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On Direct Sum of Three Fuzzy Graphs

T. Henson[1,*] and N. Devi[1]

PG & Research Department of Mathematics, St.Joseph's College of Arts and Science (Autonomous), Cuddalore, Tamilnadu, India.

Abstract: In this paper, the direct sum $G_1 \oplus G_2 \oplus G_3$ of three fuzzy graphs G_1, G_2 and $G_3$is defined. The degree of the vertices in $G_1 \oplus G_2 \oplus G_3$ is calculated. The regular property, connectedness and effectiveness on the direct sum of three fuzzy graphs are also discussed.

MSC: 05C72.

Keywords: Fuzzy graph, Direct sum, Degree of vertices in the direct sum, Regular fuzzy graph, Connected fuzzy graph, Effective fuzzy graph.

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1. Introduction

In 1975, Fuzzy graph theory was introduced by Azriel Rosenfeld. The properties of fuzzy graphs have been studied by Azriel Rosenfeld [1]. Some operations on fuzzy graphs were introduced by Mordeson. J. N and Peng. C. S [2]. Later on, Bhattacharya [3] gave some remarks on fuzzy graphs. Dr. K. Radha and Mr. S. Arumugam [4] defined the direct sum of two fuzzy graphs. In this paper, the degree of vertices in the direct sum of three fuzzy graphs is calculated with an example. The direct sum of three regular, connected and effective fuzzy graphs are discussed with an example. In this paper, the degree of vertices in the direct sum of three fuzzy graphs is calculated with an example. The direct sum of three regular, connected and effective fuzzy graphs are discussed with an example.

2. Preliminaries

Definition 2.1. *A fuzzy graph G is a pair of functions $G:(\sigma,\mu)$ where σ is a fuzzy subset of a non-empty set V and μ is a symmetric fuzzy relation on σ. The underlying crisp graph of $G:(\sigma,\mu)$ is denoted by $G^*:(V,E)$ where $E \subseteq V \times V$.*

Definition 2.2. *Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*:(V,E)$. The degree of a vertex x is defined as $d_G(x) = \sum\limits_{x \neq y} \mu(xy)$.*

Definition 2.3. *Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*:(V,E)$. If each vertex has same degree$K$, then G is said to be a regular fuzzy graph of degree K.*

Definition 2.4. *If there is a path between every pair of vertices then $G$is said to be a connected fuzzy graph.*

Definition 2.5. *Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*:(V,E)$. A fuzzy graph G is an effective fuzzy graph if$\mu(xy) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in E$.*

* *E-mail: thenson1967@mail.com*

3. Direct Sum

Definition 3.1. *Let $G_1 : (\sigma_1, \mu_1)$, $G_2 : (\sigma_2, \mu_2)$ and $G_3 : (\sigma_3, \mu_3)$ denote three fuzzy graphs with underlying crisp graphs $G_1^* : (V_1, E_1)$, $G_2^* : (V_2, E_2)$ and $G_3^* : (V_3, E_3)$ respectively. Let $V = V_1 \cup V_2 \cup V_3$ and let $E = \{xy/x, y \in V : xy \in E_1$ or $xy \in E_2$ or $xy \in E_3\}$. Define $G_1 : (\sigma, \mu)$ by*

$$\sigma(x) = \begin{cases} \sigma(x_1), & \text{if} x \in V_1 \\ \sigma(x_2), & \text{if} x \in V_2 \\ \sigma(x_3), & \text{if} x \in V_3 \\ \sigma(x_1) \vee \sigma(x_2) \vee \sigma(x_3), & \text{if} x \in V_1 \cup V_2 \cup V_3. \end{cases}$$

$$and \ \ \sigma(x) = \begin{cases} \mu_1(x, y) \leq \sigma_1(x) \cup \sigma_1(y), & \text{if} x, y \in E_1 \\ \mu_2(x, y) \leq \sigma_2(x) \cup \sigma_2(y), & \text{if} x, y \in E_2 \\ \mu_3(x, y) \leq \sigma_3(x) \cup \sigma_3(y), & \text{if} x, y \in E_3. \end{cases}$$

Therefore $G : (\sigma, \mu)$ is called the direct sum of three fuzzy graphs.

Example 3.2. *The following Figure 1 gives an example of the direct sum of three fuzzy graphs.*

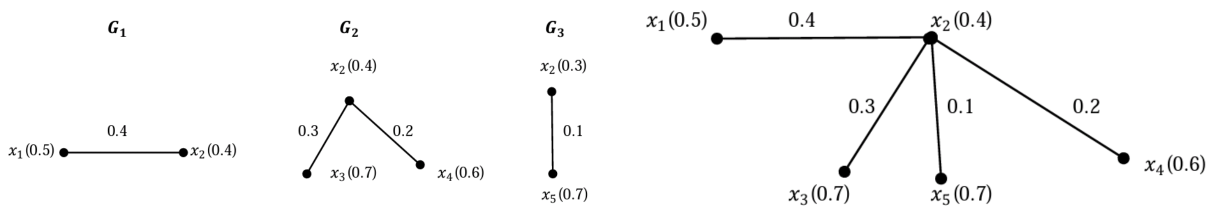


Figure 1. **Direct Sum of Three Fuzzy Graphs**

4. Degree of Vertices in $G_1 \oplus G_2 \oplus G_3$

Theorem 4.1. *The degree of a vertex in $G_1 \oplus G_2 \oplus G_3$ in terms of the degree of the vertices in G_1, G_2 and G_3 is given by,*

$$d_{G_1 \oplus G_2 \oplus G_3}(x) = \begin{cases} d_{G_1}(x), & \text{if} x \in V_1 \\ d_{G_2}(x), & \text{if} x \in V_2 \\ d_{G_3}(x), & \text{if} x \in V_3 \\ d_{G_1}(x) + d_{G_2}(x) + d_{G_3}(x), & \text{if} x \in V_1 \cup V_2 \cup V_3 \text{and} E_1 \cap E_1 \cap E_3 = \phi. \end{cases}$$

Proof. In $G_1 \oplus G_2 \oplus G_3$ for any vertex we have two cases to consider.

Case (i): If $x \in V_1$ or $x \in V_2$ or $x \in V_3$ the the edge incident at x lies in $E_1 \cap E_1 \cap E_3$.

$$(\mu_1 \oplus \mu_2 \oplus \mu_3)(x) = \begin{cases} \mu_1(x, y), & \text{if} x \in V_1 x, y \in E_1 \\ \mu_2(x, y), & \text{if} x \in V_2 x, y \in E_2 \\ \mu_3(x, y), & \text{if} x \in V_3 x, y \in E_3. \end{cases}$$

Hence,

If $x \in V_1$ then $d_{G_1 \oplus G_2 \oplus G_3}(x) = \sum_{x, y \in E_1} \mu_1(x, y) = d_{G_1}(x)$.

If $x \in V_2$ then $d_{G_1 \oplus G_2 \oplus G_3}(x) = \sum\limits_{x,y \in E_2} \mu_2(x,y) = d_{G_2}(x)$.
If $x \in V_3$ then $d_{G_1 \oplus G_2 \oplus G_3}(x) = \sum\limits_{x,y \in E_3} \mu_3(x,y) = d_{G_3}(x)$.
Case (ii): If no edge incident at x lies in $E_1 \cap E_1 \cap E_3$ but $V_1 \cap V_1 \cap V_3$. Then any edge incident at x is either E_1 in or in E_2 or in E_3 but not in $E_1 \cap E_1 \cap E_3$. Also all these edges are included in $G_1 \oplus G_2 \oplus G_3$ i given by, hence the degree of $x$in $G_1 \oplus G_2 \oplus G_3$ is given by,

$$\begin{aligned} d_{G_1 \oplus G_2 \oplus G_3}(x) &= \sum_{x,y \in E} (\mu_1 \oplus \mu_2 \oplus \mu_3)(x,y) \\ &= \sum_{x,y \in E_1} \mu_1(x,y) + \sum_{x,y \in E_2} \mu_2(x,y) + \sum_{x,y \in E_3} \mu_3(x,y) \\ &= d_{G_1}(x) + d_{G_2}(x) + d_{G_3}(x). \end{aligned}$$

□

Example 4.2. *The following Figure 2 illustrate the degree of vertices in $G_1 \oplus G_2 \oplus G_3$.*

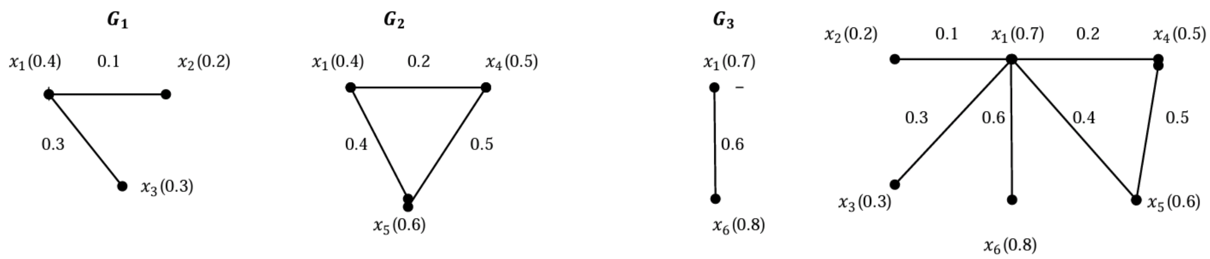


Figure 2. **Degree of Vertices in** $G_1 \oplus G_2 \oplus G_3$

The degree of the vertices in $G_1 \oplus G_2 \oplus G_3$ is as follows:

$$d_{G_1 \oplus G_2 \oplus G_3}(x_1) = 0.1 + 0.3 + 0.6 + 0.4 + 0.2 = 1.6$$
$$d_{G_1 \oplus G_2 \oplus G_3}(x_2) = 0.1$$
$$d_{G_1 \oplus G_2 \oplus G_3}(x_3) = 0.3$$
$$d_{G_1 \oplus G_2 \oplus G_3}(x_4) = 0.2 + 0.5 = 0.7$$
$$d_{G_1 \oplus G_2 \oplus G_3}(x_5) = 0.4 + 0.5 = 0.9$$
$$d_{G_1 \oplus G_2 \oplus G_3}(x_6) = 0.1$$

Now, let us find in terms of degree of the vertices in the fuzzy graphs G_1, G_2 and G_3.

$$d_{G_1 \oplus G_2 \oplus G_3}(x_1) = d_{G_1}(x_1) + d_{G_2}(x_1) + d_{G_3}(x_1) = (0.1 + 0.3) + (0.2 + 0.4) + 0.6 = 1.6$$
$$d_{G_1 \oplus G_2 \oplus G_3}(x_2) = d_{G_2}(x_2) = 0.1$$
$$d_{G_1 \oplus G_2 \oplus G_3}(x_3) = d_{G_1}(x_3) = 0.3$$
$$d_{G_1 \oplus G_2 \oplus G_3}(x_4) = d_{G_2}(x_4) = 0.2 + 0.5 = 0.7$$
$$d_{G_1 \oplus G_2 \oplus G_3}(x_5) = d_{G_2}(x_5) = 0.4 + 0.5 = 0.9$$
$$d_{G_1 \oplus G_2 \oplus G_3}(x_6) = d_{G_3}(x_6) = 0.6.$$

5. Direct Sum of Three Regular Fuzzy Graphs

Theorem 5.1. *If $G_1 : (\sigma_1, \mu_1), G_2 : (\sigma_2, \mu_2)$ and $G_3 : (\sigma_3, \mu_3)$ are regular fuzzy graphs with degrees K_1, K_2 and K_3 respectively and $V_1 \cap V_2 \cap V_3 \neq \phi$ then $G_1 \oplus G_2 \oplus G_3 : (\sigma, \mu)$ is regular if and only if $K_1 = K_2 = K_3$.*

Proof. Let $G_1 : (\sigma_1, \mu_1)$ be a K_1-regular fuzzy graph with underlying crisp graph $G_1^* : (V_1, E_1)$ and let $G_2 : (\sigma_2, \mu_2)$ be a K_2-regular fuzzy graph with underlying crisp graph $G_2^* : (V_2, E_2)$ and let $G_3 : (\sigma_3, \mu_3)$ be a K_3-regular fuzzy graph with underlying crisp graph $G_3^* : (V_3, E_3)$ respectively such that $V_1 \cap V_2 \cap V_3 \neq \phi$. Assume that $G_1 \oplus G_2 \oplus G_3 : (\sigma, \mu)$ is regular,

$$d_{G_1 \oplus G_2 \oplus G_3}(x) = \begin{cases} d_{G_1}(x), & \text{if} x \in V_1 \\ d_{G_2}(x), & \text{if} x \in V_2 \\ d_{G_3}(x), & \text{if} x \in V_3 \\ d_{G_1}(x) + d_{G_2}(x) + d_{G_3}(x), & \text{if} x \in V_1 \cap V_2 \cap V_3 \text{and} E_1 \cap E_1 \cap E_3 = \phi. \end{cases}$$

Since $V_1 \cap V_2 \cap V_3 \neq \phi$,

$$d_{G_1 \oplus G_2 \oplus G_3}(x) = \begin{cases} d_{G_1}(x) = K_1, & \text{if} x \in V_1 \\ d_{G_2}(x) = K_2, & \text{if} x \in V_2 \\ d_{G_3}(x) = K_3, & \text{if} x \in V_3 \end{cases}$$

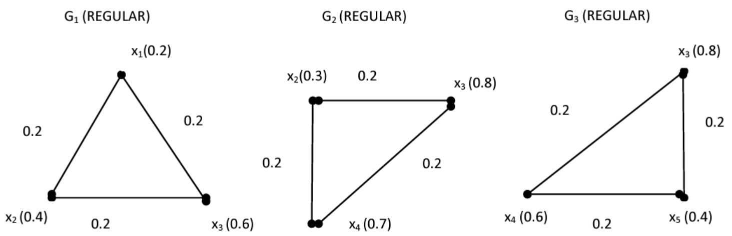
Since $G_1 \oplus G_2 \oplus G_3 : (\sigma, \mu)$ is regular, we get $K_1 = K_2 = K_3$.

Conversely, assume that $G_1 : (\sigma_1, \mu_1), G_2 : (\sigma_2, \mu_2)$ and $G_3 : (\sigma_3, \mu_3)$ are K-regular fuzzy graphs such that $V_1 \cap V_2 \cap V_3 \neq \phi$. Then the degree of any vertex in the direct sum is given by,

$$d_{G_1 \oplus G_2 \oplus G_3}(x) = \begin{cases} d_{G_1}(x) = K_1, & \text{if} x \in V_1 \\ d_{G_2}(x) = K_2, & \text{if} x \in V_2 \\ d_{G_3}(x) = K_3, & \text{if} x \in V_3 \end{cases}$$

Therefore, $d_{G_1 \oplus G_2 \oplus G_3}(x) = K$, for every $x \in V_1 \cap V_2 \cap V_3$. Hence $G_1 \oplus G_2 \oplus G_3 : (\sigma, \mu)$ is regular. □

Example 5.2. *The following Figure 3 shows the direct sum of three regular fuzzy graphs.*



$G_1 \oplus G_2 \oplus G_3$ *(Regular)*

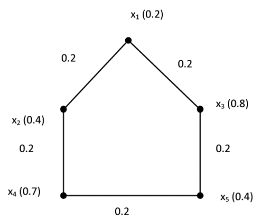
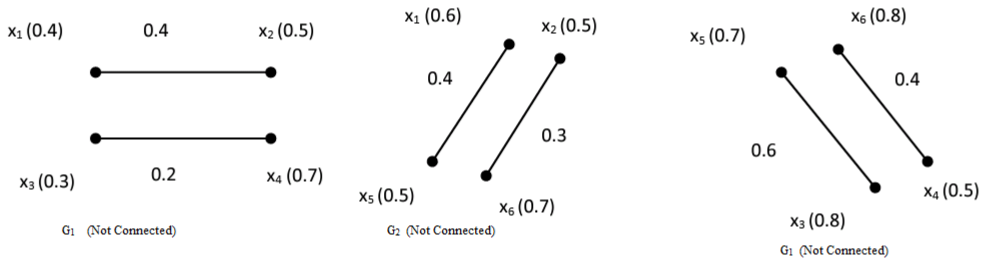


Figure 3. **The Direct Sum Of Three Regular Fuzzy Graphs**

6. Direct Sum of Three Connected Fuzzy Graphs

If $G_1 : (\sigma_1, \mu_1), G_2 : (\sigma_2, \mu_2)$ and $G_3 : (\sigma_3, \mu_3)$ is not a connect fuzzy graphs then their direct sum $G_1 \oplus G_2 \oplus G_3 : (\sigma, \mu)$ can be a connected fuzzy graph. It is illustrated with the following example.

Example 6.1.



$$G_1 \oplus G_2 \oplus G_3 : (\sigma, \mu) \ \ (Connected)$$

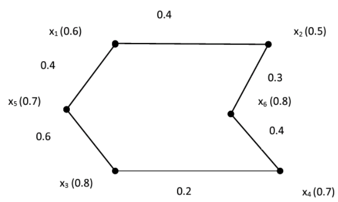
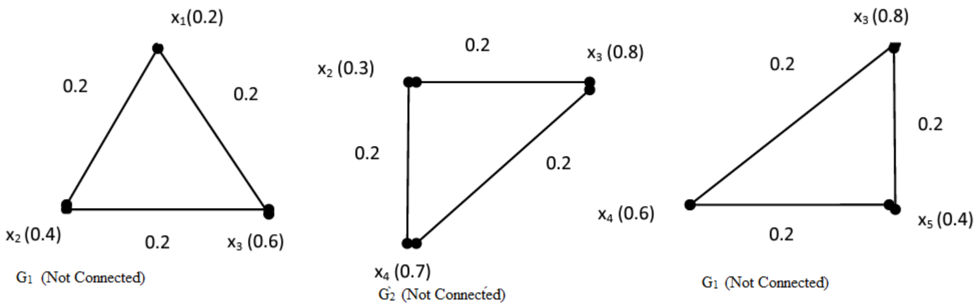


Figure 4. **The Direct Sum of Three Non-Connected Fuzzy Graphs**

Remark 6.2. *If $G_1 : (\sigma_1, \mu_1), G_2 : (\sigma_2, \mu_2)$ and $G_3 : (\sigma_3, \mu_3)$ are three connected fuzzy graphs with $n(V_1 \cap V_2 \cap V_3) = 1$ then their direct sum $G_1 \oplus G_2 \oplus G_3 : (\sigma, \mu)$ is a connected fuzzy graph. It is illustrated with the following example.*

Example 6.3.



$$G_1 \oplus G_2 \oplus G_3 : (\sigma, \mu) \ \ (Connected)$$

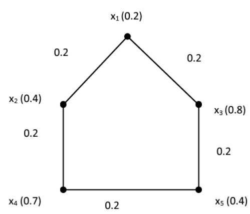


Figure 5. **The Direct Sum of Three Connected Fuzzy Graphs**

Theorem 6.4. *If $G_1 : (\sigma_1, \mu_1), G_2 : (\sigma_2, \mu_2)$ and $G_3 : (\sigma_3, \mu_3)$ are three connected fuzzy graphs with underlying crisp graphs $G_1^* : (V_1, E_1) G_2^* : (V_2, E_2)$ and $G_3^* : (V_3, E_3)$ respectively such that $E_1 \cap E_2 \cup E_3$ and $V_1 \cap V_2 \cap V_3 \neq \phi$ then their direct sum $G_1 \oplus G_2 \oplus G_3 : (\sigma, \mu)$ is connected fuzzy graphs.*

Proof. Since $G_1 : (\sigma_1, \mu_1)$) is a connected fuzzy graph,$\mu_1^\infty(x, y) > 0$ for all $(x, y) \in E_1$. $G_2 : (\sigma_2, \mu_2)$) is a connected fuzzy graph,$\mu_2^\infty(x, y) > 0$ for all $(x, y) \in E_2$. $G_3 : (\sigma_3, \mu_3)$) is a connected fuzzy graph,$\mu_3^\infty(x, y) > 0$ for all $(x, y) \in E_3$. Also $V_1 \cap V_2 \cap V_3 \neq \phi$. Therefore there exists at least one vertex which is in $V_1 \cap V_2 \cap V_3$. But there is no edge in $E_1 \cap E_2 \cup E_3$. Hence there exists a path between any two vertices in the direct sum $G_1 \oplus G_2 \oplus G_3 : (\sigma, \mu)$ of $G_1 : (\sigma_1, \mu_1), G_2 : (\sigma_2, \mu_2)$ and $G_3 : (\sigma_3, \mu_3)$. That is $\mu^\infty_{G_1 \oplus G_2 \oplus G_3}(x, y) > 0$ for all $(x, y) \in E$. This implies that $G_1 \oplus G_2 \oplus G_3 : (\sigma, \mu)$ is connected. □

7. Direct Sum of Three Effective Fuzzy Graphs

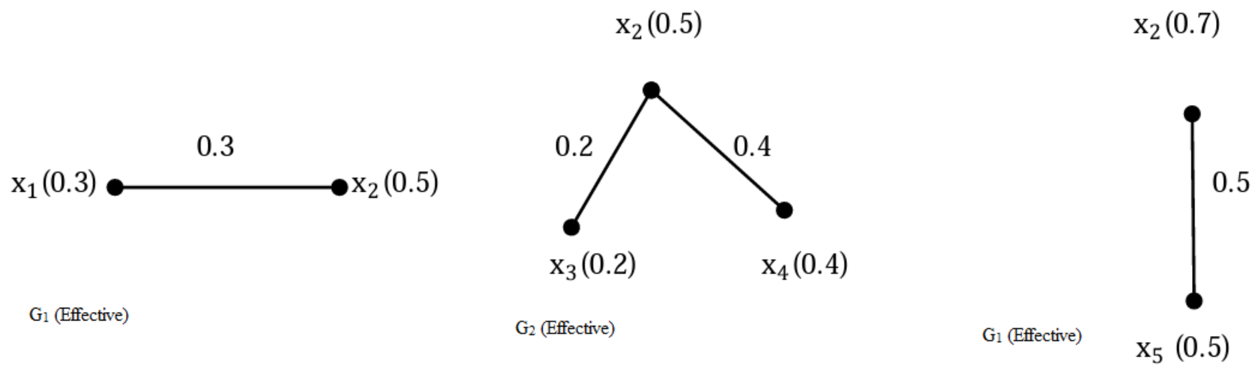
Theorem 7.1. *If G_1, G_2 and G_3 are three effective fuzzy graphs such that no edge of $G_1 \oplus G_2 \oplus G_3$ has both ends in $V_1 \cap V_2 \cap V_3$ and every edge xy of $G_1 \oplus G_2 \oplus G_3$ with one end $x \in V_1 \cap V_2 \cap V_3$ and $xy \in E(or E_2 or E_3)$such that $\sigma_1(x) \leq \sigma_1(y)$ [or $\sigma_2(x) \leq \sigma_2(y)$, or $\sigma_3(x) \leq \sigma_3(y)$] then $G_1 \oplus G_2 \oplus G_3$ effective fuzzy graph.*

Proof. Let x, y be an edge of $G_1 \oplus G_2 \oplus G_3$. We have two cases to consider.

Case (i): If $x, y \notin V_1 \cap V_2 \cap V_3$. Then $x, y \in V_1$ or $x, y \in V_2$ or$x, y \in V_3$. Therefore $\sigma(x) = \sigma_1(x), \sigma(y) = \sigma_1(y)$ and $\mu(xy) = \mu_1(xy)$. Since G_1 is an effective fuzzy graph, $\mu(xy) = \mu_1(xy) = \sigma_1(x) \wedge \sigma_1(y) = \sigma(x) \wedge \sigma(y)$. Suppose that $x, y \in V_2$. Then $xy \in E_2$. Therefore $\sigma(x) = \sigma_2(x), \sigma(y) = \sigma_2(y)$ and $\mu(xy) = \mu_2(xy)$. Since G_1 is an effective fuzzy graph. The proof is similar if $x, y \in V_3$.

Case (ii): If $x \in V_1 \cap V_2 \cap V_3$, $y \notin V_1 \cap V_2 \cap V_3$ (or vice versa). Without loss of generality, assume that $v \in V_1$. Then $\sigma(y) = \sigma_1(y)$. By hypothesis, $\sigma_1(x) \geq \sigma_1(y)$. Now $\sigma(x) = \sigma_1(x) \geq \sigma(y) = \sigma_1(y)$. So $\sigma(x) \wedge \sigma(y) = \sigma(y)$. Hence, $\mu(xy) = \mu_1(xy) = \sigma_1(x) \wedge \sigma_1(y) = \sigma(x) \wedge \sigma(y)$. Therefore $G_1 \oplus G_2 \oplus G_3$ is an effective fuzzy graph. □

Example 7.2.



$G_1 \oplus G_2 \oplus G_3$ *(Effective)*

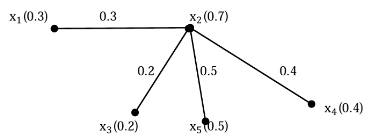
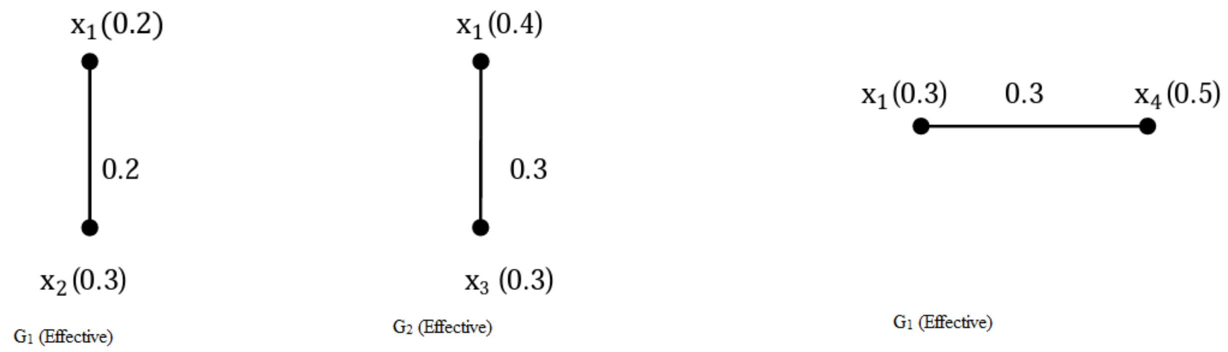


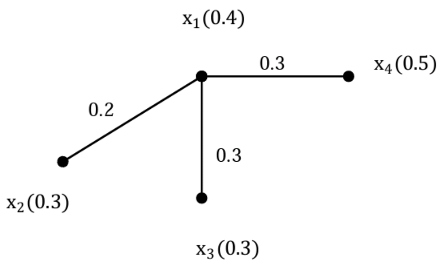
Figure 6. **The Direct Sum of Three Fuzzy Graphs**

If G_1, G_2 and G_3 are three effective fuzzy graphs , their direct sum $G_1 \oplus G_2 \oplus G_3$ need not be an effective fuzzy graph which can be seen from the following example.

Example 7.3.



G_1, G_2 and G_3(Not Effective)



8. Conclusion

In this paper, the direct sum $G_1 \oplus G_2 \oplus G_3$ of three fuzzy graphs G_1, G_2 and G_3 is defined. A formula to find the degree of vertices in the direct sum $G_1 \oplus G_2 \oplus G_3$ of three fuzzy graphs G_1, G_2 and G_3 is obtained with an example. Some of the property of the direct sum of regular, connected and effective fuzzy graphs have been illustrated. Thus operation on fuzzy graph plays an important role to consider large fuzzy graph as a combination of small fuzzy graphs. A truly tactical manoeuvre in that direction on the whole is specifically made through this paper.

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