Int. J. Math. And Appl., 6(1)(2018), 109–115. (Special Issue)
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Multiple Magic Graphs

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Abstract: Let G be a graph of order n and size m. A vertex magic total labeling is an assignment of the integers $1, 2, 3, ..., m+n$ to the vertices and the edges of G, so that at each vertex , the vertex label and the labels on the edges incident at that vertex, add to a fixed constant called the magic constant of G. Such a labeling is a-vertex multiple magic if the set of the labels of the vertices is $\{a, 2a, ..., na\}$ and is b-edge multiple magic if the set of labels of the edges is $\{b, 2b, ..., mb\}$. This article, presents properties of $a-$ vertex multiple magic graphs and $b-$ edge multiple magic graphs.

MSC: 05C78.

Keywords: $a-$vertex multiple magic graphs, $b-$edge multiple magic graphs.

Accepted on: 13th April 2018

1. Introduction

Graphs are generally finite, simple and undirected. A graph G has vertex set $V = V(G)$ and the edge set $E = E(G)$. We let $n = |V|$ and $m = |E|$. The set of neighbours of a vertex v is denoted by $N(v)$. A total labeling of G is a bijection $f : V \cup E \to \{1, 2, 3, ..., m+n\}$ and the associated weights of a vertex v_i in G is

$$w_f(v_i) = f(v_i) + \sum f(v_i v_j)$$

. If each vertex has the same weight, then the total labeling f of G is vertex magic. In this case, $w_f(v_i) = k$. The magic labeling of graphs was started by sedlacek [5], but the concept of vertex magic total labeling(VMTL) first appeared only in 2002 in [2]. A VMTL f of a graph $G = (V, E)$ is said to be $a-$ vertex consecutive $f(V) = \{a+1, a+2, ..., a+n\}$, where $a \in \{0, 1, 2, ..., m\}$. A graph which admits $a-$vertex consecutive magic total labeling is called $a-$ vertex consecutive magic. Analogously, a VMTL f of a graph $G = (V, E)$ is called $b-$edge consecutive if the set of labels of the edges is $f(E) = \{b+1, b+2, ..., b+m\}$, where $b \in \{0, 1, 2, ..., n\}$. A graph which admits a $b-$ edge consecutive magic total labeling is said to be $b-$ edge consecutive magic. The paper published in consecutive vertex magic graph is [1], which is published in 2006. Nagaraj [3] introduced the concept of an Even VMTL. A VMTL is even if $f(V(G)) = \{2, 4, 6, ..., 2n\}$. A graph G is called even vertex magic graph if it admits an even vertex magic labeling. Nagaraj [4] introduced the concept of an Odd VMTL. A VMTL is odd if $f(V(G)) = \{1, 3, 5, ..., 2n-1\}$. A graph G is called odd vertex magic graph if it admits an odd vertex magic labeling.

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2. Definition and Main Results

Definition 2.1. *A vertex magic total labeling f of a graph $G = (V, E)$ is said to be a-vertex multiple, if the set of the labels of the vertices is $f(V) = \{a, 2a, 3a, ..., na\}$, where $a \in \{1, 2, 3, ..., \lfloor \frac{m+n}{n} \rfloor\}$. A graph which admits an a-vertex multiple magic total labeling is called a-vertex multiple magic.*

Definition 2.2. *A vertex magic total labeling f of a graph $G = (V, E)$ is said to be b-edge multiple, if the set of the labels of the edges is $f(E) = \{b, 2b, 3b, ..., mb\}$, where $b \in \{1, 2, 3, ..., \lfloor \frac{m+n}{m} \rfloor\}$. A graph which admits a b-edge multiple magic total labeling is said to be b-edge multiple magic.*

Theorem 2.3. *Let G be an a-vertex multiple magic graph then the magic constant k is given by*

$$k = 2m + n + 1 + \frac{m(m+1)}{n} - \frac{a(n+1)}{2}.$$

Proof. Assume that G is an a-vertex multiple magic graph. Let f be an a-vertex multiple magic labeling of a graph G with the magic constant k. Then $f(v) = \{a, 2a, 3a, ..., na\}$ with $1 \leq a \leq \lfloor \frac{m+n}{n} \rfloor$

$$\begin{aligned}
k &= f(u) + \sum_{v \in N(u)} f(uv) \quad \forall \quad u \in V \\
nk &= [a + 2a + 3a + ... + na] + 2\{1 + 2 + 3 + ... + m + n\} - 2\{a + 2a + 3a + ... + na\} \\
&= 2\{1 + 2 + 3 + ... + m + n\} - \{a + 2a + 3a + ... + na\} \\
&= \frac{2(m+n)(m+n+1)}{2} - \frac{an(n+1)}{2} \\
nk &= (m+n)(m+n+1) - \frac{an(n+1)}{2} \\
nk &= m^2 + mn + m + mn + n^2 + n - \frac{an(n+1)}{2} \\
nk &= m^2 + 2mn + m + n^2 + n - \frac{an(n+1)}{2} \\
k &= 2m + n + 1 + \frac{m(m+1)}{n} - \frac{a(n+1)}{2}
\end{aligned}$$

□

Theorem 2.4. *Let G be any $2-$ vertex multiple magic graph with size m and magic number k. Then the maximum degree is atmost $\Delta(G) \leq \sqrt{k-2}$.*

Proof. Let us consider a vertex v with maximum degree $\Delta = \Delta(G)$. The magic number $k = w(v) \geq 2 + 1 + 3 + 5 + ... + 2\Delta - 1 \Rightarrow \Delta \leq \sqrt{k-2}$. □

Theorem 2.5. *Let G be a $2-$ vertex multiple magic graph then G cannot have isolated vertex.*

Proof. From the definition of vertex magic total graph, G cannot have more than one isolated vertex. If G has an isolated vertex, say v, then the weight of that vertex satisfies $k = w(v) \leq 2n$. Then by Theorem 2.3,

$$k = 2m + \frac{m(m+1)}{n}$$

Let G be $2-$ vertex multiple magic graph then, $m \geq n$. Therefore, $k \geq 3n + 1 > 2n$, which is a contradiction to $k \leq 2n$. Therefore G cannot have isolated vertex. □

Theorem 2.6. *Let G be a b-edge multiple magic graph then the magic constant k is given by*

$$k = m + \frac{(n+1)}{2} + \frac{m(m+1)(b+1)}{2n}.$$

Proof. Assume that G is an b-edge multiple magic graph. Let f be an b-edge multiple magic labeling of a graph G with the magic constant k. Then $f(E) = \{b, 2b, 3b, ..., mb\}$ with $1 \leq b \leq \lfloor \frac{m+n}{m} \rfloor$

$$\begin{aligned}
k &= f(u) + \sum_{v \in N(u)} f(uv) \quad \forall \quad u \in V \\
nk &= \{1+2+3+...+m+n\} - \{b+2b+3b+...+mb\} + 2\{b+2b+3b+...+mb\} \\
&= \{1+2+3+...+m+n\} + \{b+2b+3b+...+mb\} \\
&= \frac{(m+n)(m+n+1)}{2} + \frac{bm(m+1)}{2} \\
nk &= \frac{m^2+mn+m+mn+n^2+n+bm^2+bm}{2} \\
nk &= \frac{m^2}{2} + mn + \frac{m}{2} + \frac{n^2}{2} + \frac{n}{2} + \frac{bm^2}{2} + \frac{bm}{2} \\
k &= \frac{m^2}{2n} + m + \frac{m}{2n} + \frac{n}{2} + \frac{1}{2} + \frac{bm^2}{2n} + \frac{bm}{2n} \\
k &= \frac{m(m+1)}{2n} + m + \frac{n+1}{2} + \frac{bm(m+1)}{2n} \\
k &= m + \frac{n+1}{2} + \frac{m(m+1)(b+1)}{2n}
\end{aligned}$$

□

Theorem 2.7. *If G is a disconnected $a-$ vertex multiple magic graph with magic constant k, then*

$$k \leq \frac{(n^4 - 2n^3 + 7n^2 - 2an^2 - 2an - 6n + 8)}{4n}.$$

Proof. Let G be a disconnected graph then $m \leq \frac{(n-1)(n-2)}{2}$. Then, by Theorem 2.3,

$$\begin{aligned}
k &= 2m + n + 1 + \frac{m(m+1)}{n} - \frac{a(n+1)}{2} \\
k &\leq 2\frac{(n-1)(n-2)}{2} + n + 1 + \frac{(n-1)^2(n-2)^2}{4n} - \frac{a(n+1)}{2} + \frac{(n-1)(n-2)}{2n} \\
k &\leq (n-1)(n-2) + n + 1 + \frac{(n-1)^2(n-2)^2}{4n} - \frac{a(n+1)}{2} + \frac{(n-1)(n-2)}{2n} \\
k &\leq \frac{1}{4n}\left\{4n(n-1)(n-2) + 4n^2 + 4n + (n-1)^2(n-2)^2 - 2na(n+1) + 2(n-1)(n-2)\right\} \\
k &\leq \frac{1}{4n}\left\{4n(n^2-3n+2) + 4n^2 + 4n + (n^2+1-2n)(n^2+4-4n) - 2a(n^2+n) + 2n^2 - 6n + 4\right\} \\
k &\leq \frac{(n^4 - 2n^3 + 7n^2 - 2an^2 - 2an - 6n + 8)}{4n}
\end{aligned}$$

□

Theorem 2.8. *If G is a connected $a-$ vertex multiple magic graph, then $k \leq \frac{(n^3+2n^2+3n+2-2an-2a)}{4}$.*

Proof. Let G be a connected graph then $m \leq \frac{n(n-1)}{2}$. Then by Theorem 2.3,

$$\begin{aligned}
k &= 2m + n + 1 + \frac{m(m+1)}{n} - \frac{a(n+1)}{2} \\
k &\leq \frac{2n(n-1)}{2} + n + 1 + \frac{n^2(n-1)^2}{4n} + \frac{n(n-1)}{2n} - \frac{a(n+1)}{2} \\
k &\leq n^2 - n + n + 1 + \frac{n^2(n^2+1-2n)}{4n} + \frac{n(n-1)}{2n} - \frac{a(n+1)}{2}
\end{aligned}$$

$$k \le \frac{1}{4}\left\{4n^2+4+n^3+n-2n^2+2n-2-2an-2a\right\}$$
$$k \le \frac{1}{4}\left\{n^3+2n^2+3n+2-2an-2a\right\}$$

Which completes the proof. □

Theorem 2.9. *If G is a disconnected $b-$ edge multiple magic graph with magic constant k, then*

$$k \le \frac{1}{8n}\left\{n^4-2n^3+7n^2-6n+8+bn^4-6bn^3+15bn^2-18bn+8b\right\}.$$

Proof. Let G be a disconnected graph then $m\frac{(n-1)(n-2)}{2}$. By Theorem 2.4,

$$k = m+\frac{n+1}{2}+\frac{m(m+1)(b+1)}{2n}$$
$$k \le \frac{(n-1)(n-2)}{2}+\frac{n+1}{2}+\frac{(n-1)^2(n-2)^2}{8n}+\frac{(n-1)(n-2)}{4n}+\frac{b}{2n}\left\{\frac{(n-1)^2(n-2)^2}{4}+\frac{(n-1)(n-2)}{2}\right\}$$
$$k \le \frac{n^2-3n+2}{2}+\frac{n+1}{2}+\frac{(n^2+1-2n)(n^2+4-4n)}{8n}+\frac{(n^2-3n+2)}{4n}$$
$$+\frac{b}{2n}\left\{\frac{(n^2+1-2n)(n^2+4-4n)}{4}+\frac{(n^2-3n+2)}{2}\right\}$$
$$k \le \frac{1}{8n}\{4n(n^2-3n+2)+4n(n+1)+n^4+4n^2-4n^3+n^2+4-4n-2n^3-8n+8n^2+2n^2$$
$$-6n+4+b(n^4+4n^2-4n^3+n^2+4-4n-2n^3-8n+8n^2)+b(2n^2-6n+4)\}$$
$$k \le \frac{1}{8n}\left\{n^4-2n^3+7n^2-6n+8+bn^4-6bn^3+15bn^2-18bn+8b\right\}$$

□

Theorem 2.10. *If G is a connected $b-$ edge multiple magic graph with magic constant k, then $k \le \frac{1}{8n}\left\{n^4+2n^3+3n^2-6n+6n^4-2bn^3+3bn^2-2bn\right\}$.*

Proof. Let G be a connected graph then $m \le \frac{n(n-1)}{2}$. Then by Theorem 2.6,

$$k = m+\frac{(n+1)}{2}+\frac{m(m+1)(b+1)}{2n}$$
$$k \le \frac{n(n-1)}{2}+\frac{(n+1)}{2}+\frac{n^2(n-1)^2}{8n}+\frac{n(n-1)}{4n}+\frac{n^2(n-1)^2b}{8n}+\frac{n(n-1)b}{4n}$$
$$k \le \frac{1}{8n}\left\{4nn(n-1)+4n(n+1)+n^2(n-1)^2+2n(n-1)+n^2(n-1)^2b+2bn(n-1)\right\}$$
$$k \le \frac{1}{8n}\left\{4n^3-4n^2+4n^2+4n+n^2(n^2+1-2n)+2n^2-2n+n^2b(n^2+1-2n)+2bn^2-2bn\right\}$$
$$k \le \frac{1}{8n}\left\{n^4+2n^3+3n^2+2n+bn^4-2bn^3+3bn^2-2bn\right\}$$

□

Lemma 2.11. *Let G be an a-vertex multiple magic graph on n vertices and m-edges.If n is odd then n divides $m(m+1)$.*

Lemma 2.12. *Let G be b-edge multiple magic graphs on n-vertices and m-edges.If n is odd then $2n$ divides $m(m+1)(b+1)$.*

Example 2.13. *k_6 is 2-vertex multiple magic graph with magic constant $k=70$.*

-	2	4	6	8	10	12
2	-	20	17	1	9	21
4	20	-	7	13	15	11
6	17	7	-	16	19	5
8	1	13	16	-	14	18
10	9	15	19	14	-	3
12	21	11	5	18	3	-

Example 2.14. *k_7 is 2-vertex multiple magic graph with magic constant $k = 108$.*

-	2	4	6	8	10	12	14
2	-	28	22	19	1	9	27
4	28	-	3	11	25	13	24
6	22	3	-	17	23	21	16
8	19	11	17	-	26	20	7
10	1	25	23	26	-	18	5
12	9	13	21	20	18	-	15
14	27	24	16	7	5	15	-

Example 2.15. *k_5 is not 2-vertex multiple magic graph.*

For $k_5, n = 5 \Rightarrow m = 5c_2 = 10 \Rightarrow k = 42$. $f : V \to \{2, 4, 6, 8, 10\}$ and $f : E \to \{1, 3, 5, 7, 9, 11, 12, 13, 14, 15\}$. Here, in edge labels 12 and 14 only two even number. k_5 is not 2-vertex multiple magic graph.

Example 2.16. *Complete bipartite graph $k_{4,4}$ is $2-$ vertex multiple magic graph with magic constant 66.*

X/Y	4	6	12	14
2	22	24	15	3
8	1	13	23	21
10	20	18	7	11
16	19	5	9	17

Example 2.17. *$k_{3,3}$ is not 2-vertex multiple magic graph.*

For $n = 6$, $m = 9$, $m + n = 15$, $k = 23$. $f : V \to \{2, 4, 6, 8, 10, 12\}$ and $f : E \to \{1, 3, 5, 7, 9, 11, 12, 13, 14, 15\}$. Here, in edge labels 14 only one even numbers.

Example 2.18. *k_5 is $3-$ vertex multiple magic graph with magic constant 39.*

	3	6	9	12	15
3	-	14	8	4	10
6	14	-	13	5	1
9	8	13	-	7	2
12	4	5	7	-	11
15	10	1	2	11	-

Example 2.19. *$H_{4,7}$ is $2-$vertex multiple magic graph with magic constant $k = 58$.*

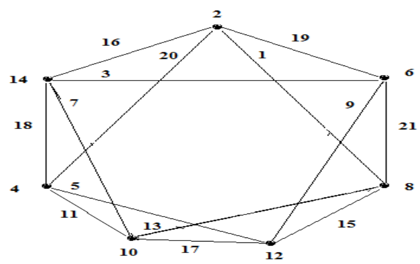

Figure 1. 2− vertex multiple magic graph. $n = 7, m = 14, k = 58$

Example 2.20. *Let G be the regular graph with* 6 *vertices and degree* 4*. Show that G is* 2− *vertex multiple magic graph with magic constant* $k = 50$.

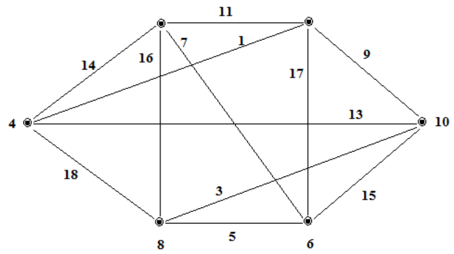

Figure 2. 2− **vertex multiple graph.** $n = 6, m = 12, k = 50$

Example 2.21. $C_3 \times C_3$ *is* 2− *vertex multiple magic graph with magic constant* $k = 70$.

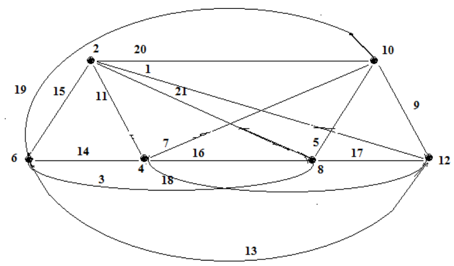

Figure 3. 2− **vertex multiple magic graph.** $n = 6, m = 15, k = 70$.

Example 2.22. $P_3 \times P_2$ *is* 2− *vertex multiple magic graph with magic constant* $k = 36$.

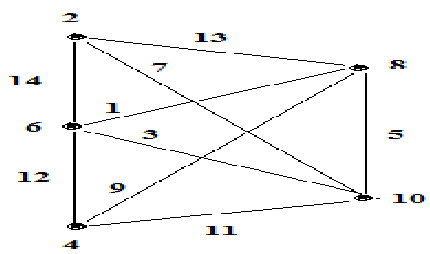

Figure 4. 2− **vertex multiple magic graph.** $n = 5, m = 9, k = 36$

Example 2.23. 2− *Edge multiple vertex magic graph with magic constant* $k = 19$

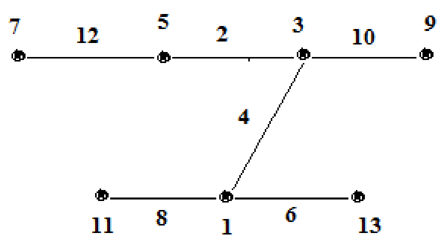

Figure 5. 2− **edge multiple vertex magic graph,** $n = 7, m = 6, k = 19$.

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