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Square Sum Labeling for Pyramid Graph and Hanging Pyramid Graph

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Abstract: In this paper we prove that square sum labeling for Py_n; $(n \geq 3)$, HPy_n, $P(r, Py_n)$, $P(r, HPy_n)$, one point union of r copies of pyramid and hanging pyramid graphs are square sum graphs by using Breadth First Search Algorithm.

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Keywords: Square sum labeling, Pyramid graph, Hanging Pyramid, Path union, One point union, BFS Algorithm.

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1. Introduction

Beineke and Hegde [2] has established a study on number theory. For recent survey on graph labeling, refer [3]. A square sum and strongly square sum was initiated by Ajitha [1]. A trees, unicyclicgraphs ${}^{m}c_n$, are square sum [4]. Some middle and total graphs are proved as square sum by [8]. Pyramid graph was introduced and it was proved that the pyramid graph is graceful labeling by [6]. We begin with simple, finite, connected and undirected graph. We refer to [5, 7] for the standard notations and terminology. We present some definitions which are useful for our work.

Definition 1.1 ([1])**.** *Graph $G(p,q)$ is said to be square sum, the existence of a one-to-one and onto $f : V(G) \rightarrow \{0, 1, 2, \ldots, p-1\}$ such that it gives rise to a function $f^* : E(G) \rightarrow N$ given by $f^*(uv) = [f(u)]^2 + [f(v)]^2$ for every $uv \in E(G)$ is injective. Any graph which admits square sum labeling is called square sum graph (SSG).*

Example 1.2. *Square sum labeling for $k_{1,8}$ and k_3 are shown in Figure 1.*

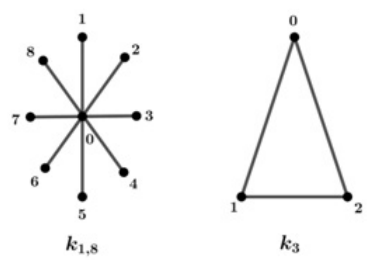

Figure 1. **Square sum labeling**

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Definition 1.3 ([6]). *A graph which procured by set out the vertices into a fixed number of rows with i vertices in i^{th} line and every line the j^{th} apex in that row is joined to the j^{th} and the $(j+1)^{th}$ vertex of the next line. We denote the pyramid with n rows by Py_n and is illustrated in Figure 2.*

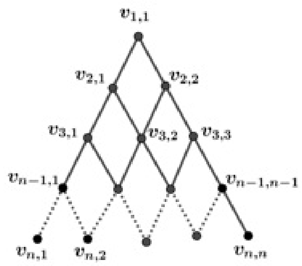

Figure 2. **Pyramid graph**

Definition 1.4. *Let graphs $G_1, G_2, \ldots, G_n$, $n \geq 2$ be all copies of a specific graph G. Adding an edge between G_i to G_{i+1} for $i = 1, 2, 3, \ldots, n-1$ is known as path union and it is denoted as $P(r, G)$ [9]. One point union of r copies represented by $G^{(r)}$.*

Definition 1.5. *Hanging pyramid graph is obtained by attaching the apex of a pyramid graph to a new pendent edge and is denoted by HPy_n.*

2. Main Results

Theorem 2.1. *Py_n admits a square sum graph.*

Proof. Let Py_n be the pyramid graph with a vertex set $V = \{v_{i,j}/1 \leq i \leq n; 1 \leq j \leq i\}$ and edge set $E = \{v_{i,j}v_{i+1,j}, v_{i,j}v_{i+1,j+1}/1 \leq i \leq n-1; 1 \leq j \leq i)\}$. Then, $|V(G)| = \frac{(n^2+n)}{2}$ and $|E(G)| = (n^2 - n)$. Start with the initial vertex, by using BFS procedure label the vertices in ascending order. Now, we define a bijection $f : V \to \{0, 1, 2, \ldots, p-1\}$as follows:

$$f(v_{i,j}) = \frac{1}{2}(i^2 - i) + (j-1) \text{ For } i = 1, 2, 3, ..., n \text{ and } j = 1, 2, 3, ..., i$$

It is obvious that from the above defined function, the condition of square sum labeling satisfies and hence the pyramid graph admits square sum. □

Example 2.2. *The Square sum labeling of Py_5 is shown in figure 3.*

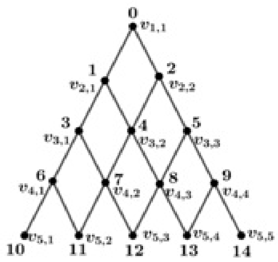

Figure 3. **Square sum of pyramid graph**

Theorem 2.3. *The Hanging Pyramid graph HPy_n admits square sum labeling.*

Proof. Consider the hanging pyramid graph with $V = V_1 \cup V_2$, where,

$$V_1 = \{v_0\}$$
$$V_2 = \{v_{i,j}\} i \text{ from } 1 \text{ to } n, j \text{ from } 1 \text{ to } i$$

and edge set $E = E_1 \cup E_2$

$$E_1 = \{V_0 V_{11}\}$$
$$E_2 = \{V_{i,j}V_{i+1,j}, V_{i,j}V_{i+1,j+1}/1 \leq i \leq n-1; 1 \leq j \leq i\}\text{for}$$

Then, $|V(G)| = \frac{1}{2}(n^2+n)+1$ and $|E(G)| = n^2 - n + 1$. Initially begin with $v_{1,1}$ of HPy_n, by Breadth First Search method label the verticesin the order in which they are visited. We define a bijection function $f : V \to \left\{0, 1, 2, \ldots, \frac{n^2+n}{2}\right\}$ as follows:

$$f(v) = 0; f(v_{i,j}) = \frac{1}{2}(i^2 - i) + j \text{ For } 1 \leq i \leq n; 1 \leq j \leq i.$$

It is clearly seen from above defined function, f is injective and all the edge labeling are in increasing sequence and are distinct. Hence the theorem is proved. □

Example 2.4. *The Square sum labeling of HPy_5 is pictorial below.*

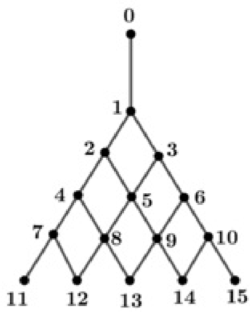

Figure 4. **Hanging pyramid graph**

Theorem 2.5. *For $n \geq 3$, $P(r, Py_n)$ is square sum graph.*

Proof. For a graph $P(r, Py_n)$ with $V = \{v_{i,j}^{(k)}/1 \leq i \leq n; 1 \leq j \leq i; 1 \leq k \leq r\}$ and $E = E_1 \cup E_2$, where,

$$E_1 = \{v_{i,j}^{(k)}v_{i+1,j}^{(k)}, v_{i,j}^{(k)}v_{i+1,j+1}^{(k)}/1 \leq i \leq n-1; 1 \leq j \leq i; 1 \leq k \leq r\} \text{ and}$$
$$E_2 = \{v_{n,n}^{(k)}v_{1,1}^{(k+1)}/1 \leq k \leq r-1; n \geq 3\}$$

Then, $|V| = r\left[\frac{(n^2+n)}{2}\right]$ and $|E| = r[n(n-1)] + (r-1)$. We begin with the vertex $v_{1,1}$ and label the vertices of pyramid graph Py_n by using Breadth first search method and the function is defined as:

$$f : V \to \left\{0, 1, 2, \ldots, r\left[\frac{n(n+1)}{2}\right] - 1\right\}$$

is bijection such that, for $1 \leq i \leq n$ and $1 \leq j \leq i$,

$$f(v_{i,j}^{(1)}) = \frac{1}{2}(i^2 - i) + (j - 1),$$

$$f(v_{i,j}^{(k)}) = f(v_{i,j}^{(k-1)}) + \frac{n(n+1)}{2}(k-1) - f(v_{1,1}^{(k-1)}), 2 \leq k \leq r$$

According to the definition of square sum labeling the above defined function f satisfies the condition of square sum. Hence $P(r, Py_n)$ is a square sum graph. □

Example 2.6. *The path union of 4 copies of Py_5 is square sum labeling is given in figure 5.*

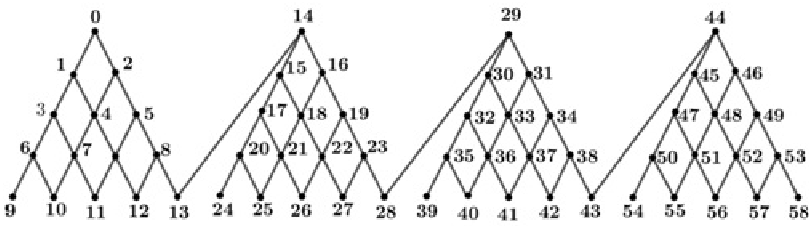

Figure 5. **Path union of 4 copies of pyramid graph**

Theorem 2.7. *For $r \geq 2$, $n \geq 3$, $P(r, HPy_n)$ is square sum graph.*

Proof. Consider $P(r, HPy_n)$ hanging pyramid graph for $n \geq 3$ with the vertex set be $V = V_1 \cup V_2$ where,

$$V_1 = \{v_k / 1 \leq k \leq r\},$$
$$V_2 = \{v_{i,j}^{(k)} / i = 1, 2, \ldots, n; j = 1, 2, \ldots, i; k = 1, 2, \ldots, r\}$$

and $E = E_1 \cup E_2 \cup E_3$, where,

$$E_1 = \{v_k v_{k+1} / 1 \leq k \leq r-1\},$$
$$E_2 = \{v_k v_{11}^{(k)} / 1 \leq k \leq r\}$$
$$E_3 = \{v_{i,j}^{(k)} v_{i+1,j}^{(k)}, v_{i,j}^{(k)} v_{i+1,j+1}^{(k)} / 1 \leq i \leq n-1; 1 \leq j \leq i; 1 \leq k \leq r\}$$

Then, $|V(G)| = r\left[\frac{n(n+1)}{2} + 1\right]$ and $|E(G)| = r(n^2 - n + 1) + (r-1)$. By BFS algorithm go through all the vertices of G in the order. Now, we define a mapping

$$f : V \to \left\{0, 1, 2, \ldots, r\left[\frac{(n^2+n)}{2} + 1\right] - 1\right\} \text{ as follows:}$$

For $1 \leq k \leq r$,

$$f(v^k) = r - k$$
$$f(v_{1,1}^{(k)}) = (r + k - 1) \text{ For } 1 \leq k \leq r$$
$$f(v_{i,j}^{(k)}) = f(v_{i-1,i-1}^{(r)}) + (k-1)i + j \text{ For } 2 \leq i \leq n; 1 \leq j \leq i.$$

The edge labeling are in increasing order and distinct. Therefore, the path union of r copies of hanging pyramid graph is a square sum. □

Example 2.8. *The $P(4, HPy_5)$ is square sum graph is shown in figure 6.*

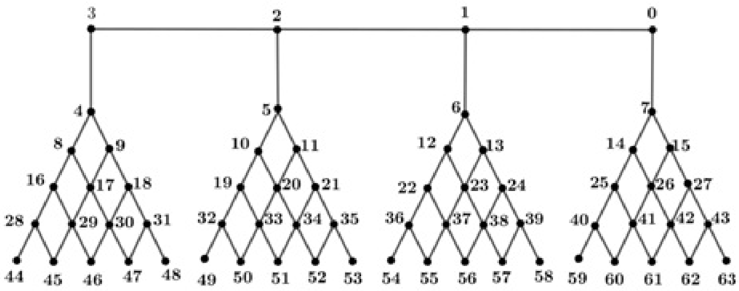

Figure 6. **Path union of 4 copies of hanging pyramid graph**

Theorem 2.9. *$Py_n^{(r)}$ admits square sum labeling for $n \geq 3$.*

Proof. Let G be $Py_n^{(r)}$ with $V = V_1 \cup V_2$, where $V_1 = \{v_{1,1} : i = j = 1\}$, $V_2 = \{v_{i,j}^{(k)} : 1 \leq i \leq n; 1 \leq j \leq i; 1 \leq k \leq r\}$ and edge set

$$E = \{v_{i,j}^{(k)} v_{i+1,j}^{(k)}, v_{i,j}^{(k)} v_{i+1,j+1}^{(k)} : 1 \leq i \leq n-1; 1 \leq j \leq i; 1 \leq k \leq r\}$$

Then, $|V(G)| = r\left[\frac{n(n+1)}{2}\right] - r + 1$ and $|E(G)| = rn(n-1)$. With the initial vertex $v_{1,1}$ by breadth first search visit and label all the vertices in non decreasing order. We define a bijection $f : V \to \left\{0, 1, 2, \ldots r\left[\frac{n(n+1)}{2}\right] - r\right\}$ as follows:

$$f(v_{1,1}) = 0, \text{ for } i = j = 1$$

$$f(v_{2,j}^{(k)}) = (k-1)i + j \text{ For } i = 2; 1 \leq j \leq i$$

$$f(v_{i,j}^{(k)}) = f(v_{i-1,i-1}^{(r)}) + (k-1)i + j \text{ For } 3 \leq i \leq n; 1 \leq j \leq i; 1 \leq k \leq r$$

We observe from above defined function, f is injective and all the edge labeling are in increasing order. Therefore, $Py_n^{(r)}$ is a square sum graph. □

Example 2.10. *$Py_4^{(4)}$ is SSG.*

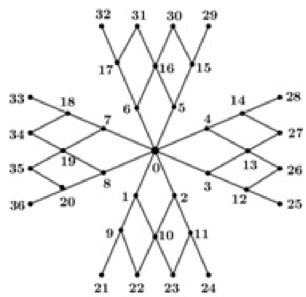

Figure 7. **One point union of 4 copies of pyramid graph**

Theorem 2.11. *The $HPy_n^{(r)}$ is square sum graph for $n \geq 3$.*

Proof. A graph with the vertex set $V = V_1 \cup V_2$ of $HPy_n^{(r)}$, where, $V_1 = \{v_0\}$, $V_2 = \{v_{i,j}^{(k)} : 1 \leq i \leq n; 1 \leq j \leq i; 1 \leq k \leq r\}$ and edge set $E = E_1 \cup E_2$ where, $E_1 = \{v_0 v_{11}^{(k)} : 1 \leq k \leq r\}$, $E_2 = \{v_{i,j}^{(k)} v_{i+1,j}^{(k)}, v_{i,j}^{(k)} v_{i+1,j+1}^{(k)} : 1 \leq i \leq n-1; 1 \leq j \leq i; 1 \leq k \leq r\}$. Then, $|V| = r\left[\frac{n(n+1)}{2}\right] + 1$ and $|E| = r(n^2 - n + 1)$. By BFS method the vertices are labeled from the root vertex in a ascending sequence. Now we define a bijection $f : V \to \left\{0, 1, 2, \ldots r\left[\frac{n(n+1)}{2}\right]\right\}$ as follows:

$$f(v_0) = 0$$

$$f(v_{1,1}^{(k)}) = k\left(\frac{i+1}{2}\right) \text{ for } i = 1, j = 1; 1 \le k \le r$$

$$f(v_{i,j}^{(k)}) = f(v_{i-1,i-1}^{(r)}) + (k-1)i + j \text{ for } 2 \le i \le n; 1 \le j \le i; 1 \le k \le r$$

According to the definition of square sum labeling, the above defined function f satisfies the condition. Therefore $HPy_n^{(r)}$ is a square sum graph. □

Example 2.12. *$HPy_4^{(8)}$ is square sum graph is shown in figure 8.*

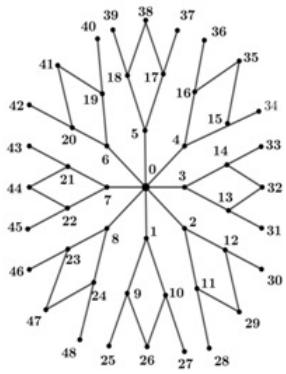

Figure 8. **One point union of 8 copies of hanging pyramid graph**

3. Conclusion

We have proved that the pyramid graph, hanging pyramid graphs, $P(r, Py_n)$, $P(r, HPy_n)$, $Py_n^{(r)}$, $HPy_n^{(r)}$ are square sum graph. To study and establish equivalent results for different graph families can be done for further work.

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