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Sub Super Mean Labeling on a General Three Star Graph

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Abstract: In this paper, we consider a three star graph $G = K_{1,\ \ell} \cup K_{1,\ m} \cup K_{1,\ n},\ \ \ell \leq m \leq n$ with a $SSML\,(V(4), E(2))$, for $g = \left[\dfrac{n+\ell-m}{2}\right]$, and find the omissions and repetitions in terms of g for two different cases.

MSC: 05C78.

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1. Introduction

Here our discussion is on a three star graph which is finite, simple and undirected one. For notations and terminology, we follow [1]. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. So G (V, E) is a graph with p vertices and q edges.

1.1. Literature Review

Much work is done by many researchers on Mean labeling and Super mean labeling, applying them on a variety of graphs [2–10]. The concept of $SSML\,(V(4), E(2))$ on a two star graph was introduced by Uma Maheswari et al [8]. After a discussion on $SSML\,(V(4), E(2))$ on a two star graph, the researchers were inspired to apply $SSML\,(V(4), E(2))$ on a three star graph and hence this paper. A study on omissions and repetitions of numbers that can be assigned to a three star graph which admits $SSML\,(V(4), E(2))$ is done and the formula is obtained in terms of ℓ, m and n in this paper.

2. Definitions

Definition 2.1. *Let G be a (p, q) graph and $f:\ V(G) \to \{1,2,3,,p+q\}$ be an injection. For each edge e=uv, let $f^*(e)=\dfrac{f(u)+f(v)}{2}$ if $f(u)+f(v)$ is even and $f^*(e)=\dfrac{f(u)+f(v)+1}{2}$ if $f(u)+f(v)$ is odd. Then f is called super mean labeling if $f(V) \cup \{f^*(e) : e \in E(G)\} = \{1,2,3,\cdots,p+q\}$. A graph that admits a super mean labeling is called a super mean graph.*

Definition 2.2. *The three star graph is the disjoint union of $K_{1,\ell}, K_{1,m}$ and $K_{1,n}$. It is denoted by $K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$.*

Example 2.3. $G = K_{1,\ 2} \cup K_{1,\ 3} \cup K_{1,\ 4}, p = 12, q = 9, p+q = 21.$

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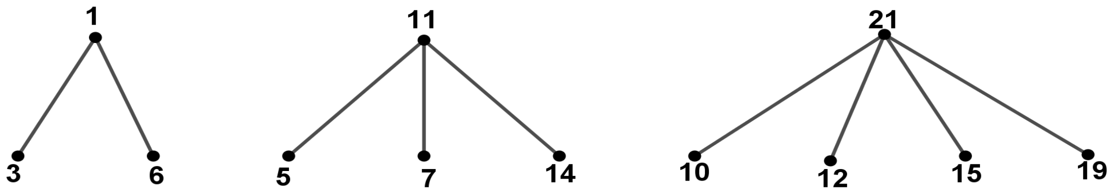


Figure 1. **Super Mean Labeling**

Definition 2.4. *Let G be a (p,q) graph and $f: V(G) \rightarrow \{1,2,3,,p+q\}$ be an injection. For each edge $e=uv$, let $f^*(e)=\dfrac{f(u)+f(v)}{2}$ if $f(u)+f(v)$ is even and $f^*(e)=\dfrac{f(u)+f(v)+1}{2}$ if $f(u)+f(v)$ is odd. Then f is called sub super mean labeling if $f(V) \cup \{f^*(e): e \in E(G)\} \subset \{1,2,3,,p+q\}$. A graph that admits a sub super mean labeling is called a sub super mean graph.*

Definition 2.5. *A specific SSML defined on a three star graph where the numbers assigned to the pendant vertices differ by four and the corresponding edge values differ by two, almost everywhere is named as $SSML\,(V(4), E(2))$ aptly.*

Example 2.6. $G = K_{1,\,2} \cup K_{1,\,3} \cup K_{1,\,4}, p = 12, q = 9, p+q = 21$

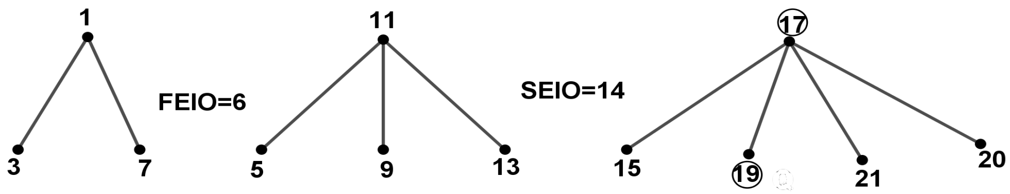


Figure 2. **Super Mean Labeling**

Definition 2.7. *If the number of repetitions = number of omissions = $(g-1), (g)\,(g+1)$ or $(g+2)$ with respect to $SSML\,(V(4), E(2))$, then the three star graph G is called $(g-1), (g)\,(g+1)$ or $(g+2)$ RO sub super mean graph where $g = \left[\dfrac{n+\ell-m}{2}\right]$.*

Definition 2.8. *For any real number x, the greatest integer $\leq x$ is denoted by $[x]$, is the integral part of x. If x is an integer $[x] = x$. If x is not an integer $[x] < x$.*

Definition 2.9. *While assigning numbers to the pendant vertices using $SSML\,(V(4), E(2))$, an even integer gets omitted moving from $K_{1,\,\ell}$ to $K_{1,\,m}$. This even integer is the first even integer omitted and is denoted by $FEIO = 2\ell + 2$.*

Definition 2.10. *While assigning numbers to the pendant vertices using $SSML\,(V(4), E(2))$, an even integer gets omitted in moving from $K_{1,\,m}$ to $K_{1,\,n}$. This even integer is the second even integer omitted and is denoted by $SEIO = 2\ell + 2m + 4$.*

Definition 2.11. *While assigning numbers to the pendant vertices using, an odd integer gets omitted in moving from This odd integer is the first odd integer omitted and is denoted by $FOIO = 4\ell + 7$.*

3. Discussion and Findings

Discussion 3.1. *The discussion of $SSML\,(V(4), E(2))$ on $G = K_{1,\,\ell} \cup K_{1,\,m} \cup K_{1,\,n}$, $\ell \leq m \leq n$, for all values of ℓ, m and n is provided below as it is required for the theorem. Let $G = K_{1,\,\ell} \cup K_{1,\,m} \cup K_{1,\,n}$, $p = 3 + \ell + m + n$, $q = \ell + m + n$, $p + q = 3 + 2\ell + 2m + 2n$. In the first copy of $G = K_{1,\,\ell} \cup K_{1,\,m} \cup K_{1,\,n}$ the labeling is done as follows: $f(u) = 1, f(u_i) = 3 + 4(i-1),\ 1 \leq i \leq \ell$. The corresponding edge labeling is given by $f^*(uu_i) = 2i,\ 1 \leq i \leq \ell$.*

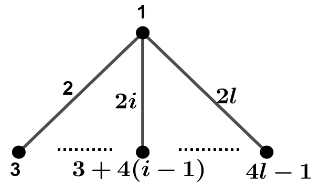


Figure 3. $K_{1,\ \ell}$

In the second copy of $G = K_{1,\ \ell} \cup K_{1,\ m} \cup K_{1,\ n}$ the labeling is done as follows: Define: $f(v) = f(u_\ell)+4 = (4\ell-1)+4 = 4\ell+3$; $f(v_j) = 5 + 4(j-1)\,,\quad 1 \le j \le m$. The corresponding edge labeling is given by

$$f^*(vv_j) = \frac{(4\ell+3)+5+4(j-1)}{2} = 2\ell+2j+2,\ 1 \le j \le m;\ FEIO = 2\ell+2.$$

The pendant vertices of the first and second copies differ by two correspondingly. As the ℓ^{th} pendant vertex of the first copy $= 4\ell-1$, the ℓ^{th} pendant vertex of the second copy is $4\ell+1$, the $\ell^{th}, (\ell+1)^{st}, (\ell+2)^{nd} \cdots$ pendant vertices of the second copy are $4\ell+1, 4\ell+5, 4\ell+9, \cdots$. Up to the ℓ^{th} place, no odd integers are omitted and $4\ell+3$ is allotted to $f(v)$.So, FOIO is $4\ell+7$, this should be the first pendant vertex of the third copy of $G = K_{1,\ \ell} \cup K_{1,\ m} \cup K_{1,\ n}$.

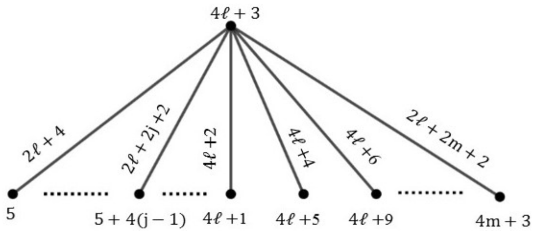
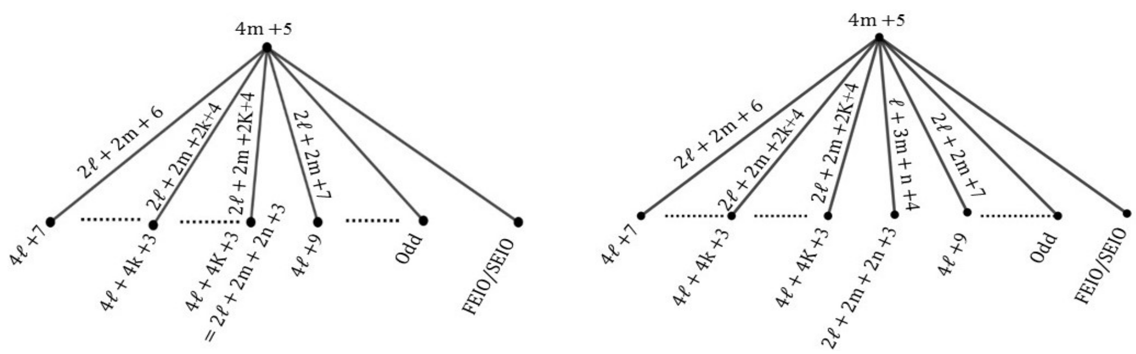


Figure 4. $K_{1,\ m}$

In the third copy of $G = K_{1,\ \ell} \cup K_{1,\ m} \cup K_{1,\ n}$ the labeling is done as follows: Define $f(w) = f(v_m)+4 = (4m+1)+4 = 4m+5$; $f(w_k) = (4\ell+7)+4(k-1),\quad 1 \le k \le K$.

To find K: As $p+q = 2\ell+2m+2n+3$, $(4\ell+7)+4(k-1) \le 2\ell+2m+2n+3$, $4k \le 2m+2n-2\ell$, $k \le \left[\frac{n+m-\ell}{2}\right]$, $K = \left[\frac{n+m-\ell}{2}\right]$, $f(w_{K+1}) = 2\ell+2m+2n+3\ or\ 4\ell+9$, $f(w_{K+2})\ to\ f(w_n)$ are allotted all the remaining odd integers, FEIO and SEIO if required, the corresponding edge labeling is given by $f^*(ww_k) = \frac{(4m+5)+(4\ell+7)+4(k-1)}{2} = 2\ell+2m+2k+4,\quad 1 \le k \le K$. The edge values $f^*(ww_{K+1})\ to\ f^*(ww_n)$ assume only odd integral values (proof given below) whether the end vertices are odd or FEIO or SEIO.



Case(A) Case(B)

Figure 5. $K_{1,\ n}$

Theorem 3.1. *If* $G = K_{1,\ \ell} \cup K_{1,\ m} \cup K_{1,\ n}$ *admits* $SSML\,(V(4), E(2))$ *with* $2 \leq \ell$, *or* $2 \leq (n-m)+\ell$ *and* $g = \left[\dfrac{n+\ell-m}{2}\right]$, *then*

(A). If $p+q = 2\ell+2m+2n+3 = 4J+3,\ J = \ell+K, K = \left[\dfrac{n+m-\ell}{2}\right]$, *the graph is a* $(g-1)\,RO, g\,RO$ *and* $(g+1)\,RO$ *graph according as both FEIO,SEIO accommodated, any one is accommodated and both not accommodated respectively.*

(B). If $(p+q) = 2\ell+2m+2n+3 > 4J+3$, *then the graph is* $g\,RO, (g+1)\,RO$ *and* $(g+2)\,RO$ *respectively as stated in (A).*

Claim (i): *If* a *is any odd integer,* b, c, d *and* e *are consecutive odd integers such that* $b < c < d < e$ *and if* $\dfrac{a+b}{2}$ *is an even integer then* $\dfrac{a+d}{2}$ *is also an even integer and* $\dfrac{a+c}{2}$ and $\dfrac{a+e}{2}$ *are odd integers, where* $|b-c| = 2,\ |b-d| = 4,\ |c-d| = 2,\ |c-e| = 4,\ \dfrac{a+b}{2} = 2k$ *(even)* $a = 4k-b$; $\dfrac{a+d}{2} = \dfrac{4k-b+d}{2} = \dfrac{4k+4}{2} = 2k+2$ *(even);* $\dfrac{a+c}{2} = \dfrac{4k-b+c}{2} = \dfrac{4k+2}{2} = 2k+1$ *(odd);* $\dfrac{a+e}{2} = \dfrac{4k-b+e}{2} = \dfrac{4k+6}{2} = 2k+3$ *(odd)*

Claim (ii): *The edge values from the* $(K+1)^{st}$ *position of the third copy of* $G = K_{1,\ \ell} \cup K_{1,\ m} \cup K_{1,\ n}$ *are odd integers. Here* $(K+1)^{st}$ *position of the pendant vertex of the third copy is an odd integer* $2\ell+2m+2n+3$ *or* $4\ell+9$ *from case (B) and case (A) respectively. The odd integer assigned to the* K^{th} *pendant vertex is* $4\ell+4K+3$. *The difference between the numbers allotted to the pendant vertices of the* K^{th} *position,* $(4\ell+4K+3)$ *and* $(K+1)^{st}$ *position* $(2\ell+2m+2n+3)$ *from case (B) cannot be one or three as both are odd, (adding 1 or 3 to* $4\ell+4K+3$ *makes* $2\ell+2m+2n+3$ *even, not possible). If the difference is four,* $2\ell+2m+2n+3$ *corresponds to the* K^{th} *pendant vertex from case (A). Therefore, the difference is two and so* $\dfrac{f(w)+(2\ell+2m+2n+3)}{2}$ *is an odd integer from Claim (i). Also if* $4\ell+9$ *is the pendant vertex of* $(K+1)^{st}$ *or* $(K+2)^{nd}$ *from case(A) and case(B), then the edge value is odd from claim(i). So the edge values corresponding to the* $(K+1)^{st}$ *position onwards assume odd integers from Claim (i). For* $\dfrac{f(w)+4\ell+7}{2}$ *=even,* $\dfrac{f(w)+4\ell+9}{2}$ *=odd.*

Claim (iii): *FEIO is a pendant vertex for the third copy of* $G = K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$ *if* $\ell+2m+4$ *is odd. Here FEIO is* $2\ell+2$.

The following observations are made

(1). The edge values in the first copy, second copy and upto the K^{th} *position of the third copy are even integers and they are in the increasing order, omitting FEIO, SEIO and few integers.*

(2). The pendant vertices are odd integers for the same, not omitting any odd integers.

(3). $FEIO = 2\ell+2$, *that is FEIO cannot be an edge value upto the* K^{th} *position of the third copy. As it is an even integer, it cannot be a pendant vertex for the same.*

(4). If $2\ell+2$ *is an edge value for some integer* x *of the third copy, then* $\dfrac{(4m+5)+x}{2} = 2\ell+2 \Rightarrow 4m+5+x = 4\ell+4 \Rightarrow x = 4(\ell-m)-1, x$ *is a negative integer, this is not possible.*

(5). If $2\ell+2$ *is a pendant vertex of the third copy, then* $\dfrac{(4m+5)+(2\ell+2)}{2} = \ell+2m+4$ *is the corresponding edge value.*

(6). If $\ell+2m+4 \leq 2\ell+2m+2$ *(is the largest edge value of the second copy)* $\Rightarrow 2 \leq \ell$*(acceptable)* $\ell+2m+4 > 2\ell+2m+2 \Rightarrow 2 > \ell \Rightarrow \ell = 1$ *(not acceptable)*

(7). Further if $\ell+2m+4$ *is even, it should be an edge value of the second copy. Repetition is not allowed, as in this case,* $\ell+2m+4$ *occurs twice as the edge value.*

(8). So, we conclude that FEIO is a pendant vertex only if $\ell+2m+4$ *is an odd integer. Hence claim (iii) is established. Proceeding in the same way claim (iv) is also established.*

Claim (iv): *SEIO is a pendant vertex for the third copy of* $G = K_{1,\ \ell} \cup K_{1,\ m} \cup K_{1,\ n}$ *if* $\ell + 3m + 5$ *is odd, here SEIO is* $2\ell+2m+4$. *It can be a pendant vertex only for the third copy and the corresponding edge value is* $\dfrac{(2\ell+2m+4)+(4m+5)}{2} = \ell+3m+5$ $\ell+3m+5$ *cannot be even. For if it is even as* $\ell+3m+5 <$ *SEIO, it should be an edge value of the second copy, which is not possible. So SEIO is a pendant vertex only if* $\ell+3m+5$ *is odd.*

Claim (v): *The result of the theorem is valid only if* $2 \leq \ell$ *or* $2 \leq (n-m)+\ell$. *The edge value corresponding to the* K^{th} *position which is even given by,*

$$\frac{(4m+5)+(4\ell+4K+3)}{2} = \frac{(4m+5)+(4\ell+4\left[\dfrac{m+n-\ell}{2}\right]+3)}{2} = \frac{2\ell+6m+2n+8}{2}$$

$(\ell+3m+n+4) \leq 2\ell+2m+2n+2$ $(2\ell+2m+2n+2\,)$ *is the largest even integer available* $m+2 \leq \ell+n \Rightarrow 2 \leq (n-m)+\ell$.

Proof. Now a counting scheme is provided for the number of odd and even integers assigned and omitted. $p+q = 2\ell+2m+2n+3$ is an odd integer. So, number of odd integers is one more than the number of even integers. $p+q = (\ell+m+n+2)+(\ell+m+n+1)$.

The total number of odd integers is $(\ell+m+n+2)$ and the total number of even integers is $(\ell+m+n+1)$. $\rightarrow (a)$

The labeling of the first copy and the second copy are filled up for the edge values and pendant vertices. So concentrate only on the third copy. In the third copy of $G = K_{1,\ \ell} \cup K_{1,\ m} \cup K_{1,\ n}$ the labeling is done as follows:

Define $f(w) = f(v_m)+4 = (4m+1)+4 = 4m+5$; $f(w_k) = (4\ell+7)+4(k-1)$, $\ 1 \leq k \leq K$. The number of odd integers used up to the K^{th} position from case (A) is

$= (1+\ell)+(1+m)+(1+K) = 3+\ell+m+K = 3+(\ell+m)+\left[\dfrac{m+n-\ell}{2}\right] = \left[\dfrac{6+\ell+3m+n}{2}\right]$ $\rightarrow (b)$

The number of odd integers yet to be assigned to the rest of the end positions from $(K+1)$ to n is

$= (\ell+m+n+2) - \left[\dfrac{6+\ell+3m+n}{2}\right] = \left[\dfrac{\ell-m+n-2}{2}\right]$ $\rightarrow (c)$

The edge values and the pendant vertices yet to be allotted is in number,

$2(n-K) = 2\left\{n - \left[\dfrac{m+n-\ell}{2}\right]\right\} = [n-m+\ell]$ $\rightarrow (d)$

$\left[\dfrac{\ell-m+n-2}{2}\right] < (n-K)$, is true, for, on substitution of K on the RHS we get $\left[\dfrac{\ell-m+n-2}{2}\right] < \left[\dfrac{n-m+\ell}{2}\right]$, which is true. From (c) and (d), we conclude that, some pendant vertices may have to assume even integers. Now we count the number of even integers to be assigned. In the first copy ℓ even integers, in the second copy m even integers, and in the third copy K even integers are used for edge values. Total number of even integers available $= \ell+m+n+1$. The difference is found out.

$$\begin{aligned}(\ell+m+n+1) - \left\{\ell+m+\left[\frac{n+m-\ell}{2}\right]\right\} &= (n+1) - \left[\frac{n+m-\ell}{2}\right]\\ &= \left[\frac{2n+2-n-m+\ell}{2}\right]\\ &= \left[\frac{n-m+\ell+2}{2}\right]\\ &= \left[\frac{n-m+\ell}{2}\right]+1\\ &= g+1,\end{aligned}$$

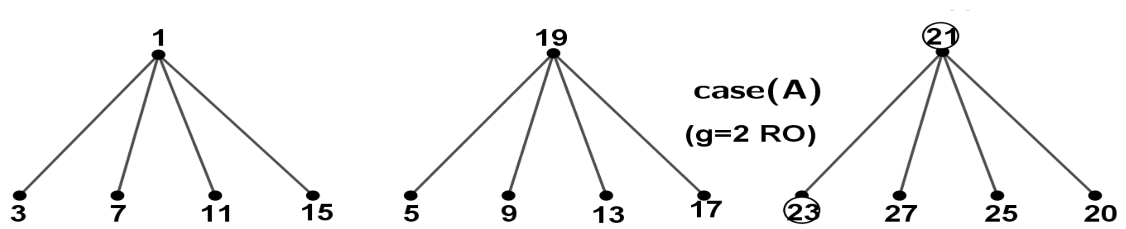
where $g = \left[\dfrac{n-m+\ell}{2}\right]$. From Claim (iii) and (iv), if $\ell+2m+4$ is odd then FEIO is accepted and if $\ell+3m+5$ is odd SEIO is accepted as pendant vertices.

Case (i): when anyone FEIO or SEIO is accommodated, the number of even integers omitted is $(g+1)-1 = g$, so the graph is a g RO graph.

Case (ii): when both FEIO and SEIO are accommodated, the number of even integers omitted is $(g+1)-2=(g-1)$, so the graph is a $(g-1)\,RO$ graph.

Case (iii): when both FEIO and SEIO are not accommodated, the number of even integers omitted is $(g+1)$, so the graph is a $(g+1)$ RO graph. When $4(\ell+K)+3<(p+q)=2\ell+2m+2n+3,\ \textit{then}\ (K+1)^{st}$ pendant vertex is given the odd integer $2\ell+2m+2n+3$. So, one extra position for an odd integer is allotted (no change in the edge values, as they are odd otherwise also) which leads to omission of one more even integer. Hence, RO discussed in case (A) is increased by one, so case (B) is established. □

Example 3.2.



$K_{1,\,4}\cup K_{1,\,4}\cup K_{1,\,4}, p=18, q=15, p+q=33$

Example 3.3.

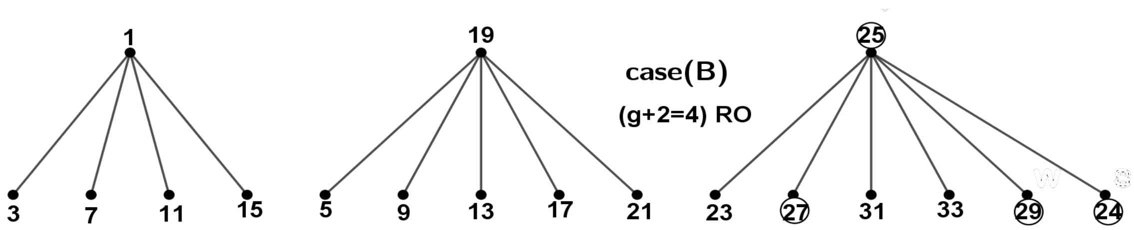


Figure 6. $K_{1,\,4}\cup K_{1,\,5}\cup K_{1,\,6}, p=18, q=15, p+q=33$

Note 3.4. *In the discussion of the above theorem we require two conditions on ℓ, m and n.*

(i). $2\leq\ell$ *(Claim iii)*

(ii). $2\leq(n-m)+\ell$ *(Claim v)*

We now prove that $2\leq\ell\Rightarrow\ 2\leq(n-m)+\ell$. If $n=m, 2\leq(n-m)+\ell$ is true as $2\leq\ell$. If $n\neq m, (n-m)\geq 1,\ (n-m)+\ell>2$. Therefore $2\leq\ell\Rightarrow 2\leq(n-m)+\ell$, for all values of m and n, but $2\leq(n-m)+\ell$ need not imply $2\leq\ell$. For if $n\neq m$ and $\ell=1, 2\leq(n-m)+\ell$ holds. But as $\ell=1, 2\leq\ell$ fails. When $\ell=1, m=2$ and $n=3$ the condition $2\leq(n-m)+\ell$ is true but $2\leq\ell$ fails, the theorem holds good for the three star $K_{1,\,1}\cup K_{1,\,2}\cup K_{1,\,3}$. When $\ell=m=n=1$ the condition $2\leq(n-m)+\ell$ and $2\leq\ell$ both fail. It is found that the labeling itself is not available for $4\ell+7$, the first pendant vertex of the third copy exceeds $(p+q)$. So, on this graph SSM $L(V(4), E(2))$ cannot be assigned and g cannot be used in this case. When $\ell=1$, the theorem may or may not hold.

4. Conclusion

We have proved that a three star graph $G=K_{1,\,\ell}\cup K_{1,\,m}\cup K_{1,\,n},\ \ell\leq m\leq n$ for various values of ℓ, m *and* n with $2\leq\ell$, is $(g-1), (g)\,(g+1)$ or $(g+2)$ RO sub super mean graphs where $g=\left\lceil\dfrac{n+\ell-m}{2}\right\rceil$ according as both FEIO and SEIO are accommodated, anyone of FEIO and SEIO is accommodated and both FEIO and SEIO not accommodated respectively with respect to two different positions of $(p+q)$ as a pendant vertex.

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