G) = D(G) + A(G)$, then the signless Laplacian energy $LE^+(G)$ of the graph G is defined in [1] as

$$LE^+(G) = \sum_{i=1}^{n} \mid \gamma_i - \frac{2m}{n} \mid$$

For details on the properties of Laplacian and signless Laplacian energies, we refer to [2–6]. Computation of graph energy for the graphs obtained by applying graph operations has interested many researchers. Some of the results obtained on the same are: H. Ma and X.Liu in [11] determined the computational formulas for the energy of the duplication graph, line graph, subdivision graph and total graph of a regular graph G. In terms of binary operation, they proved that the energy of direct product graph $G_1 \times G_2$ is equal to the product of the energies of graphs G_1 and G_2. H. Ramane et.al. [10] proved that if two graphs G_1 and G_2 have equal average vertex degrees, then $LE(G_1 \cup G_2) = LE(G_1) + LE(G_2)$. The study on signless Laplacian energy of the product graphs is limited and hence we focus on determining the signless Laplacian energy of a particular graph product called direct product of graphs.

Another variation of graph energy called maximum degree energy of a graph was introduced by C. Adiga and M. Smitha [12] and they showed that if the maximum degree energy of a graph is rational then it must be an even integer. Based on this study, A. M. Naji and N. D. Soner [7] introduced the concept of maximum eccentricity matrix and defined the maximum eccentricity energy $EM_e(G)$ of a graph G . For a graph $G(V,E)$ with n vertices $v_1, v_2, \ldots, v_n$, let $e(v_i)$ be the eccentricity of a vertex $v_i, 1 \leq i \leq$ n, then, the maximum eccentricity matrix [7] of G is defined as $M_e(G) = [e_{ij}]$ where

$$e_{ij} = \begin{cases} max\{e(v_i), e(v_j)\}, & \text{if } v_iv_j \in E(G) \\ 0, & \text{otherwise} \end{cases}$$

Let $\delta_1, \delta_2, \ldots, \delta_n$ be the eigen values of $M_e(G)$, then, the maximum eccentricity energy is defined as

$$EM_e(G) = \sum_{i=1}^{n} \mid \delta_i \mid$$

As an extension of this study, we introduce the concept of signless Laplacian maximum eccentricity matrix $LM_e^+(G)$ of a graph G and obtain the value of signless Laplacian maximum eccentricity energy $LM_e^+E(G)$ for some classes of graphs. Further,we compare $LE^+(G)$ and $LM_e^+E(G)$. All the graphs considered in this paper are simple, connected and undirected. The eigen values are computed using Python, the spectra of matrices is generalised by observing the pattern of eigen values for different graphs and the graphs are drawn using LaTeXDraw.

2. Signless Laplacian Maximum Eccentricity Energy

For the graph $G(n,m)$, let $D(G)$ be the diagonal matrix and $M_e(G)$ be the maximum eccentricity matrix of G, then, we define signless Laplacian maximum eccentricity matrix as $LM_e^+(G) = D(G) + M_e(G)$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigen values of the signless Laplacian maximum eccentricity matrix $LM_e^+(G)$, then , the signless Laplacian maximum eccentricity energy of the graph G is defined as

$$LM_e^+E(G) = \sum_{i=1}^{n} | \lambda_i - \frac{2m}{n} |$$

Let us illustrate the concept of signless Laplacian maximum eccentricity energy by the following example.

Example 2.1. *Consider the complete graph K_5.*

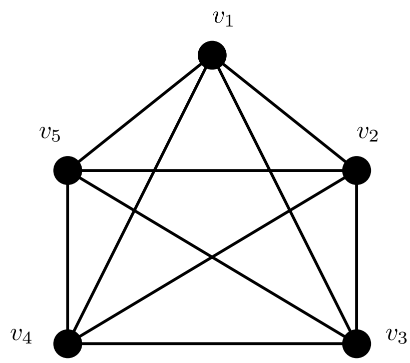

Figure 1. **Complete graph K_5**

The maximum eccentricity matrix $M_e(K_5)$ and the diagonal matrix $D(K_5)$ of the graph K_5 is

$$M_e(K_5) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}_{5\times5}$$

and

$$D(K_5) = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}_{5\times5}$$

Then, the signless Laplacian maximum eccentricity matrix $LM_e^+(K_5)$ is

$$LM_e^+(K_5) = \begin{bmatrix} 4 & 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}_{5\times5}$$

The eigen values of $LM_e^+(K_5)$ are $\lambda_1 = 8, \lambda_2 = 3, \lambda_3 = 3, \lambda_4 = 3, \lambda_5 = 3$. Therefore,

$$LM_e^+E(K_5) = \sum_{i=1}^{n} | \lambda_i - 4 |= 8.$$

2.1. Signless Laplacian maximum eccentricity energy of some standard graphs

We now proceed to determine the exact values of the signless Laplacian maximum eccentricity energies of complete bipartite graph, complete graph and friendship graph.

Theorem 2.2. *For the complete bipartite graph $K_{r,r}$ of order 2r, $r \geq 2$, the signless Laplacian maximum eccentricity energy is 4r.*

Proof. Let $K_{r,r}$, r $\geq$ 2 be the complete bipartite graph of order 2r. Then, $D(K_{r,r})$ and $M_e(K_{r,r})$ is given by

$$D(K_{r,r}) = \begin{bmatrix} r & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & r & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & r & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & r & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & r & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & r & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & r \end{bmatrix}_{2r\times 2r}$$

and

$$M_e(K_{r,r}) = \begin{bmatrix} 0 & 0 & \dots & 0 & 2 & 2 & \dots & 2 \\ 0 & 0 & \dots & 0 & 2 & 2 & \dots & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 2 & 2 & \dots & 2 \\ 2 & 2 & \dots & 2 & 0 & 0 & \dots & 0 \\ 2 & 2 & \dots & 2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \dots & 2 & 0 & 0 & \dots & 0 \end{bmatrix}_{2r\times 2r}$$

Then, the signless Laplacian maximum eccentricity matrix $LM_e^+(K_{r,r})$ is

$$LM_e^+(K_{r,r}) = \begin{bmatrix} r & 0 & \dots & 0 & 2 & 2 & \dots & 2 \\ 0 & r & \dots & 0 & 2 & 2 & \dots & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r & 2 & 2 & \dots & 2 \\ 2 & 2 & \dots & 2 & r & 0 & \dots & 0 \\ 2 & 2 & \dots & 2 & 0 & r & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \dots & 2 & 0 & 0 & \dots & r \end{bmatrix}_{2r\times 2r}$$

The spectrum of $LM_e^+(K_{r,r})$ is $\begin{pmatrix} -r & 3r & r \\ 1 & 1 & 2r-2 \end{pmatrix}$. Therefore, the signless Laplacian maximum eccentricity energy of $K_{r,r}$ is

$$LM_e^+E(K_{r,r}) = \sum_{i=1}^{2r} \mid \lambda_i - r \mid$$

$$=\mid -r-r\mid+\mid 3r-r\mid+(2r-2)\mid r-r\mid= 4r$$

□

Theorem 2.3. *The signless Laplacian maximum eccentricity energy of the complete graph K_n is $2(n-1)$.*

Proof. Let K_n be the compete graph with vertex set $\{v_1, v_2, ..., v_n\}$ then, each vertex v_i has eccentricity 1 and degree n-1. The signless Laplacian maximum eccentricity matrix of K_n is given by

$$LM_e^+(K_n) = \begin{bmatrix} n-1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & n-1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & n-1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & n-1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & 1 & n-1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & \dots & n-1 & 1 \\ 1 & 1 & 1 & 1 & 1 & \dots & 1 & n-1 \end{bmatrix}_{n\times n}$$

The spectrum of $LM_e^+(K_n)$ is $\begin{pmatrix} 2n-2 & n-2 \\ 1 & n-1 \end{pmatrix}$. Therefore, the signless Laplacian maximum eccentricity energy of K_n is

$$LM_e^+E(K_n) = \sum_{i=1}^{n} \mid \lambda_i - (n-1)\mid$$
$$=\mid 2n-2-(n-1)\mid +(n-1)\mid n-2-(n-1)\mid$$
$$= 2(n-1)$$

□

Friendship graph [13] denoted by F_r , r $\geq$ 2, is the graph constructed by joining r copies of K_3 graph with a common vertex. F_r has order 2r+1 and size 3r.

Theorem 2.4. *For the friendship graph F_r, the signless Laplacian maximum eccentricity energy is* $\dfrac{12r(r+1)}{2r+1}$.

Proof. Let F_r be the friendship graph with vertex set v_1,v_2,...,v_{2r+1}. Then, the signless Laplacian maximum eccentricity matrix of F_r is

$$LM_e^+(F_r) = \begin{bmatrix} 2r & 2 & 2 & 2 & 2 & \dots & 2 & 2 \\ 2 & 2 & 2 & 0 & 0 & \dots & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & \dots & 0 & 0 \\ 2 & 0 & 0 & 2 & 2 & \dots & 0 & 0 \\ 2 & 0 & 0 & 2 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 0 & 0 & 0 & 0 & \dots & 2 & 2 \\ 2 & 0 & 0 & 0 & 0 & \dots & 2 & 2 \end{bmatrix}_{(2r+1)\times(2r+1)}$$

The spectrum of $LM_e^+(F_r)$ is $\begin{pmatrix} 0 & 2r+4 & 4 \\ r+1 & 1 & r-1 \end{pmatrix}$. Therefore, the signless Laplacian maximum eccentricity energy of F_r is

$$LM_e^+E(F_r) = \sum_{i=1}^{2r+1} \mid \lambda_i - \frac{6r}{2r+1}\mid$$

$$= (r+1) \mid -\frac{6r}{2r+1} \mid + \mid 2r+4-\frac{6r}{2r+1} \mid +(r-1) \mid 4-\frac{6r}{2r+1} \mid$$
$$= \frac{12r(r+1)}{2r+1}$$

□

2.2. Relation between Signless Laplacian Energy and Signless Laplacian Maximum Eccentricity Energy

Conjecture 2.5. *Signless Laplacian energy of a graph G is less than or equal to signless Laplacian maximum eccentricity energy of G:*

$$LE^{+}(G) \leq LM_e^{+}E(G)$$

Bound is sharp for complete graphs.

3. Signless Laplacian Energy of Direct Product of graphs

Direct product [9] of two graphs G and H is a graph, denoted by $G \times H$, whose vertex set is $V(G) \times V(H)$ and for which two vertices (g_1, h_1) and (g_2, h_2) are precisely adjacent if and only if $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$ where $E(G)$ and $E(H)$ are edge sets of graphs G and H, respectively. Let $G(n_1, m_1)$ and $H(n_2, m_2)$ be two graphs, then $G \times H$ is a graph with order n_1n_2 and size $2m_1m_2$.

Example 3.1. *Consider two paths P_3 and P_2*

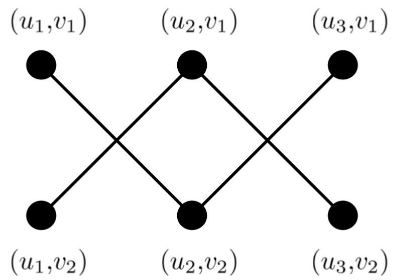

Figure 2. $P_3 \times P_2$

We now determine the signless Laplacian energy of direct product of a complete graph of order n and complete graph of order 2.

Theorem 3.2. *The signless Laplacian energy of direct product of complete graphs K_n and K_2 is 4n-4.*

Proof. Let K_n be a complete graph of order n and size $\frac{n(n-1)}{2}$, then, $K_n \times K_2$ has order 2n and size $n(n-1)$. The signless Laplacian matrix of $K_n \times K_2$ is

$$L^+(K_n \times K_2) = \begin{bmatrix} n-1 & 0 & \dots & 0 & 0 & 1 & \dots & 1 \\ 0 & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & n-1 & 1 & \dots & 1 & 0 \\ 0 & 1 & \dots & 1 & n-1 & 0 & \dots & 0 \\ 1 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & 0 \\ 1 & \dots & 1 & 0 & 0 & \dots & 0 & n-1 \end{bmatrix}_{2n\times 2n}$$

The spectrum of $L^+(K_n \times K_2)$ is $\begin{pmatrix} 0 & 2n-2 & n-2 & n \\ 1 & 1 & n-1 & n-1 \end{pmatrix}$. Therefore, the signless Laplacian energy of $K_n \times K_2$ is

$$\begin{aligned} LE^+(K_n \times K_2) &= \sum_{i=1}^{2n} \mid \gamma_i - (n-1) \mid \\ &= \mid -n+1 \mid + \mid 2n-2-n+1 \mid +(n-1) \mid n-2-n+1 \mid +(n-1) \mid n-n+1 \mid \\ &= 4n-4 \end{aligned}$$

□

Lemma 3.3 ([5]). *The signless Laplacian energy of the complete graph K_n is $2n-2$.*

Corollary 3.4. $LE^+(K_n \times K_2) = LE^+(K_n) \times LE^+(K_2)$.

Proof. From the Lemma 3.1, it follows that $LE^+(K_n)$ = 2n - 2 and $LE^+(K_2) = 2$. By Theorem 3.1, $LE^+(K_n \times K_2) =$ 4n-4. Therefore, $LE^+(K_n \times K_2) = LE^+(K_n) \times LE^+(K_2)$. □

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