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Star Colouring of Line Graph Formed From the Cartesian Product of cycle and Path graphs

L. Jethruth Emelda Mary[1,*] and T. Monica[1]

1 PG & Research Department of Mathematics, St.Joseph's College of Arts and Science, Cuddalore, Tamil Nadu, India.

Abstract: In this paper we determined the star chromatic number of the line graph which is formed from the Cartesian product of path and cycle graphs. Also the relationship between the chromatic number and star chromatic number are analysed and illustrated with some examples.

MSC: 05C90.

Keywords: Path graph, Cycle graph, Cartesian product of simple graph, Line graph, Proper colouring, Chromatic number, Star-colouring, Star chromatic number.

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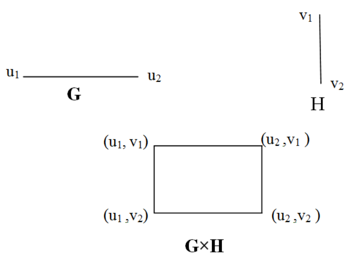
1. Introduction

In this paper, we have taken the graph to be undirected, finite and simple graph.The concept of Star Colouring was introduced by Grunbaum in 1973. In 2003, Nesetril and Ossona de Mendez proved that the star chromatic number to be bounded on each proper minor closed class [2]. The complexity of star colouring was exhibited by Albertson et al in 2004. Cartesian product of graphs have been described by Whitehead and Russel in 1912 according to Imrich and Klavzar [5]. The concept of line graph was invented by H. Whitney in 1932. Here we constructed the Line graph $L(G)$ formed from the Cartesian product of cycle and path graphs and we obtained the bounds for $L(G)$ using star colouring for various non-negative values of m and n. We begin with some basic definitions and notations.

Definition 1.1. *The Cartesian product of simple graphs G and H is denoted by $G \times H$ whose vertex set is $V(G) \times V(H)$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if $u_1 = v_1$ and u_2 is adjacent to v_2 in H or u_1 is adjacent to v_1 in G and $u_2 = v_2$ in H.*

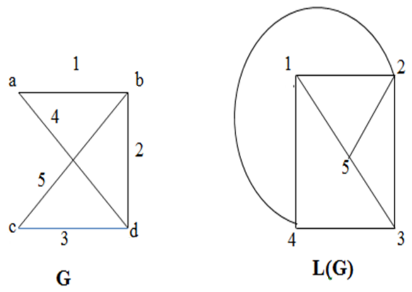
Example 1.2.



* *E-mail: jethruth15@ gmail.com*

Definition 1.3. *The line graph of a simple graph G is obtained by means of associating a vertex with every edge of the graph and connecting two vertices with an edge if and only if the corresponding edges of G have a vertex in common. The Line graph of G is denoted by $L(G)$.*

Example 1.4.



Definition 1.5. *Let G be a graph and let $V(G)$ be the set of all vertices of G and let $\{1, 2, 3 \ldots ..k\}$ be denote the set ofall colours which are assigned to each vertex of G. A proper vertex colouring of a graph G is a mapping $c : V(G) \to \{1, 2, 3 \ldots k\}$ such that $c(u) \neq c(v)$ for all arbitrary adjacent vertices $u, v \in V(G)$.*

Definition 1.6. *If G has a proper vertex colouring then the chromatic number of G is the minimum number of colours needed to colour G. The chromatic number of G is denoted by $\chi(G)$.*

Definition 1.7. *A proper vertex colouring of a graph G is called star colouring [2], if every path of G on four vertices i.e., every path on length 3 is not bicoloured.*

Definition 1.8. *The star chromatic number is the minimum number of colours needed to star colour G and is denoted by $\chi_s(G)$.*

Note 1.9.

(1). $\delta(G)$-Minimum degree of a graph G.

(2). $\Delta(G)$-Maximum degree of a graph G.

2. Star Chromatic Number of $L(C_m \times C_n)$

In this section we constructed the line graph of G formed from the Cartesian product of cycle graphs and we obtained the bounds for the line graphs using the concept of star colouring.

Theorem 2.1. *Let C_m and C_n be two cycle graphs of order m and n respectively and let $G = C_m \times C_n$ be the Cartesian product of two cycle graphs and let $L(G)$ be the line graph then the star chromatic number of $L(G)$ is given by $X_s[L(G)] = 5$, $\forall\ m, n = 3$.*

Proof. Let C_m and C_n be the cycle graphs of order m and n respectively. Let $G = C_m \times C_n$ be the Cartesian product of two cycle graphs. Let $L(G)$ be the line graph. Let $U = \{u_1, u_2, \ldots, u_m\}$ be the vertex set of C_m and $W = \{w_1, w_2, \ldots, w_n\}$ be the vertex set in C_n. The graph G has mn vertices and $2mn$ edges. By definition of line graph, $L(G)$ has $2mn$ vertices and it has $6mn$ edges. This theorem can be proved using two cases:

Case 1: When $m = n = even$ i.e., when $m = n = 2k + 2, \forall\ k = 1, 2, 3 \ldots.$ Then,

$$V(G) = \left\{ \begin{array}{cccc} (u_1, w_1) & (u_1, w_2) & . \ . \ . & (u_1, w_{2k+2}) \\ (u_2, w_1) & (u_2, w_2) & . \ . \ . & (u_2, w_{2k+2}) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ (u_{2k+2}, w_1) & (u_{2k+2}, w_2) & & (u_{2k+2}, w_{2k+2}) \end{array} \right\}$$

be the vertex set of order $4k^2 + 8k + 4$ and $E(G) = \{e_1, e_2, ..., e_{8k^2+16k+8}\}$ be the edge set of order $8k^2 + 16k + 8$. By definition of line graph,

$$V[L(G)] = \{v1 = e_1, v_2 = e_2, ..., v_{8k^2+16k+8} = e_{8k^2+16k+8}\}$$

with $8k^2 + 16k + 8$ vertices and $24k^2 + 48k + 24$ edges. In particular when $m = 4, n = 4$, the graph G has 16 vertices and 32 edges and the graph $L(G)$ has 32 vertices and 96 edges. Here $V[L(G)] = \{v_1, v_2, \ldots, v_{32}\}$. Now we star colour the graph $L(G)$ as follows.

Assign colour 1 to the vertices $v_1, v_5, v_{10}, v_{12}, v_{15}, v_{19}, v_{24}, v_{31}$ and the vertices $v_2, v_4, v_7, v_{14}, v_{17}, v_{18}, v_{21}$ are non-adjacent and are assigned colour 2 and the vertices $v_3, v_9, v_{11}, v_{13}, v_{16}, v_{23}$ are assigned with colour 3 and the vertices $v_6, v_8, v_{20}, v_{22}, v_{26}, v_{28}$ are assigned with colour 4.The remaining vertices v_{27}, v_{29}, v_{30} and v_{18} are non-adjacent so we assign same colour 5 to these vertices which satisfies the condition of star colouring. Hence the minimum number of colour needed to star colour the graph $L(G)$ is 5 i.e., $\chi_s[L(G)] = 5$, for $m = n = 4$ and $k = 1$. Proceeding in this way, for m and n are equal and even, we have the star chromatic number of $L(G)$ is 5 i.e., $\chi_s[L(G)] = 5$, when $m = n = 2k + 2,\ \forall\ k = 1, 2, \ldots.$

Case 2: When m and n are odd and distinct i.e., if $m = 2k + 1, n = 2k + 3 \forall k = 1, 2, \ldots.$

$$V(G) = \left\{ \begin{array}{cccc} (u_1, w_1) & (u_1, w_2) & . \ . \ . & (u_1, w_{2k+3}) \\ (u_2, w_1) & (u_2, w_2) & . \ . \ . & (u_2, w_{2k+3}) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ (u_{2k+1}, w_1) & (u_{2k+1}, w_2) & & (u_{2k+1}, w_{2k+3}) \end{array} \right\}$$

be the vertex set of order $4k^2 + 8k + 3$ and $E(G) = \{e_1, e_2, ..., e_{8k^2+16k+6}\}$ be the edge set of order $8k^2 + 16k + 6$. By definition of line graph,

$$V[L(G)] = \{v_1 = e_1, v_2 = e_2, ..., v_{8k^2+16k+6} = e_{8k^2+16k+6}\}$$

with $8k^2 + 16k + 6$ vertices and $24k^2 + 48k + 18$ edges. Suppose $m = 3, n = 5$ we have the graph G has 15 vertices and 30 edges and $L(G)$ has 30 vertices and 90. Here $V[L(G)] = \{v_1, v_2, \ldots v_{30}\}$ edges. We assign the colours as follows:

Assign colour 1 to the vertex set $\{v_1, v_3, v_7, v_{11}, v_{14}, v_{17}, v_{21}\}$ and colour 2 to the set of vertices $\{v_2, v_4, v_6, v_9, v_{13}, v_{19}, v_{23}\}$ and the set of vertices $\{v_5, v_8, v_{10}, v_{12}, v_{15}\}$ and $\{v_{16}, v_{18}, v_{20}, v_{22}, v_{24}\}$ are assigned with colour 3 and 4 respectively and remaining vertex set $\{v_{26}, v_{27}, v_{28}, v_{29}, v_{30}\}$ is assigned with colour 5 in which no path with 4 vertices are bi-coloured in $L(G)$. Hence we need five colour for proper star colouring. Therefore $\chi_s[L(G)] = 5$. Proceeding in this manner, for m and n are odd and distinct we have the star chromatic number of $L(G)$ is 5 i.e., $\chi_s[L(G)] = 5$ for $m = 2k + 1, n = 2k + 3 \forall k = 1, 2, \ldots.$ From Case 1 and 2 we have, $\chi_s[L(G)] = 5,\ \forall\ m, n \geq 3.$ □

Example 2.2. *When $m = 3$ and $n = 3$. The Cartesian product of $G = C_3 \times C_3$ is given in figure 1.*

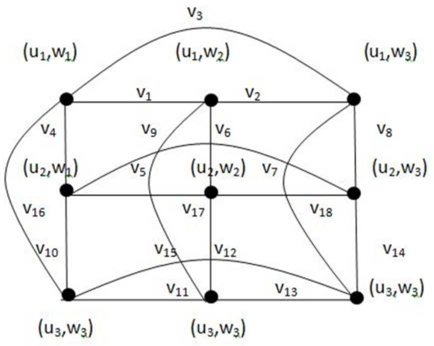


Figure 1. $G = C_3 \times C_3$

The line graph of $G = C_3 \times C_3$ is denoted by $L(G)$ is shown in fig 2.

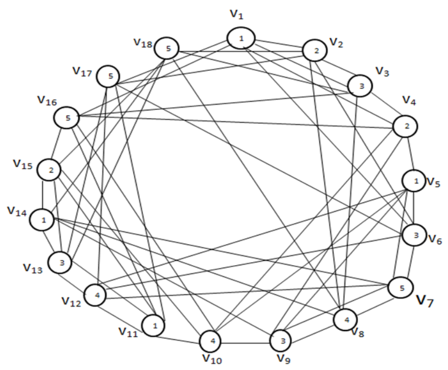


Figure 2. $L(G)$

From figures 1 and 2 the graph G has 9 vertices and 18 edges and the graph $L(G)$ has 18 vertices and 54 edges. By proper star colouring we have, $c(v_1) = c(v_5) = c(v_1 1) = c(v_1 4) = 1$, $c(v_2) = c(v_4) = c(v_1 5) = 2$, $c(v_3) = c(v_6) = c(v_9) = c(v_1 3) = 3$, $c(v_8) = c(v_1 0) = c(v_1 2) = 4$, $c(v_7) = c(v_1 6) = c(v_1 7) = c(v_1 8) = 5$ i.e., $\chi_s[L(G)] = 5$ and $\chi[L(G)] = 5$.

Result 2.3.

(1). When $m = 4k$ and $n = 2k + 1\ \forall\ k = 1, 2, \ldots.$ $\chi[L(G)] = 4$ and $\chi_s[L(G)] = 5$.

(2). For all $m, n > 2$, $\chi[L(G)] = \chi_s[L(G)] = 5$.

Observation 2.4. *From the above theorem we observed that,*

- *$L(G)$ is a cyclic graph but not a tree*
- *$L(G)$ is not a complete graph but a regular graph.*
- *$L(G)$ is not both Euler graph and Hamiltonian graph.*

Observation 2.5.

- *When m is even and n is odd $\chi[L(C_m \times C_n)] < \chi_s[L(C_m \times C_n)] = \delta[L(C_m \times C_n)] > \Delta[L(C_m \times C_n)]$*
- *For every $m, n > 2$, $\chi[L(C_m \times C_n)] = \chi_s[L(C_m \times C_n)] = \delta[L(C_m \times C_n)] > \Delta[L(C_m \times C_n)]$.*

3. Star Chromatic Number Of $L(P_m \times P_n)$

In this section we constructed the line graph of G formed from the Cartesian product of path graphs and we obtained the bounds for the line graphs using the concept of star colouring.

Theorem 3.1. *Let P_m and P_n be two path graphs of order m and n respectively and let $G = P_m \times P_n$ be the Cartesian product of two path graphs and let $L(G)$ be the line graph then the star chromatic number of $L(G)$ is given by $\chi_s[L(G)] = 5, \forall n, m = 3$.*

Proof. Let P_m and P_n be the path graphs of order m and n respectively. Let $G = P_m \times P_n$ be the Cartesian product of two path graphs. Let $L(G)$ be the line graph. The vertex set of P_m be $U = \{u_1, u_2, \ldots, u_m\}$ be the vertex set in P_m and the vertex set of P_n be $W = \{w_1, w_2, \ldots, w_n\}$. The graph G has nm vertices and $2mn - (m + n)$ edges and the line graph $L(G)$ has $2mn - (m + n)$ vertices. This theorem can be proved in different cases:

Case 1: When $m = n = odd$ i.e., $m = n = 2k + 1\ \forall k = 1, 2, \ldots.$

$$V(G) = \left\{\begin{array}{cccc} (u_1, w_1) & (u_1, w_2) & \cdots & (u_1, w_{2k+1}) \\ (u_2, w_1) & (u_2, w_2) & \cdots & (u_2, w_{2k+1}) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ (u_{2k+1}, w_1) & (u_{2k+1}, w_2) & & (u_{2k+1}, w_{2k+1}) \end{array}\right\}$$

be the vertex set and $E(G) = \{e_1, e_2, \ldots.., e_{8k^2+8k-4}\}$ be the edge set in G. By the definition of line graph, $V[L(G)] = \{v_1 = e_1, v_2 = e_2, \ldots.., v_{4k^2+4k-4} = e_{8k^2+8k-4}\}$ with $8k^2 + 8k - 4$ vertices. In particular, for $m = n = 3$, the graph G has 9 vertices and 12 edges and the line graph $L(G)$ has 12 vertices. Here $V[L(G)] = \{v_1, v_2, \ldots v_{12}\}$. Now we star colour $L(G)$ as follows. Assign colour 1 to the vertices v_1, v_7, v_9 and colour 2 to the vertices v_2, v_3, v_6, v_{11} and colour 3 to the vertices v_5, v_8, v_{12} and colour 4 is assigned to only v_{10} and assign colour 5 to the vertex v_4 which satisfies the star colouring concept. Therefore $\chi_s[L(G)] = 5$ for $m = n = 3$ and $k = 1$. Proceeding in this way for m and n are equal and odd, we have the star chromatic number of $L(G)$ is 5 i.e., $\chi_s[L(G)] = 5$ when $m = n = 2k + 1\ \forall\ k = 1, 2, \ldots.$

Case 2: When m is even and n is odd i.e., $m = 2k + 2, n = 2k + 1\ \forall\ k = 1, 2, 3 \ldots.$

$$V(G) = \left\{\begin{array}{cccc} (u_1, w_1) & (u_1, w_2) & \cdots & (u_1, w_{2k+1}) \\ (u_2, w_1) & (u_2, w_2) & \cdots & (u_2, w_{2k+1}) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ (u_{2k+2}, w_1) & (u_{2k+2}, w_2) & & (u_{2k+2}, w_{2k+1}) \end{array}\right\}$$

be the vertex set of order $4k^2 + 4k + 2$ and $E(G) = \{e_1, e_2, \ldots, e_{8k^2-1}\}$ be the edge set. By the definition of line graph, $V[L(G)] = \{v_1 = e_1, v_2 = e_2, \ldots, v_{8k^2-1} = e_{8k^2-1}\}$ with $8k^2 - 1$ vertices. Suppose $m = 4, n = 3$, the graph G has 12 vertices and 17 edges and the line graph $L(G)$ has 17 vertices. Here, $V[L(G)] = \{v_1, v_2, \ldots v_{17}\}$. Now we assign colour as follows. Colour 1 is assigned to the vertices $v_1, v_4, v_7, v_14, v_17$ and colour 2 is assigned to the vertices v_6, v_{11}, v_{16}, and assigned colour 3 to the vertices v_5, v_{13}, v_{15}, and assigned colour 4 to the vertices v_8, v_{10}, v_{12} and colour 5 is assigned to the remaining vertices in which no path on four vertices are bi-coloured. □

This is a proper star colouring and hence $\chi_s[L(G)] = 5 \; \forall \; n = 4, m = 3$ and $k = 1$. Proceeding in this way for m is even and n is odd, the star chromatic number of $L(G)$ is 5 i.e.,$\chi_s[L(G)] = 5$ when $m = 2k + 2, n = 2k + 1 \; \forall k = 1, 2, \ldots$. From these two cases we have, $\chi_s[L(G)] = 5 \; \forall \; n, m = 3$.

Example 3.2. *When $n = 5, m = 3$ The Cartesian product of $G = P_5 \times P_3$ is given in figure 3.*

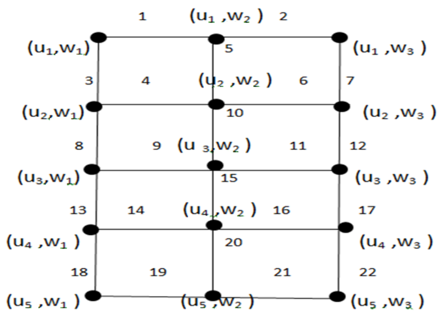


Figure 3. $G = P_5 \times P_3$

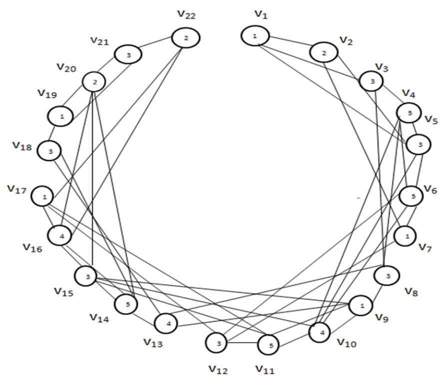


Figure 4. $G = P_5 \times P_3$

The line graph of $G = P5 \times P3$ is shown in figure 4. From figure 3 and figure 4 the graph G has 15 vertices and 22 edges and the line graph $L(G)$ has 22 vertices. By proper star colouring we have, $c(v_1) = c(v_7) = c(v_9) = c(v_{17}) = c(v_{19}) = 1$, $c(v_2) = c(v_3) = c(v_{20}) = c(v_{22}) = 2$, $c(v_5) = c(v_8) = c(v_{12}) = c(v_{15}) = c(v_{18}) = c(v_{21}) = 3$, $c(v_{10}) = c(v_{13}) = c(v_{16}) = 4$, $c(v_4) = c(v_6) = c(v_{11}) = c(v_{14}) = 5$ i.e., $\chi_s[L(G)] = 5$ and $\chi[L(G)] = 5$.

Theorem 3.3. *Let P_m and P_n be two path graphs of order m and n respectively and let $G = P_m \times P_n$ be the Cartesian product of two path graphs and let $L(G)$ be the line graph then the star chromatic number of $L(G)$ is given by $\chi_s[L(G)] = 4, \; \forall \; m, n = 2$.*

Proof. Let P_m and P_n be the two path graphs of order m and n respectively. Let $G = P_m \times P_n$ be the Cartesian product of two path graphs. Let $L(G)$ be the line graph. Let $U = \{u_1, u_2, \ldots, u_m\}$ be the vertex set of C_m and $W = \{w_1, w_2, \ldots, w_n\}$ be the vertex set of C_n. The graph G has nm vertices. We prove this theorem by the following cases:

Case 1: When m is even and n is odd i.e., if $m = 2k, n = 2k + 1 \forall k = 1, 2, 3, \ldots$. $V(G) = \{(u_i, w_j)/1 \leq i \leq 2k; 1 \leq j \leq 2k + 1; \forall \; k = 1, 2, \ldots\}$ be the vertex set and the edge set of G is $E(G) = \{e_1, e_2, ..., e_{4k^2+4k-1}\}$. By the definition of line graph, $V[L(G)] = \{v_1 = e_1, v_2 = e_2, ..., v_{4k^2+4k-1} = e_{4k^2+4k-1}\}$ with $4k^2 + 4k - 1$ vertices. In particular, when $m = 2, n = 5$, the graph G has 10 vertices and 13 edges and the graph $L(G)$ has 13 vertices. Here $V[L(G)] = \{v_1, v_2, \ldots v_{13}\}$. Now we star colour the graph$L(G)$ as follows:

Assign colour 1 to the vertices v_1, v_3, v_{10}, v_{13}, and colour 2 to is assigned to the vertices $v_2, v_4, v_5, v_8, v_{12}$ and the vertices v_7, v_{11} are assigned with colour 3 and v_6, v_9 are assigned with colour 4. This is a proper star colouring and hence required four colours. Therefore $\chi_s[L(G)] = 4$ for all $m = 2, n = 5$ when $k = 1$. In general, for m and n are equal and odd we have the star chromatic number of $L(G)$ is 4 i.e.,$\chi_s[L(G)] = 4$ when $m = 2k, n = 2k + 1 \forall k = 1, 2, \ldots.$

Case 2: When m and n are even and distinct i.e., if $m = 2k, n = 2k + 2 \forall k = 1, 2, 3, \ldots.$ $V(G) = \{(u_i, wj)/1 \le i \le 2k; 1 \le j \le 2k + 2;\ \forall\ k = 1, 2, \ldots..\}$ be the vertex set and the edge set of G is $E(G) = \{e_1, e_2, \ldots.., e_{4k^2+6k}\}$. By the definition of line graph, $V[L(G)] = \{v_1 = e_1, v_2 = e_2, \ldots.., v_{4k^2+6k} = e_{4k^2+6k}\}$ with $4k^2 + 6k$ vertices. For $m = 4, n = 2$; we have the graph G has 8 vertices and 10 edges and the line graph $L(G)$ has 10 vertices i.e., $V[L(G)] = \{v_1, v_2, \ldots v_{10}\}$. By proper star colouring we have, the vertices v_1, v_3, v_9 are assigned with colour 1 and assigned colour 2 to the vertices v_2, v_8 and v_{10} and assign colour 3 to the vertices v_5 and v_7 and assigned colour 4 to the vertices v_4 and v_6 which satisfies the condition of star colouring and the minimum number of colour needed to star colour the $L(G)$ graph is 4. Therefore $\chi_s[L(G)] = 4$ for all $m = 4, n = 2$ and $k = 1$. Proceeding in this way for all m and n are even and distinct, the star chromatic number of $L(G)$ is 4 i.e., $\chi_s[L(G)] = 4$ when $m = 2k, n = 2k + 2 \forall k = 1, 2, \ldots.$ Hence $\chi_s[L(G)] = 4,\ \forall\ m, n = 2.$ □

Example 3.4. *When $m = 4, n = 2$. The Cartesian product of $G = P_4 \times P_2$ is given in figure 5.*

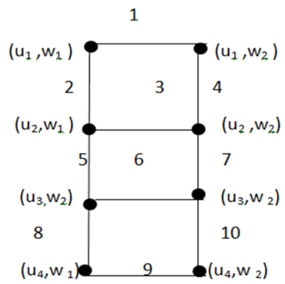


Figure 5. $G = P_4 \times P_2$

From figure 5, the graph G has 8 vertices and 10 edges. The line graph of figure 5 is shown in figure 6.

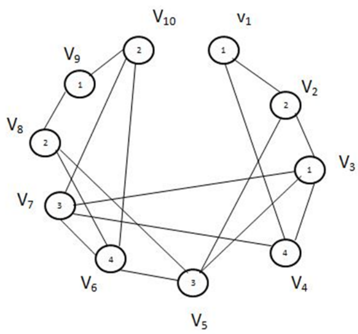


Figure 6. $L(G)$

From figure 6, the line graph $L(G)$ has 10 vertices. Therefore by proper star colouring, $\chi_s[L(G)] = 4$ and $\chi[L(G)] = 4$.

Result 3.5.

(1). For all $m, n = 2, \chi[L(G)] = \chi_s[L(G)] = 4$.

(2). For all $m, n > 2, \chi[L(G)] = \chi_s[L(G)] = 5$.

Observation 3.6.

- *For all values of m and when $n = 2$,*

$$\chi[L(P_m \times P_n)] = \delta[L(P_m \times P_n)]$$

$$\chi_s[L(P_m \times P_n)] < \chi[L(P_m \times P_n)] < \Delta[L(P_m \times P_n)].$$

- *For $m = n = 2$,*

$$\chi[L(P_m \times P_n)] < \chi_s[L(P_m \times P_n)] = \delta[L(P_m \times P_n)] < \Delta[L(P_m \times P_n)]$$

- *For all m and $n = 3$,*

$$\chi[L(P_m \times P_n)] = \chi_s[L(P_m \times P_n)] > \delta[L(P_m \times P_n)].$$

$$\chi[L(P_m \times P_n)] < \chi_s[L(P_m \times P_n)] < \Delta[L(P_m \times P_n)].$$

4. Star Chromatic Number Of $L(P_m \times C_n)$

In this section we obtained the line graph of G formed from the Cartesian product of path and cycle graphs and we determined the bounds for the line graphs using the concept of star colouring.

Theorem 4.1. *Let P_m and C_n be two graphs of vertices m and n respectively and let $G = P_m \times C_n$ be the Cartesian product of two graphs and let $L(G)$ be the line graph then the star chromatic number of $L(G)$ is given by $\chi_s[L(G)] = 5$, for all $n = 2$ and $m > 2$.*

Proof. Let P_m and C_n be two graphs of order m and n respectively. Let $G = P_m \times C_n$ be the Cartesian product of two graphs. Let $L(G)$ be the line graph. Let $U = \{u_1, u_2, \ldots, u_m\}$ be the vertex set in C_m and $W = \{w_1, w_2, \ldots, w_n\}$ be the vertex set in C_n. The graph G has mn vertices and $2mn - n$ edges. The line graph $L(G)$ has $2mn - n$ vertices. We shall prove this theorem in different cases:

Case 1: When m and n are even and distinct i.e., $m = 2k, n = 2k + 2 \forall k = 1, 2, 3, \ldots$ $V(G) = \{(u_i, w_j)/1 \leq i \leq 2k; 1 \leq j \leq 2k + 2; \forall k = 1, 2, \ldots\}$ be the vertex set and the edge set of G is $E(G) = \{e_1, e_2, \ldots.., e_{8k^2+6k-2}\}$. By the definition of line graph, $V[L(G)] = \{v_1 = e_1, v_2 = e_2, \ldots.., v_{8k^2+6k-2} = e_{8k^2+6k-2}\}$ with $8k2 + 6k - 2$ vertices. In particular, when $m = 2, n = 4$ and when $k = 1$; we have G has 8 vertices and 12 edges and $L(G)$ has 12 vertices. i.e., $V[L(G)] = \{v_1, v_2, \ldots v_{12}\}$. Now we assign star colour as follows:

The vertices v_1, v_6, v_{10} are assigned colour 1. Next assign colour 2 to the vertices v_2, v_4, v_8 and colour 3 to the vertices v_5, v_{11} colour 4 to the vertices v_3, v_7, and the vertices v_9, v_12 are assigned with colour 5. Hence$\chi_s[L(G)] = 5$ for all $m = 2, n = 4$ and $k = 1$. Proceeding in this manner for n and m are even and distinct we have the star chromatic number of $L(G)$ is 5 i.e.,$\chi_s[L(G)] = 5$ when $m = 2k, n = 2k + 2 \forall k = 1, 2, \ldots.$

Case 2: When m is even and n is odd i.e.,$m = 2k, n = 2k + 1$; $V(G) = \{(u_i, w_j)/1 \leq i \leq 2k; 1 \leq j \leq 2k + 2; \forall k = 1, 2, \ldots\}$ be the vertex set and the edge set of G is $E(G) = \{e_1, e_2, \ldots.., e_{8k^2+2k-1}\}$. By the definition of line graph, $V[L(G)] = \{v_1 = e_1, v_2 = e_2, \ldots.., v_{8k^2+2k-1} = e_{8k2+2k-1}\}$ with $8k^2 + 2k - 1$ vertices. In particular, suppose $m = 4, n = 3$ and $k = 1$; the graph G has 12 vertices and 21 edges and the graph $L(G)$ has 21 vertices. i.e., $V[L(G)] = \{v_1, v_2, \ldots v_{21}\}$. Now we star colour $L(G)$ as follows:

Assign colour 1 to the vertices v_1, v_8, v_{12}, v_{20} and colour 2 to the vertices v_2, v_4, v_{15}, v_{21} and colour 3 to the vertices $v_5, v_{11}, v_{14}, v_{17}$ and the vertices $v_3, v_9, v_{13}, v_{16}, v_{19}$ are assigned colour 4 and the remaining vertices $v_6, v_{10}, v_{18}, v_{21}$ are assigned with colour 5. Hence the minimum number of colours needed to star colour this graph is 5. Therefore $\chi_s[L(G)] = 5$ for all $m = 4$ and $n = 3$ and $\forall m = 2, n > 2$. Proceeding in this way,for m is even and n is odd we have the star chromatic number of $L(G)$ is 5 i.e.,$\chi_s[L(G)] = 5$ when $m = 2k, n = 2k + 1 \forall k = 1, 2, \ldots$. □

Example 4.2. *When $m = 3, n = 3$. The Cartesian product of $G = P_3 \times C_3$ is shown in figure 7.*

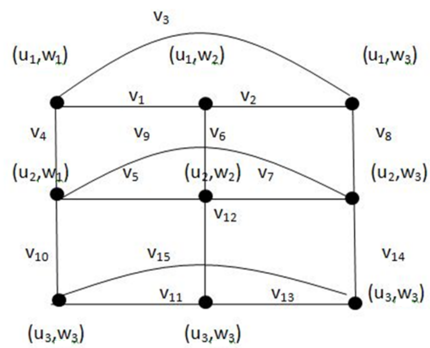


Figure 7. $P3 \times C3$

The line graph of figure 7 is shown in figure 8.

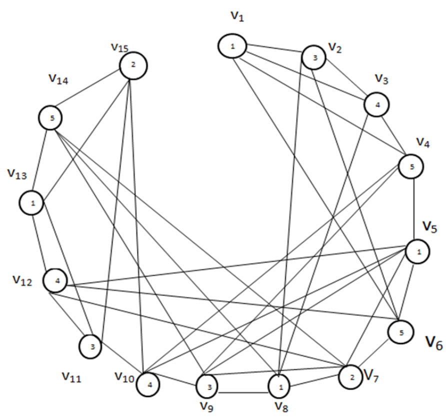


Figure 8. $L(G)$

From figure 7 and figure 8 the graph G has 9 vertices and 15 edges and the line graph $L(G)$ has 32 vertices. Therefore by proper star colouring, $c(v_1) = c(v_5) = c(v_8) = c(v_{13}) = 1, c(v_7) = c(v_{15}) = 2, c(v_2) = c(v_9) = c(v_{11}) = 3, c(v_3) = c(v_{10}) = c(v_{12}) = 4, c(v_4) = c(v_6) = c(v_{14}) = 5$. Then the number of vertices is given by i.e., $\chi_s[L(G)] = 5$ and $\chi[L(G)] = 4$.

Result 4.3.

(1). When $m = n = 3, \chi[L(G)] = 4$ and $\chi_s[L(G)] = 5$.

(2). When $m = 2$ and for all values of $n, \chi[L(G)] = 3$ and $\chi_s[L(G)] = 4$.

(3). For every $n, m > 2, \chi[L(G)] = \chi_s[L(G)] = 5$.

Observation 4.4. *When $G = P_m \times P_n$ and $G = P_m \times C_n$ we observed that $L(G)$ has the following properties:*

(1). It is a cyclic graph.

(2). It is not both complete graph and regular graph.

(3). It is not both Euler graph and Hamiltonian graph.

Observation 4.5.

- *When m =2 and for n>2,*

$$\chi[L(P_m \times C_n)] < \chi_s[L(P_m \times C_n)] < \delta[L(P_m \times C_n)] = \Delta[L(P_m \times C_n)]$$

- *When* $m, n = 3$,

$$\chi[L(P_m \times C_n)] < \chi_s[L(P_m \times C_n)] = \delta[L(P_m \times C_n)] < \Delta[L(P_m \times C_n)].$$

- *For every* $n, m > 2$.

$$\chi[L(P_m \times C_n)] = \chi_s[L(P_m \times C_n)] > \delta[L(P_m \times C_n)]).$$

$$\chi[L(P_m \times C_n)] = \chi_s[L(P_m \times C_n)] < \Delta[L(P_m \times C_n)].$$

5. Conclusion

In this paper we have established the chromatic and star chromatic number of line graphs formed from the Cartesian product of path and cycle. We dissected the relationship between the star chromatic number and chromatic number with different parameters. Subsequently this work can be additionally extended to simple graphs formed from various graph products.

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