International Journal of Mathematics And its Applications
Volume 4, Issue 4 (2016), 16–32. (Special Issue)
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Generalized Hyers - Ulam Stability of Additive - Quadratic - Cubic - Quartic Functional Equation in Fuzzy Normed Spaces Using Fixed Point Method

Research Article*

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Abstract: In this paper, the authors investigate the generalized Hyers-Ulam-stability of AQCQ functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)-4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y)$$

in fuzzy normed spaces using fixed point method.

MSC: 39B52, 39B82, 26E50, 46S50.

Keywords: Fuzzy normed space, Additive, quadratic, cubic and quartic functional equation, Generalized Hyers-Ulam-Rassias stability.

1. Introduction and Preliminaries

The stability of various functional equations in fuzzy normed spaces was have been extensively investigated by a number of mathematicians in referenes (see [3–5, 16, 17, 28–31, 37]). In 2003, V. Radu [32] introduced a new method, successively developed in ([8–10]), to obtaining the existence of the exact solutions and the error estimations, based on the fixed point alternative. A.K. Katsaras [19] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [13, 21, 40]. In particular, T. Bag and S.K. Samanta [6], following S.C. Cheng and J.N. Mordeson [11], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [20]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [7]. We use the definition of fuzzy normed spaces given in [6] and [24–27].

Definition 1.1. *Let X be a real linear space. A function $N : X\times\mathbb{R}\to[0,1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y\in X$ and all $s, t\in\mathbb{R}$,*

$(F1)$ $N(x,c)=0$ *for* $c\le 0$;

$(F2)$ $x=0$ *if and only if* $N(x,c)=1$ *for all* $c>0$;

* *Proceedings : National Conference on* `Recent Trends in Applied Mathematics` *held on 22 & 23.07.2016, organized by Department of Mathematics, St. Joseph's College of Arts & Science, Manjakuppam, Cuddalore (Tamil Nadu), India.*

(F3) $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;

(F4) $N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;

(F5) $N(x,\cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $lim_{t\to\infty}N(x,t) = 1$;

(F6) *for* $x \neq 0, N(x,\cdot)$ *is (upper semi) continuous on* $\mathbb{R}$.

The pair (X,N) *is called a fuzzy normed linear space. One may regard* $N(X,t)$ *as the truth-value of the statement the norm of* x *is less than or equal to the real number* t*'.*

Example 1.2. *Let* $(X, ||\cdot||)$ *be a normed linear space. Then*

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 1.3. *Let* (X,N) *be a fuzzy normed linear space. Let* x_n *be a sequence in* X. *Then* x_n *is said to be convergent if there exists* $x \in X$ *such that* $\lim_{n\to\infty} N(x_n - x, t) = 1$ *for all* $t > 0$. *In that case,* x *is called the limit of the sequence* x_n *and we denote it by* $N - \lim_{n\to\infty} x_n = x$.

Definition 1.4. *A sequence* x_n *in* X *is called Cauchy if for each* $\epsilon > 0$ *and each* $t > 0$ *there exists* n_0 *such that for all* $n \geq n_0$ *and all* $p > 0$, *we have* $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

Definition 1.5. *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

Definition 1.6. *A mapping* $f : X \to Y$ *between fuzzy normed spaces* X *and* Y *is continuous at a point* x_0 *if for each sequence* $\{x_n\}$ *covering to* x_0 *in* X, *the sequence* $f\{x_n\}$ *converges to* $f(x_0)$. *If* f *is continuous at each point of* $x_0 \in X$ *then* f *is said to be continuous on* X.

In this paper, the authors investigate the generalized Hyers-Ulam-Aoki-Rassias stability of AQCQ functional equation

$$f(x+2y) + f(x-2y) = 4f(x+y) - 4f(x-y) - 6f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y) \tag{1}$$

in fuzzy normed vector space by fixed point method.

2. Stability Results: Fixed Point Method

In this section, the authors presented the generalized Ulam - Hyers stability of the functional equation (1) in fuzzy normed space using fixed point method. Now we will recall the fundamental results in fixed point theory.

Theorem 2.1 (Banach's contraction principle)**.** *Let* (X,d) *be a complete metric space and consider a mapping* $T : X \to X$ *which is strictly contractive mapping, that is*

(A1) $d(Tx,Ty) \leq Ld(x,y)$ *for some (Lipschitz constant)* $L < 1$. *Then,*

(i) The mapping T *has one and only fixed point* $x^* = T(x^*)$;

(ii)The fixed point for each given element x^* *is globally attractive, that is*

(A2) $lim_{n\to\infty}T^n x = x^*$, *for any starting point* $x \in X$;

(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L} \; d(T^n x, T^{n+1}x), \forall \; n \geq 0, \forall \; x \in X;$

(A4) $d(x, x^*) \leq \frac{1}{1-L} \; d(x, x^*), \forall \; x \in X.$

Theorem 2.2 (The alternative of fixed point [22]). *Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*

$(B1)$ $$d(T^n x, T^{n+1}x) = \infty \quad \forall \; n \geq 0,$$

or

$(B2)$ *there exists a natural number n_0 such that:*

(i) $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$ *;*

(ii) The sequence $(T^n x)$ *is convergent to a fixed point* y^* *of* T

(iii) y^* *is the unique fixed point of* T *in the set* $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L} \; d(y, Ty)$ *for all* $y \in Y$.

In order to prove the stability results, we define the following:

δ_i is a constant such that

$$\delta_i = \begin{cases} 2 & if \quad i = 0, \\ \frac{1}{2} & if \quad i = 1 \end{cases}$$

and Ω is the set such that

$$\Omega = \{g \mid g : X \to Y, g(0) = 0\}.$$

Theorem 2.3. *Let $f : X \to Y$ be a mapping for which there exist a function $\alpha : X^2 \to Z$ with the condition*

$$\lim_{k\to\infty} N'\left(\alpha\left(\delta_i^k x, \delta_i^k y\right), \delta_i^k r\right) = 1, \quad \forall \; x, y \in X, r > 0 \tag{2}$$

and satisfying the functional inequality

$$N\left(D \; f(x, y), r\right) \geq N'\left(\alpha(x, y), r\right), \; \forall \; x, y \in X, r > 0. \tag{3}$$

If there exists $L = L(i)$, such that the function

$$x \to N'\left(\beta(y), r\right) = min\left\{N'\left(\alpha\left(\frac{y}{2}, \frac{y}{2}\right), \frac{r}{8}\right), N'\left(\alpha\left(y, \frac{y}{2}\right), \frac{r}{2}\right)\right\},$$

has the property

$$N'\left(L\frac{1}{\delta_i}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right), \; \forall \; x \in X, r > 0. \tag{4}$$

Then there exists unique additive function $A : X \to Y$ satisfying the functional equation (1) and

$$N\left(f(2y) - 8f(y) - A(y), r\right) \geq min\left\{N'\left(\alpha(y, y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y, y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right)\right\} \tag{5}$$

Proof. Let d be a general metric on Ω, such that

$$d(g, h) = inf\left\{K \in (0, \infty) | N\left(g(x) - h(x), r\right) \geq N'\left(\beta(x), Kr\right), x \in X, r > 0\right\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $Tg(x) = \frac{1}{\delta_i} g(\delta_i x)$, for all $x \in X$. For $g, h \in \Omega$, we have $d(g, h) \leq K$

$$\begin{aligned}
\Rightarrow &\quad N\left(g(x) - h(x), r\right) \geq N'\left(\beta(x), Kr\right) \\
\Rightarrow &\quad N\left(\frac{g(\delta_i x)}{\delta_i} - \frac{h(\delta_i x)}{\delta_i}, r\right) \geq N'\left(\beta(\delta_i x), K\delta_i r\right) \\
\Rightarrow &\quad N\left(Tg(x) - Th(x), r\right) \geq N'\left(\beta(x), LKr\right) \\
\Rightarrow &\quad d\left(Tg(x), Th(x)\right) \leq KL \\
\Rightarrow &\quad d\left(Tg, Th\right) \leq Ld(g, h) \qquad (6)
\end{aligned}$$

for all $g, h \in \Omega$. There fore T is strictly contractive mapping on Ω with Lipschitz constant L. Replacing (x, y) by (y, y) in (3), we get

$$N\left(f(3y) - 4f(2y) + 5f(y), r\right) \geq N'\left(\alpha(y, y), r\right) \qquad (7)$$

for all $y \in X$ and all $r > 0$. Replacing x by $2y$ in (3), we obtain

$$N\left(f(4y) - 4f(3y) + 6f(2y) - 4f(y), r\right) \geq N'\left(\alpha(2y, y), r\right) \qquad (8)$$

for all $y \in X$ and all $r > 0$. Now, from (7) and (8), we have

$$\begin{aligned}
N\left(f(4y) - 10f(2y) + 16f(y), r\right) &\geq min\left\{N\left(4\left(f(3y) - 4f(2y) + 5f(y)\right), \frac{r}{2}\right), N\left(f(4y) - 4f(3y) + 6f(2y) - 4f(y), \frac{r}{2}\right)\right\} \\
&\geq min\left\{N'\left(\alpha(y, y), \frac{r}{8}\right), N'\left(\alpha(2y, y), \frac{r}{2}\right)\right\} \qquad (9)
\end{aligned}$$

for all $y \in X$ and all $r > 0$. Let $a : X \to Y$ be a mapping defined by $a(y) = f(2y) - 8f(y)$. Then we conclude that

$$N\left(a(2y) - 2a(y), r\right) \geq min\left\{N'\left(\alpha(y, y), \frac{r}{8}\right), N'\left(\alpha(2y, y), \frac{r}{2}\right)\right\} \qquad (10)$$

for all $y \in X$ and all $r > 0$. Using (F3) in (10), we arrive

$$N\left(\frac{a(2y)}{2} - a(y), r\right) \geq min\left\{N'\left(\alpha(y, y), \frac{2r}{8}\right), N'\left(\alpha(2y, y), \frac{2r}{2}\right)\right\} \qquad (11)$$

for all $y \in X, r > 0$, with the help of (4) when $i = 0$, it follows from (11), we get

$$N\left(\frac{a(2y)}{2} - a(y), r\right) \geq N'\left(\beta(y), Lr\right) \Rightarrow d(Tf, f) \leq L = L^{1-i} \qquad (12)$$

for all $y \in X, r > 0$. Letting $y = \frac{y}{2}$ in (10), we obtain

$$N\left(a(y) - 2a\left(\frac{y}{2}\right), r\right) \geq min\left\{N'\left(\alpha\left(\frac{y}{2}, \frac{y}{2}\right), \frac{r}{8}\right), N'\left(\alpha\left(y, \frac{y}{2}\right), \frac{r}{2}\right)\right\} \qquad (13)$$

for all $y \in X, r > 0$, with the help of (4) when $i = 1$, it follows from (13), we get

$$N\left(a(y) - 2a\left(\frac{y}{2}\right), r\right) \geq N'\left(\beta(x), r\right) \Rightarrow d(f, Tf) \leq 1 = L^0 = L^{1-i} \qquad (14)$$

Then from (12) and (14) we can conclude,

$$d(f,Tf) \leq L^{1-i} < \infty$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point A of T in Ω such that

$$A(y) = N - \lim_{k\to\infty} \frac{f(2^k y)}{2^k}, \qquad \forall y \in X, r > 0. \tag{15}$$

Replacing (x,y) by $(\delta_i x, \delta_i y)$ in (3), we arrive

$$N\left(\frac{1}{\delta_i} Df(\delta_i x, \delta_i y), r\right) \geq N'\left(\alpha(\delta_i x, \delta_i), \delta_i r\right) \tag{16}$$

for all $r > 0$ and all $x, y \in X$. It is easy to prove the function $A : X \to Y$ satisfies the functional equation (1). By fixed point alternative, since A is unique fixed point of T in the set

$$\Delta = \{f \in \Omega | d(f,A) < \infty\},$$

therefore A is a uniqe function such that

$$N\left(a(y) - A(y), r\right) \geq min\left\{N'\left(\alpha(y,y), Kr\right), N'\left(\alpha(2y,y), Kr\right)\right\} \tag{17}$$

for all $y \in X, r > 0$ and $K > 0$. Again using the fixed point alternative, we obtain

$$\begin{aligned} & d(f,A) \leq \frac{1}{1-L} d(f,Tf) \\ \Rightarrow\quad & d(f,A) \leq \frac{L^{1-i}}{1-L} \\ \Rightarrow\quad & N\left(a(y) - A(y), r\right) \geq N'\left(\beta(y), \frac{L^{1-i}}{1-L} r\right), \end{aligned} \tag{18}$$

for all $y \in X$ and $r > 0$. This completes the proof of the theorem. □

From Theorem 2.3, we obtain the following corollary concerning the stability for the functional equation (1).

Corollary 2.4. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x,y), r\right) \quad \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon\left\{||x||^s + ||y||^s\right\}, r\right), & s \neq 1; \\ N'\left(\epsilon\left\{||x||^s ||y||^s\right\}, r\right), & s \neq \frac{1}{2}; \\ N'\left(\epsilon\left(||x||^s ||y||^s + ||x||^{2s} + ||y||^{2s}\right), r\right), & s \neq \frac{1}{2}; \end{cases} \tag{19}$$

for all $x, y \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$N\left(f(2y) - 8f(y) - A(y), r\right) \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{|2|r}{8}\right), N'\left(\epsilon, \frac{|2|r}{4}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{4|2^s-2|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{r}{2|2^s-2|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{4|2^{2s}-2|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{2|2^{2s}-2|}\right)\right\} \\ min\left\{N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{4|2^{2s}-2|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{r}{2|2^{2s}-2|}\right)\right\} \end{cases} \tag{20}$$

for all $y \in X$ and all $r > 0$.

Proof. Setting

$$\alpha(x,y)=\begin{cases}\epsilon,\\ \epsilon\left(||x||^s+||y||^s\right),\\ \epsilon\left(||x||^s||y||^s\right)\\ \epsilon\left(||x||^s||y||^s+||x||^{2s}+||y||^{2s}\right)\end{cases}$$

for all $x,y\in X$. Then,

$$N'\left(\alpha(\delta_i^k x,\delta_i^k y),\delta_i^k r\right)=\begin{cases}N'\left(\epsilon,\delta_i^k r\right)\\ N'\left(\epsilon\left(||\delta_i^k x||^s+||\delta_i^k y||^s\right),\delta_i^k r\right)\\ N'\left(\epsilon\left(||\delta_i^k x||^s||\delta_i^k y||^s\right),\delta_i^k r\right)\\ N'\left(\epsilon\left(||\delta_i^k x||^s||\delta_i^k y||^s+||\delta_i^k x||^{2s}+||\delta_i^k y||^{2s}\right),\delta_i^k r\right)\end{cases}$$

$$=\begin{cases}N'\left(\epsilon,\delta_i^k r\right)\\ N'\left(\epsilon\left(||x||^s+||y||^s\right),\delta_i^{(1-s)k} r\right)\\ N'\left(\epsilon\left(||x||^s||y||^s\right),\delta_i^{(1-2s)k} r\right)\\ N'\left(\epsilon\left(||x||^s||y||^s+||x||^{2s}+||y||^{2s}\right),\delta_i^{(1-2s)k} r\right)\end{cases}\quad\begin{cases}\to 1 \text{ as } k\to\infty\\ \to 1 \text{ as } k\to\infty\\ \to 1 \text{ as } k\to\infty\\ \to 1 \text{ as } k\to\infty.\end{cases}$$

Thus, (2) is holds. But we have $N'\left(\beta(y),r\right)=min\left\{N'\left(\alpha\left(\frac{y}{2},\frac{y}{2}\right),\frac{r}{8}\right),N'\left(\alpha\left(y,\frac{y}{2}\right),\frac{r}{2}\right)\right\}$, has the property

$$N'\left(L\frac{1}{\delta_i}\beta(\delta_i y),r\right)\geq N'\left(\beta(y),r\right)\;\;\forall\; y\in X, r>0.$$

Hence

$$N'\left(\beta(y),r\right)=min\left\{N'\left(\alpha\left(\frac{y}{2},\frac{y}{2}\right),\frac{r}{8}\right),N'\left(\alpha\left(y,\frac{y}{2}\right),\frac{r}{2}\right)\right\}$$

$$=\begin{cases}min\left\{N'\left(\epsilon,\frac{r}{8}\right),N'\left(\epsilon,\frac{r}{4}\right)\right\}\\ min\left\{N'\left(\frac{\epsilon}{2^s}||y||^s,\frac{r}{8}\right),N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s,\frac{r}{4}\right)\right\}\\ min\left\{N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s},\frac{r}{8}\right),N'\left(\frac{\epsilon}{2^s}||y||^{2s},\frac{r}{4}\right)\right\}\\ min\left\{N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s},\frac{r}{8}\right),N'\left(\epsilon\left(\frac{1+2^s}{2^s}+\frac{1}{2^{2s}}\right)||y||^{2s},\frac{r}{4}\right)\right\}\end{cases}$$

Now,

$$N'\left(\frac{1}{\delta_i}\beta(\delta_i y),r\right)=\begin{cases}min\left\{N'\left(\frac{\epsilon}{\delta_i},\frac{r}{8}\right),N'\left(\frac{\epsilon}{\delta_i},\frac{r}{4}\right)\right\}\\ min\left\{N'\left(\frac{\epsilon}{2^s\delta_i}||\delta_i y||^s,\frac{r}{8}\right),N'\left(\epsilon\frac{1+2^s}{2^s\delta_i}||\delta_i y||^s,\frac{r}{4}\right)\right\}\\ min\left\{N'\left(\frac{\epsilon}{2^{2s}\delta_i}||\delta_i y||^{2s},\frac{r}{8}\right),N'\left(\frac{\epsilon}{2^s\delta_i}||\delta_i y||^{2s},\frac{r}{4}\right)\right\}\\ min\left\{N'\left(\frac{3\epsilon}{2^{2s}\delta_i}||\delta_i y||^{2s},\frac{r}{8}\right),N'\left(\frac{\epsilon}{\delta_i}\left(\frac{1+2^s}{2^s}+\frac{1}{2^{2s}}\right)||\delta_i y||^{2s},\frac{r}{4}\right)\right\}\end{cases}$$

$$=\begin{cases}min\left\{N'\left(\epsilon,\frac{\delta_i r}{8}\right),N'\left(\epsilon,\frac{\delta_i r}{4}\right)\right\}\\ min\left\{N'\left(\frac{\epsilon}{2^s}||y||^s,\frac{\delta_i^{1-s}r}{8}\right),N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s,\frac{\delta_i^{1-s}r}{4}\right)\right\}\\ min\left\{N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s},\frac{\delta_i^{1-2s}r}{8}\right),N'\left(\frac{\epsilon}{2^s}||y||^{2s},\frac{\delta_i^{1-2s}r}{4}\right)\right\}\\ min\left\{N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s},\frac{\delta_i^{1-2s}r}{8}\right),N'\left(\epsilon\left(\frac{1+2^s}{2^s}+\frac{1}{2^{2s}}\right)||y||^{2s},\frac{\delta_i^{1-2s}r}{4}\right)\right\}\end{cases}$$

Now from (5), we desired (20). □

Theorem 2.5. *Let $f : X \to Y$ be a mapping for which there exist a function $\alpha : X^2 \to Z$ with the condition*

$$\lim_{k\to\infty} N'\left(\alpha\left(\delta_i^k x, \delta_i^k y\right), \delta_i^{3k} r\right) = 1, \quad \forall\ x, y \in X, r > 0 \tag{21}$$

and satisfying the functional inequality

$$N\left(D\ f(x,y), r\right) \geq N'\left(\alpha(x,y), r\right), \ \forall\ x, y \in X, r > 0. \tag{22}$$

If there exists $L = L(i)$ such that the function

$$x \to N'\left(\beta(y), r\right) = min\left\{N'\left(\alpha\left(\frac{y}{2}, \frac{y}{2}\right), \frac{r}{8}\right), N'\left(\alpha\left(y, \frac{y}{2}\right), \frac{r}{2}\right)\right\},$$

has the property

$$N'\left(L\frac{1}{\delta_i^3}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right), \ \forall\ x \in X, r > 0. \tag{23}$$

Then there exists unique cubic function $C : X \to Y$ satisfying the functional equation (1) and

$$N\left(f(2y) - 2f(y) - C(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right)\right\} \tag{24}$$

Proof. It is easy to see from (9) that

$$\begin{aligned} N\left(f(4y) - 2f(2y) - 8f(2y), r\right) &\geq min\left\{N\left(4\left(f(3y) - 4f(2y) + 5f(y)\right), \frac{r}{2}\right), N\left(f(4y) - 4f(3y) + 6f(2y) - 4f(y), \frac{r}{2}\right)\right\} \\ &\geq min\left\{N'\left(\alpha(y,y), \frac{r}{8}\right), N'\left(\alpha(2y,y), \frac{r}{2}\right)\right\} \end{aligned} \tag{25}$$

for all $y \in X$ and all $r > 0$. Let $h : X \to Y$ be a mapping defined by $h(y) = f(2y) - 2f(y)$. Then we conclude that

$$N\left(h(2y) - 8h(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{r}{8}\right), N'\left(\alpha(2y,y), \frac{r}{2}\right)\right\} \tag{26}$$

for all $y \in X$ and all $r > 0$. The rest of the proof is similar to that of Theorem 2.3. □

The following corollary is an immediate consequence of Theorem 2.5 concerning the Ulam-Hyers stability of the functional equation(1).

Corollary 2.6. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x,y), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon\left\{||x||^s + ||y||^s\right\}, r\right), & s \neq 3; \\ N'\left(\epsilon\left\{||x||^s ||y||^s\right\}, r\right), & s \neq \frac{3}{2}; \\ N'\left(\epsilon\left(||x||^s ||y||^s + ||x||^{2s} + ||y||^{2s}\right), r\right), & s \neq \frac{3}{2}; \end{cases} \tag{27}$$

for all $x, y \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$N\left(f(2y) - 2f(y) - C(y), r\right) \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{r}{|7|}\right), N'\left(\epsilon, \frac{2r}{|7|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{|2^s - 2^3|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{2r}{|2^s - 2^3|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{|2^{2s} - 2^3|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{2|2^{2s} - 2^3|}\right)\right\} \\ min\left\{N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{|2^{2s} - 2^3|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{2r}{|2^{2s} - 2^3|}\right)\right\} \end{cases} \tag{28}$$

for all $y \in X$ and all $r > 0$.

Theorem 2.7. *Let $f : X \to Y$ be a mapping for which there exist a function $\alpha : X^2 \to Z$ with the conditions*

$$\lim_{k\to\infty} N'\left(\alpha\left(\delta_i^k x, \delta_i^k y\right), \delta_i^k r\right) = 1, \ \lim_{k\to\infty} N'\left(\alpha\left(\delta_i^k x, \delta_i^k y\right), \delta_i^{3k} r\right) = 1, \quad \forall\ x, y \in X, r > 0 \tag{29}$$

and satisfying the functional inequality

$$N\left(D\ f(x,y), r\right) \geq N'\left(\alpha(x,y), r\right), \ \forall\ x, y \in X, r > 0. \tag{30}$$

If there exists $L = L(i)$ such that the function

$$x \to N'\left(\beta(y), r\right) = min\left\{N'\left(\alpha\left(\frac{y}{2}, \frac{y}{2}\right), \frac{r}{8}\right), N'\left(\alpha\left(y, \frac{y}{2}\right), \frac{r}{2}\right)\right\},$$

has the property

$$N'\left(L\frac{1}{\delta_i}\beta(\delta_i y), r\right) = N'\left(\beta(y), r\right), \ N'\left(L\frac{1}{\delta_i^3}\beta(\delta_i y), r\right) = N'\left(\beta(y), r\right), \ \forall\ y \in X, r > 0. \tag{31}$$

Then there exists unique additive function $A : X \to Y$ and unique cubic function $C : X \to Y$ satisfying the functional equation (1) and

$$\begin{aligned} N\left(f(y) - A(y) - C(y), r\right) \geq \min\Bigg\{ & N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right), \\ & N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right)\Bigg\} \end{aligned} \tag{32}$$

Proof. By Theorems 2.3 and 2.5, there exists a unique additive function $A_1 : X \to Y$ and a unique cubic function $C_1 : X \to Y$ such that

$$N\left(f(2y) - 8f(y) - A_1(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right)\right\} \tag{33}$$

for all $y \in X$ and all $r > 0$ and

$$N\left(f(2y) - 2f(y) - C_1(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right)\right\} \tag{34}$$

for all $y \in X$ and all $r > 0$. Now from (33) and (34), one can see that

$$\begin{aligned} N\left(f(y) + \frac{1}{6}A_1(y) - \frac{1}{6}C_1(y), 2r\right) &= N\left(-\frac{f(2y)}{6} + \frac{8}{6}f(y) + \frac{1}{6}A_1(y) + \frac{f(2y)}{6} - \frac{2}{6}f(y) - \frac{1}{6}C_1(y), \frac{2r}{6}\right) \\ &\geq \min\left\{N\left(\frac{f(2y)}{6} - \frac{8}{6}f(y) - \frac{1}{6}A_1(y), \frac{r}{6}\right), N\left(\frac{f(2y)}{6} - \frac{2}{6}f(y) - \frac{1}{6}C_1(y), \frac{r}{6}\right)\right\} \\ &\geq \min\left\{N\left(f(2y) - 8f(y) - A_1(y), r\right), N\left(f(2y) - 2f(y) - C_1(y), r\right)\right\} \\ &\geq \min\Bigg\{N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right), \\ &\qquad N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right)\Bigg\} \end{aligned}$$

for all $y \in X$ and all $r > 0$. Thus we obtain (32) by defining $A(y) = \frac{-1}{6}A_1(y)$ and $C(y) = \frac{1}{6}C_1(y)$ for all $y \in X$ and all $r > 0$. □

Corollary 2.8. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x,y),r\right) \quad \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon\left\{||x||^s + ||y||^s\right\}, r\right), & s \neq 1,3; \\ N'\left(\epsilon\left\{||x||^s||y||^s\right\}, r\right), & s \neq \frac{1}{2}, \frac{3}{2}; \\ N'\left(\epsilon\left(||x||^s||y||^s + ||x||^{2s} + ||y||^{2s}\right), r\right), & s \neq \frac{1}{2}, \frac{3}{2}; \end{cases} \tag{35}$$

for all $x, y \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique Cubic mapping $C : X \to Y$ such that

$$N\left(f(x) - A(x) - C(x), r\right) \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{|2|r}{8}\right), N'\left(\epsilon, \frac{|2|r}{4}\right), N'\left(\epsilon, \frac{r}{|7|}\right), N'\left(\epsilon, \frac{2r}{|7|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{4|2^s-2|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{r}{2|2^s-2|}\right), \right. \\ \qquad \left. N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{|2^s-2^3|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{2r}{|2^s-2^3|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{4|2^{2s}-2|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{2|2^{2s}-2|}\right), \right. \\ \qquad \left. N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{|2^{2s}-2^3|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{2|2^{2s}-2^3|}\right)\right\} \\ min\left\{N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{4|2^{2s}-2|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{r}{2|2^{2s}-2|}\right), \right. \\ \qquad \left. N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{|2^{2s}-2^3|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{2r}{|2^{2s}-2^3|}\right)\right\} \end{cases} \tag{36}$$

for all $y \in X$ and all $r > 0$.

Theorem 2.9. *Let $f : X \to Y$ be a mapping for which there exist a function $\alpha : X^2 \to Z$ with the condition*

$$\lim_{k\to\infty} N'\left(\alpha\left(\delta_i^k x, \delta_i^k y\right), \delta_i^{2k} r\right) = 1, \quad \forall\ x, y \in X, r > 0 \tag{37}$$

and satisfying the functional inequality

$$N\left(D\ f(x,y), r\right) \geq N'\left(\alpha(x,y), r\right), \ \forall\ x, y \in X, r > 0. \tag{38}$$

If there exists $L = L(i)$ such that the function

$$x \to N'\left(\beta(y), r\right) = min\left\{N'\left(\alpha\left(\frac{y}{2}, \frac{y}{2}\right), \frac{r}{8}\right), N'\left(\alpha\left(y, \frac{y}{2}\right), \frac{r}{2}\right)\right\},$$

has the property

$$N'\left(L\frac{1}{\delta_i^2}\beta(\delta_i y), r\right) = N'\left(\beta(y), r\right), \ \forall\ y \in X, r > 0. \tag{39}$$

Then there exists unique quadratic function $Q_2 : X \to Y$ satisfying the functional equation (1) and

$$N\left(f(2y) - 16f(y) - Q_2(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right)\right\} \tag{40}$$

Proof. It is easy to see from (9) that

$$N\left(f(3y) - 6f(2y) + 15f(y), r\right) \geq N'\left(\alpha\left(y, y\right), r\right) \tag{41}$$

for all $y \in X$ and all $r > 0$. Replacing x by $2y$ in (9), we obtain

$$N\left(f(4y) - 4f(3y) + 4f(2y) + 4f(y), r\right) \geq N'\left(\alpha\left(2y, y\right), r\right) \tag{42}$$

for all $y \in X$ and all $r > 0$. It follows from (41) and (42) that

$$\begin{aligned} N\left(f(4y) - 20f(2y) + 64f(2y), r\right) &\geq min\left\{N\left(4\left(f(3y) - 24f(2y) + 60f(y)\right), \frac{r}{2}\right), N\left(f(4y) - 4f(3y) + 4f(2y) + 4f(y), \frac{r}{2}\right)\right\} \\ &\geq min\left\{N'\left(\alpha(y,y), \frac{r}{8}\right), N'\left(\alpha(2y,y), \frac{r}{2}\right)\right\} \end{aligned} \tag{43}$$

for all $y \in X$ and all $r > 0$. Let $q_2 : X \to Y$ be a mapping defined by $q_2(x) = f(2x) - 16f(x)$. Then we conclude that

$$N\left(q_2(2y) - 4q_2(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \frac{r}{8}\right), N'\left(\alpha(2y,y), \frac{r}{2}\right)\right\} \tag{44}$$

for all $y \in X$ and all $r > 0$. The rest of the proof is similar to that of Theorem 2.3.

□

The following corollary is an immediate consequence of Theorem 2.9 concerning the Ulam-Hyers stability of the functional equation(1).

Corollary 2.10. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x,y), r\right) \quad \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon\left\{||x||^s + ||y||^s\right\}, r\right), & s \neq 2; \\ N'\left(\epsilon\left\{||x||^s ||y||^s\right\}, r\right), & s \neq 1; \\ N'\left(\epsilon\left(||x||^s ||y||^s + ||x||^{2s} + ||y||^{2s}\right), r\right), & s \neq 1; \end{cases} \tag{45}$$

for all $x, y \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q_2 : X \to Y$ and a such that

$$N\left(f(2y) - 16f(y) - Q_2(y), r\right) \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{r}{2|-3|}\right), N'\left(\epsilon, \frac{r}{|-3|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{2|2^s - 2^2|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{r}{|2^s - 2^2|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{2|2^{2s} - 2^2|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{|2^{2s} - 2^2|}\right)\right\} \\ min\left\{N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{2|2^{2s} - 2^2|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{r}{|2^{2s} - 2^2|}\right)\right\} \end{cases} \tag{46}$$

for all $y \in X$ and all $r > 0$.

Theorem 2.11. *Let $f : X \to Y$ be a mapping for which there exist a function $\alpha : X^2 \to Z$ with the condition*

$$\lim_{k\to\infty} N'\left(\alpha\left(\delta_i^k x, \delta_i^k y\right), \delta_i^{4k} r\right) = 1, \quad \forall\ x, y \in X, r > 0 \tag{47}$$

and satisfying the functional inequality

$$N\left(D\ f(x,y), r\right) \geq N'\left(\alpha(x,y), r\right), \ \forall\ x, y \in X, r > 0. \tag{48}$$

If there exists $L = L(i)$ such that the function

$$x \to N'\left(\beta(y), r\right) = min\left\{N'\left(\alpha\left(\frac{y}{2}, \frac{y}{2}\right), \frac{r}{8}\right), N'\left(\alpha\left(y, \frac{y}{2}\right), \frac{r}{2}\right)\right\},$$

has the property

$$N'\left(L\frac{1}{\delta_i^4}\beta(\delta_i y), r\right) = N'\left(\beta(y), r\right), \ \forall\ y \in X, r > 0. \tag{49}$$

Then there exists unique quartic function $Q_4 : X \to Y$ satisfying the functional equation (1) and

$$N\left(f(2y) - 4f(y) - Q_4(y), r\right) \geq min\left\{N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right)\right\} \tag{50}$$

Proof. It is easy to see from (43)

$$N\left(f(4y)-4f(2y)-16f(2y),r\right)\geq min\left\{N'\left(\alpha(y,y),\frac{r}{8}\right),N'\left(\alpha(2y,y),\frac{r}{2}\right)\right\} \tag{51}$$

for all $y\in X$ and all $r>0$. Let $q_4:X\to Y$ be a mapping defined by $q_4(x)=f(2x)-4f(x)$. Then we conclude that

$$N\left(q_4(2y)-16q_4(y),r\right)\geq min\left\{N'\left(\alpha(y,y),\frac{r}{8}\right),N'\left(\alpha(2y,y),\frac{r}{2}\right)\right\} \tag{52}$$

for all $y\in X$ and all $r>0$. The rest of the proof is similar to that of Theorem 2.3. □

The following corollary is an immediate consequence of Theorem 2.11 concerning the Ulam-Hyers stability of the functional equation(1).

Corollary 2.12. *Suppose that a function $f:X\to Y$ satisfies the inequality*

$$N\left(Df(x,y),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), & \\ N'\left(\epsilon\left\{||x||^s+||y||^s\right\},r\right), & s\neq 4; \\ N'\left(\epsilon\left\{||x||^s||y||^s\right\},r\right), & s\neq 2; \\ N'\left(\epsilon\left(||x||^s||y||^s+||x||^{2s}+||y||^{2s}\right),r\right), & s\neq 2; \end{cases} \tag{53}$$

for all $x,y\in X$ and all $r>0$, where ϵ,s are constants with $\epsilon>0$. Then there exists a unique quartic mapping $Q_4:X\to Y$ such that

$$N\left(f(2y)-4f(y)-Q_4(y),r\right)\geq \begin{cases} min\left\{N'\left(\epsilon,\frac{2r}{|15|}\right),N'\left(\epsilon,\frac{4r}{|15|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^s}||y||^s,\frac{2r}{|2^s-2^4|}\right),N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s,\frac{4r}{|2^s-2^4|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s},\frac{2r}{|2^{2s}-2^4|}\right),N'\left(\frac{\epsilon}{2^s}||y||^{2s},\frac{4r}{|2^{2s}-2^4|}\right)\right\} \\ min\left\{N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s},\frac{2r}{|2^{2s}-2^4|}\right),N'\left(\epsilon\left(\frac{1+2^s}{2^s}+\frac{1}{2^{2s}}\right)||y||^{2s},\frac{4r}{|2^{2s}-2^4|}\right)\right\} \end{cases} \tag{54}$$

for all $y\in X$ and all $r>0$.

Theorem 2.13. *Let $f:X\to Y$ be a mapping for which there exist a function $\alpha:X^2\to Z$ with the conditions*

$$\lim_{k\to\infty}N'\left(\alpha\left(\delta_i^kx,\delta_i^ky\right),\delta_i^{2k}r\right)=1,\ \lim_{k\to\infty}N'\left(\alpha\left(\delta_i^kx,\delta_i^ky\right),\delta_i^{4k}r\right)=1,\quad\forall\ x,y\in X,r>0 \tag{55}$$

and satisfying the functional inequality

$$N\left(D\ f(x,y),r\right)\geq N'\left(\alpha(x,y),r\right),\ \forall\ x,y\in X,r>0. \tag{56}$$

If there exists $L=L(i)$ such that the function

$$x\to N'\left(\beta(y),r\right)=min\left\{N'\left(\alpha\left(\frac{y}{2},\frac{y}{2}\right),\frac{r}{8}\right),N'\left(\alpha\left(y,\frac{y}{2}\right),\frac{r}{2}\right)\right\},$$

has the property

$$N'\left(L\frac{1}{\delta_i}\beta(\delta_i^2y),r\right)=N'\left(\beta(y),r\right),\ N'\left(L\frac{1}{\delta_i^4}\beta(\delta_iy),r\right)=N'\left(\beta(y),r\right),\ \forall\ y\in X,r>0. \tag{57}$$

Then there exists unique quadratic function $Q_2 : X \to Y$ and unique quartic function $Q_4 : X \to Y$ satisfying the functional equation (1) and

$$N(f(y) - Q_2(y) - Q_4(y), r) \geq \min\left\{N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right),\right.$$
$$\left. N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right)\right\} \tag{58}$$

Proof. By Theorems 2.9 and 2.11, there exists a unique quadratic function $Q_2 : X \to Y$ and a quartic function $Q_4 : X \to Y$ such that

$$N(f(2y) - 16f(y) - Q_{2_1}(y), r) \geq min\left\{N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right)\right\} \tag{59}$$

for all $y \in X$ and all $r > 0$ and

$$N(f(2y) - 4f(y) - Q_{4_1}(y), r) \geq min\left\{N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right)\right\} \tag{60}$$

for all $y \in X$ and all $r > 0$. Now from (59) and (60), one can see that

$$\begin{aligned}
&N\left(f(y) + \frac{1}{12}Q_{2_1}(y) - \frac{1}{12}Q_{4_1}(y), 2r\right)\\
&\geq \min\left\{N\left(\frac{f(2y)}{12} - \frac{16}{12}f(y) - \frac{1}{12}Q_{2_1}(y), \frac{r}{12}\right), N\left(\frac{f(2y)}{12} - \frac{4}{12}f(y) - \frac{1}{12}Q_{4_1}(y), \frac{r}{12}\right)\right\}\\
&\geq \min\left\{N(f(2y) - 16f(y) - Q_{2_1}(y), r), N(f(2y) - 4f(y) - Q_{4_1}(y), r)\right\}\\
&\geq \min\left\{N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right), N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right)\right\}
\end{aligned}$$

for all $y \in X$ and all $r > 0$. Thus we obtain (58) by defining $Q_2(y) = \frac{-1}{12}Q_{2_1}(y)$ and $Q_4(y) = \frac{1}{12}Q_{4_1}(y)$ for all $y \in X$ and all $r > 0$. □

The following corollary is an immediate consequence of Theorem 2.13 concerning the Ulam-Hyers stability of the functional equation(1).

Corollary 2.14. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N(Df(x,y), r) \quad \geq \begin{cases} N'(\epsilon, r), & \\ N'(\epsilon\{||x||^s + ||y||^s\}, r), & s \neq 2,4; \\ N'(\epsilon\{||x||^s ||y||^s\}, r), & s \neq 1,2; \\ N'(\epsilon(||x||^s ||y||^s + ||x||^{2s} + ||y||^{2s}), r), & s \neq 1,2; \end{cases} \tag{61}$$

for all $x, y \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q_2 : X \to Y$ and a unique quartic mapping $Q_4 : X \to Y$ such that

$$N(f(y) - Q_2(y) - Q_4(y), r) \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{r}{2|3|}\right), N'\left(\epsilon, \frac{r}{|3|}\right), N'\left(\epsilon, \frac{2r}{|15|}\right), N'\left(\epsilon, \frac{4r}{|15|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{2|2^s-2^2|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{r}{|2^s-2^2|}\right),\right. \\ \qquad \left. N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{2r}{|2^s-2^4|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{4r}{|2^s-2^4|}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{2|2^{2s}-2^2|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{|2^{2s}-2^2|}\right),\right. \\ \qquad \left. N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{2r}{|2^{2s}-2^4|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{4r}{|2^{2s}-2^4|}\right)\right\} \\ min\left\{N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{2|2^{2s}-2^2|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{r}{|2^{2s}-2^2|}\right),\right. \\ \qquad \left. N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{2r}{|2^{2s}-2^4|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{4r}{|2^{2s}-2^4|}\right)\right\} \end{cases} \tag{62}$$

for all $y \in X$ and all $r > 0$.

Theorem 2.15. *Let $f : X \to Y$ be a mapping for which there exist a function $\alpha : X^2 \to Z$ with the conditions*

$$\lim_{k\to\infty} N'\left(\alpha\left(\delta_i^k x, \delta_i^k y\right), \delta_i^k r\right) = 1, \ \lim_{k\to\infty} N'\left(\alpha\left(\delta_i^k x, \delta_i^k y\right), \delta_i^{2k} r\right) = 1,$$
$$\lim_{k\to\infty} N'\left(\alpha\left(\delta_i^k x, \delta_i^k y\right), \delta_i^{3k} r\right) = 1, \ \lim_{k\to\infty} N'\left(\alpha\left(\delta_i^k x, \delta_i^k y\right), \delta_i^{4k} r\right) = 1, \quad \forall\ x, y \in X, r > 0 \tag{63}$$

and satisfying the functional inequality

$$N\left(D\ f(x,y), r\right) \geq N'\left(\alpha(x,y), r\right), \ \forall\ x, y \in X, r > 0. \tag{64}$$

If there exists $L = L(i)$ such that the function

$$x \to N'\left(\beta(y), r\right) = min\left\{N'\left(\alpha\left(\frac{y}{2}, \frac{y}{2}\right), \frac{r}{8}\right), N'\left(\alpha\left(y, \frac{y}{2}\right), \frac{r}{2}\right)\right\},$$

has the property

$$N'\left(L\frac{1}{\delta_i}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right), \ N'\left(L\frac{1}{\delta_i^2}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right),$$
$$N'\left(L\frac{1}{\delta_i}\beta(\delta_i^3 x), r\right) = N'\left(\beta(x), r\right), \ N'\left(L\frac{1}{\delta_i^4}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right), \ \forall\ x \in X, r > 0. \tag{65}$$

Then there exists a unique additive mapping $A : X \to Y$, a unique quadratic mapping $Q_2 : X \to Y$, a unique cubic cubic mapping $C : X \to Y$ and unique quartic mapping $Q_4 : X \to Y$ satisfying the functional equation (1) and

$$\begin{aligned}
&N\left(f(y) - A(y) - Q_2(y) - C(y) - Q_4(y), r\right)\\
&\geq \min\left\{N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{16}\right), N'\left(\alpha(-y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{16}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right),\right.\\
&\quad N'\left(\alpha(-2y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{16}\right), N'\left(\alpha(-y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{16}\right),\\
&\quad N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(-2y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{16}\right),\\
&\quad N'\left(\alpha(-y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{16}\right), N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(-2y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right),\\
&\quad N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{16}\right), N'\left(\alpha(-y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{16}\right),\\
&\quad \left.N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(-2y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right)\right\}
\end{aligned} \tag{66}$$

Proof. Let $f_{ac}(y) = \frac{f_o(y) - f_o(-y)}{2}$ for all $y \in X$. Then $f_{ac}(0) = 0$ and $f_o(-y) = -f_o(y)$ for all $y \in X$. Hence

$$N\left(Df_{ac}(x,y), r\right) \geq min\left\{N'\left(\alpha(x,y), \frac{r}{2}\right), N'\left(\alpha(-x,-y), \frac{r}{2}\right)\right\} \tag{67}$$

for all $y \in X$ and all $r > 0$. By Theorem (2.7), there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\begin{aligned}
N\left(f_{ac}(y) - A(y) - C(y), r\right) \geq \min&\left\{N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(-y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right),\right.\\
&N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right), N'\left(\alpha(-2y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right),\\
&N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(-y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right),\\
&\left.N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right), N'\left(\alpha(-2y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right)\right\}
\end{aligned} \tag{68}$$

for all $y \in X$ and all $r > 0$. Also, let $f_{qq}(y) = \frac{f_e(y)+f_e(-y)}{2}$ for all $y \in X$. Then $f_{qq}(0) = 0$ and $f_o(-y) = f_o(y)$ for all $y \in X$. Hence

$$N\left(Df_{qq}(x,y), r\right) \geq min\left\{N'\left(\alpha(x,y), \frac{r}{2}\right), N'\left(\alpha(-x,-y), \frac{r}{2}\right)\right\} \tag{69}$$

for all $y \in X$ and all $r > 0$. By Theorem 2.13, there exists a unique quadratic mapping $Q_2 : X \to Y$, and a unique quartic mapping $Q_4 : X \to Y$ such that

$$\begin{aligned} N\left(f_{qq}(y) - Q_2(y) - Q_4(y), r\right) \geq \min\Bigg\{ & N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(-y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), \\ & N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right), N'\left(\alpha(-2y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right), \\ & N'\left(\alpha(y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\alpha(-y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), \\ & N'\left(\alpha(2y,y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right), N'\left(\alpha(-2y,-y), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{4}\right)\Bigg\} \end{aligned} \tag{70}$$

for all $y \in X$ and all $r > 0$. Define a function $f(y)$ by

$$f(y) = f_{ac}(y) + f_{qq}(y) \tag{71}$$

for all $y \in X$. Combining (71), (68) and (70), we arrive our result. □

The following corollary is an immediate consequence of Theorem 2.15 concerning the Ulam-Hyers stability of the functional equation(1).

Corollary 2.16. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x,y), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon\left\{||x||^s + ||y||^s\right\}, r\right), & s \neq 1, 3, 2, 4; \\ N'\left(\epsilon\left\{||x||^s ||y||^s\right\}, r\right), & s \neq \frac{1}{2}, \frac{3}{2}, 2, 4; \\ N'\left(\epsilon\left(||x||^s ||y||^s + ||x||^{2s} + ||y||^{2s}\right), r\right), & s \neq \frac{1}{2}, \frac{3}{2}, 2, 4; \end{cases} \tag{72}$$

for all $x, y \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique Cubic mapping $C : X \to Y$, a unique quadratic mapping $Q_2 : X \to Y$ and a unique quartic mapping $Q_4 : X \to Y$ such that

$$\begin{aligned} & N\left(f(x) - A(x) - Q_2(x) - C(x) - Q_4(x), r\right) \\ \geq & \begin{cases} (i) N'\left(\epsilon, \frac{|2|r}{8}\right), N'\left(\epsilon, \frac{|2|r}{4}\right), N'\left(\epsilon, \frac{r}{|7|}\right), N'\left(\epsilon, \frac{2r}{|7|}\right), N'\left(\epsilon, \frac{r}{2|3|}\right), N'\left(\epsilon, \frac{r}{|3|}\right), \\ \quad N'\left(\epsilon, \frac{2r}{|15|}\right), N'\left(\epsilon, \frac{4r}{|15|}\right) \\ (ii) N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{4|2^s-2|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{r}{2|2^s-2|}\right), N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{|2^s-2^3|}\right), \\ \quad N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{2r}{|2^s-2^3|}\right), N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{r}{2|2^s-2^2|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{r}{|2^s-2^2|}\right), \\ \quad N'\left(\frac{\epsilon}{2^s}||y||^s, \frac{2r}{|2^s-2^4|}\right), N'\left(\epsilon\frac{1+2^s}{2^s}||y||^s, \frac{4r}{|2^s-2^4|}\right) \\ (iii) N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{4|2^{2s}-2|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{2|2^{2s}-2|}\right), N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{|2^{2s}-2^3|}\right), \\ \quad N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{2|2^{2s}-2^3|}\right), N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{2|2^{2s}-2^2|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{r}{|2^{2s}-2^2|}\right), \\ \quad N'\left(\frac{\epsilon}{2^{2s}}||y||^{2s}, \frac{2r}{|2^{2s}-2^4|}\right), N'\left(\frac{\epsilon}{2^s}||y||^{2s}, \frac{4r}{|2^{2s}-2^4|}\right) \\ (iv) N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{4|2^{2s}-2|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{r}{2|2^{2s}-2|}\right), \\ \quad N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{|2^{2s}-2^3|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{2r}{|2^{2s}-2^3|}\right), \\ \quad N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{r}{2|2^{2s}-2^2|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{r}{|2^{2s}-2^2|}\right), \\ \quad N'\left(\frac{3\epsilon}{2^{2s}}||y||^{2s}, \frac{2r}{|2^{2s}-2^4|}\right), N'\left(\epsilon\left(\frac{1+2^s}{2^s} + \frac{1}{2^{2s}}\right)||y||^{2s}, \frac{4r}{|2^{2s}-2^4|}\right) \end{cases} \end{aligned} \tag{73}$$

for all $y \in X$ *and all* $r > 0$.

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