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On the Star Coloring of Graphs Formed From the Cartesian Product of Some Simple Graphs

Research Article*

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Abstract: Let $G = (V, E)$ be an undirected simple graph. The star chromatic number of a graph G is the least number of colors needed to color the path on four vertices with three distinct colors. The purpose of this paper is to study the star coloring of some graph families formed from the Cartesian product of some simple graphs.

Keywords: Proper coloring, Chromatic number, Star coloring, Star chromatic number, Cartesian product of graphs, Prism graph, Barbell graph, Fan graph, Windmill graph, Lollipop graph.

1. Introduction and Preliminaries

In this paper, we have taken the graphs to be finite, undirected and loopless. Let us acknowledge the terminology of graph theory in [2]. The path P of a graph is a walk in which no vertices are repeated. In 1973, the concept of star coloring was introduced by Grünbaum [4] and also he introduced the notion of star chromatic number. His works were developed further by Bondy and Hell [3]. According to them the star coloring is the proper coloring on the paths with four vertices by giving 3-distinct colors on it. The Cartesian product of a graph $G = G_1 \Box G_2$ is the graph with vertex set $V = V_1 \times V_2$ and $u = (u_1, u_2)$ adjacent $v = (v_1, v_2)$ with whenever $u_1 = v_1$ and u_2 adj v_2 or $u_2 = v_2$ and u_1 adj v_1. In this paper we have obtained the star chromatic number of some graphs formed from the Cartesian product of two simple graphs.

Definition 1.1 (Vertex Coloring)**.** *Let G be a graph and let $V(G)$ be the set of all vertices of G and let $\{1, 2, 3, ..., k\}$ denotes the set of all colors which are assigned to each vertex of G. A proper vertex coloring of a graph is a mapping $c : V(G) \to \{1, 2, 3, \ldots, k\}$ such that $c(u) \neq c(v)$ for all arbitrary adjacent vertices u, $v \in V(G)$.*

Definition 1.2 (Chromatic Number)**.** *If G has a proper vertex coloring then the chromatic number of G is the minimum number of colors needed to color G. The chromatic number of G is denoted by $\chi(G)$.*

Definition 1.3 (Star Coloring)**.** *A proper vertex coloring of a graph G is called star coloring [10], if every path of G on four vertices is not 2-colored.*

Definition 1.4 (Star Chromatic Number)**.** *The star chromatic number is the minimum number of colors needed to star color G [1] and is denoted by $\chi_s(G)$.*

Let us consider the following example:

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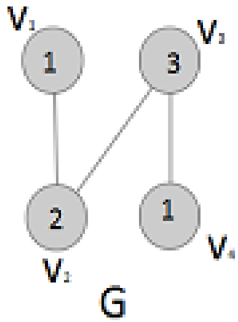
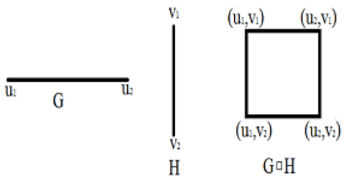


Let G be a path graph and V(G) =$\{v_1,v_2,v_3,v_4\}$. Here $c(v_1) = c(v_4) = 1$, $c(v_2) = 2$, $c(v_3) = 3$. Then the path $P_4$of G is $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4$, $c(P_4) = c(v_1) \rightarrow c(v_2) \rightarrow c(v_3) \rightarrow c(v_4)$= 1-2 - 3 - 1. $\chi_s(P_4) = 3 \Longrightarrow \chi_s(G) = 3$.

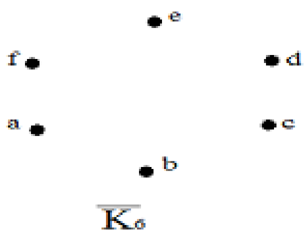
Definition 1.5 (Cartesian Product of Graphs). *The Cartesian product of simple graphs G and H is the G?H [6] whose vertex set is V(G) × V(H) and whose edge set is the set of all pairs* (u_i,v_i), (u_j,v_j) *such that* $u_i,u_j \in V(G)$ *and* $v_i,v_j \in V(H)$.

Definition 1.6 (Empty Graph). *An empty graph [10] on n vertices consists of n isolated n vertices with no edges, such graphs are sometimes also called edgeless graph and is commonly denoted by* $\overline{K}_n$.

Example 1.7.



(a) Cartesian Product of Graphs

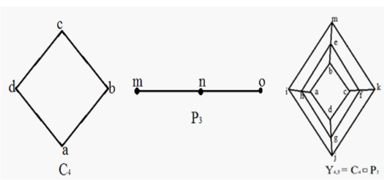


(b) Empty Graph

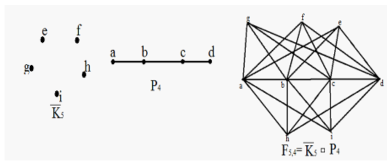
Definition 1.8 (Prism Graph). *A generalized prism graph* $Y_{n,m}$ *[5] is a simple graph obtained by the Cartesian product of two graphs say cycle* C_n *and path* P_m.

Definition 1.9 (Fan Graph). *A fan graph* $F_{m,n}$ *[5] is defined as the graph by joining* $\overline{K}_m$*and* P_n *graphs where* $\overline{K}_m$ *is the empty graph on m vertices and* P_n*is the path graph on n vertices.*

Example 1.10.



(a) Prism Graph



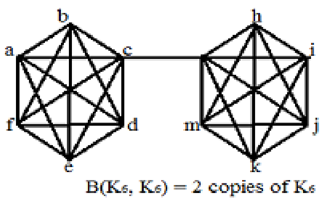
(b) Fan Graph

Definition 1.11 (Barbell Graph). *The n-barbell graph [6] is the simple graph obtained by connecting two copies of a complete graph* K_n *by a bridge and is denoted by* $B(K_n,\ K_n)$.

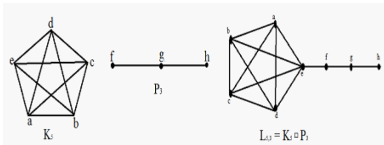
Definition 1.12 (Lollipop Graph). *The (m,n)-lollipop graph [6] is the graph obtained by joining a complete graph K_m to a path graph $P_n$with a bridge and is denoted by $L_{m,n}$.*

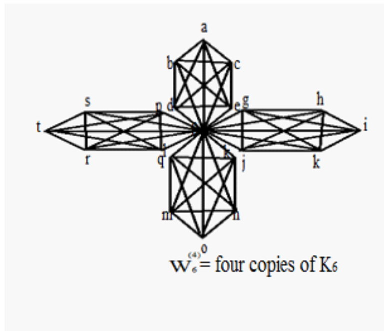
Definition 1.13 (Windmill Graph). *The windmill graph ${W_n}^{(m)}$ [7] is the graph obtained by taking m copies of the complete graph K_n with a vertex in common.*

Example 1.14.



(a) Barbell Graph


(b) Lollipop Graph



(c) Windmill Graph

Theorem 1.15 (Star Coloring of Cycle [4]). *Let C_n be the cycle with $n \geq 3$ vertices. Then* $\chi_s(C_n) = \begin{cases} 4, & when\ n = 5 \\ 3, & otherwise \end{cases}$.

Theorem 1.16 (Star Coloring of Complete graph [4]). *If K_n is the complete graph with n vertices. Then $\chi_s(k_n)= n$, $\forall\ n \geq 3$.*

Theorem 1.17 (Star Coloring of path graph [4]). *Let P_n be the path graph with n vertices. Then $\chi_s(P_n)= 3$, $\forall\ n \geq 4$.*

2. Star Coloring of Some Graphs Formed From the Cartesian Product of Simple Graphs

In this section we have obtained the star chromatic number of various graphs formed from the Cartesian product of simple graphs

Theorem 2.1 (Star coloring of prism graph). *Let $Y_{n,m}$ be the prism graph then the star chromatic number of $Y_{n,m}$ is given by*

$$\chi_s(Y_{n,m}) = \begin{cases} 4, & when\ n = 5 \\ 3, & otherwise \end{cases},\ when\ m = 1,\ \ \forall\ n \geq 3\ and$$

$$\chi_s(Y_{n,m})= \begin{cases} 6, & when\ n = 5 \\ 5, & otherwise \end{cases},\ when\ m \geq 2,\ \ \forall\ n \geq 3.$$

Proof. Let $Y_{n,m} = C_n \Box P_m$ be the prism graph is formed by the Cartesian product of C_n and P_m [5]. The star coloring of $Y_{n,m}$ are discussed in two cases

<u>*Case 1:*</u> For m=1, the prism graph $Y_{n,1}$is a cycle graph C_n with $n \geq 3$ also we know that [4]

$$\chi_s(C_n) = \begin{cases} 4, & \text{when } n = 5 \\ 3, & \text{otherwise.} \end{cases}$$

Therefore

$$\chi_s(Y_{n,1}) = \begin{cases} 4, & \text{when } n = 5 \\ 3, & \text{otherwise.} \end{cases} \tag{1}$$

Case 2: for $m \geq 2$, the prism graph $\mathrm{Y}_{n,m}$ is a m-concentric copies of cycle graph C_n in which all the corresponding vertices are adjoined. When $n = 5$, assign colors 1, 2, 3, 4 consecutively to the vertices of the innermost cycle of the graph $\mathrm{Y}_{5,m}$. Now assign colors1, 2, 3, 4, 5, 6 to all the vertices of the remaining cycle of $\mathrm{Y}_{5,m}$ in such a way that the star coloring condition is satisfied. When n≠5, assign colors 1, 2, 3 consecutively to the vertices of the innermost cycle of the graph $\mathrm{Y}_{n,m}$. now all the vertices of the remaining cycle of $\mathrm{Y}_{n,m}$ are assigned with colors 1, 2, 3, 4, 5 such that there is no possibility for all the paths on four vertices to be bicolored. Therefore this is a proper star coloring. Hence

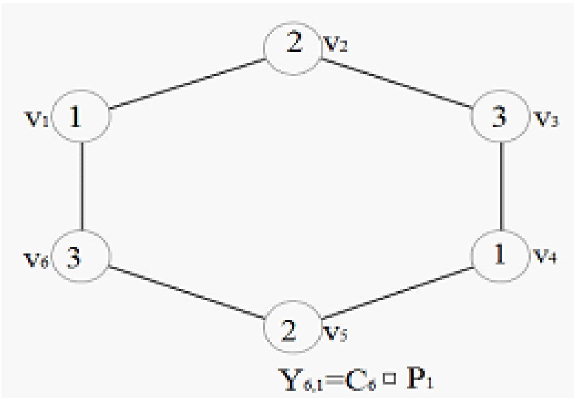
$$\chi_s(Y_{n,m}) = \begin{cases} 6, & \text{when } n = 5 \\ 5, & \text{otherwise.} \end{cases} \tag{2}$$

From (1) and (2) we get,

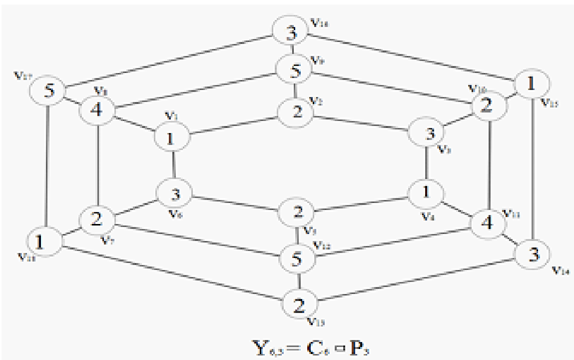
$\chi_s(\mathrm{Y}_{n,m}) = \begin{cases} 4, & when\ n = 5 \\ 3, & otherwise \end{cases}$, when $m = 1,\ \forall\, n \geq 3$ and

$\chi_s(\mathrm{Y}_{n,m}) = \begin{cases} 6, & when\ n = 5 \\ 5, & otherwise \end{cases}$, when $m \geq 2,\ \forall\, n \geq 3$. □

Example 2.2.



$c(v_1) = 1,\ c(v_2) = 2, \ldots,\ c(v_6) = 3$. *Here no path on four vertices is bicolored. Therefore* $\chi_s(Y_{6,3}) = 5$.



$c(v_1) = c(v_4) = c(v_{18}) = c(v_{15}) = 1,\ c(v_2) = c(v_5) = c(v_{13}) = c(v_7) = c(v_{10}) = 2,\ c(v_3) = c(v_6) = c(v_{14}) = c(v_{14}) = c(v_{16}) = 3,\ c(v_{11}) = c(v_8) = 4,\ c(v_{17}) = c(v_{12}) = c(v_9) = 5$. *Here no path on four vertices is bicolored. Therefore* $\chi_s(Y_{6,3}) = 5$.

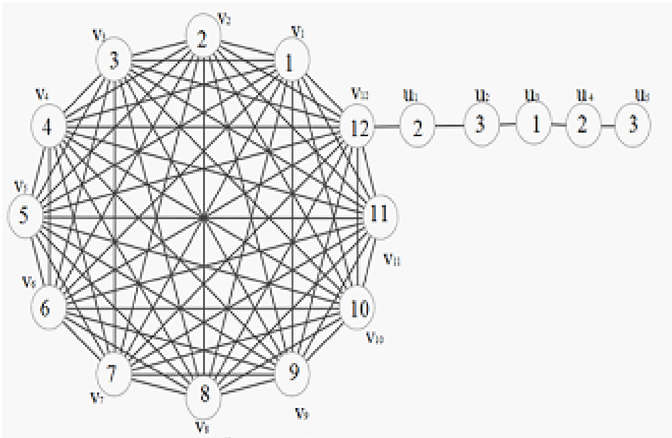
Theorem 2.3 (Star coloring of lollipop graph)**.** *Let $L_{m,n}$ be the lollipop graph, then the star chromatic number of $L_{m,n}$ is given by $\chi_s(L_{m,n}) = m,\ \forall\ m,\ n \geq 3$.*

Proof. By the definition of lollipop graph, it is obtained by joining a complete graph K_m to a path graph P_n [6]. Let A denote the vertex set of the complete graph K_m (i.e.) A= {v_1, v_2, $v_3 \ldots$ v_m}. Let B denote the vertex set of the path graph P_n (i.e.) B= {u_1, u_2, $u_3 \ldots$ u_n}. Let us now star color the graph $L_{m,n}$ by the following procedure:

- Assign colors 1, 2, 3, ..., m to all the vertices of the vertex set A [4] (since all the vertices v_i are adjacent to each other).
- Assign colors 1, 2, 3 consecutively to all the vertices of the vertex set B [4].

Thus by combining the above steps we get no paths on four vertices is bicolored. This is a proper star coloring. Hence $\chi_s(L_{m,n}) = m, \ \forall \ m, \ n \geq 3$. □

Example 2.4.



$L_{12,5} = K_{12} \square P_5$

$c(v_1) = c(u_3) = 1, \ c(v_2)c(u_1) = c(u_4) = 2, \ c(v_3) = c(u_2) = c(u_5) = 3, \ c(v_4) = 4 \ldots, \ c(v_{12}) = 12$. *Here no path on four vertices is bicolored. Therefore* $\chi_s(L_{12,5}) = 12$.

Theorem 2.5 (Star coloring of barbell graph)**.** *Let $B(K_n, \ K_n)$ be the barbell graph, then the star chromatic number of $B(K_n, \ K_n)$ is given by $\chi_s(B(K_n, \ K_n)) = n+1$. $\forall \ n \geq 3$.*

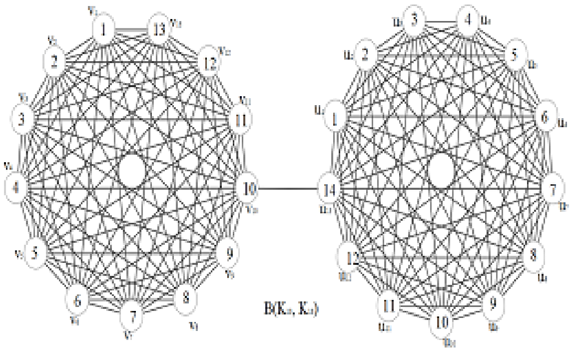
Proof. We know that a barbell graph is obtained by connecting two copies of complete graph K_n by a bridge [6]. Let A be the first copy of the complete graph K_n and let B be the second copy of the complete graph K_n. Let V= {v_1, v_2, $v_3 \ldots$ v_n} be the vertices of A and let U={u_1, u_2, $u_3 \ldots$ u_n} be the vertices of B. The procedure to star color the graph B(K_n, K_n) is as follows:

- Assign colors 1, 2, 3,..., n to v_1, v_2, $v_3 \ldots$ $v_n$consecutively [4]
- Assign color n+1 to the particular vertex u_j of B which is adjacent to the particular vertex v_i of A.
- Assign colors 1, 2, 3, ... n-1 to the remaining vertices of B

Thus there is no possibility for all the paths to be bicolored hence this is a proper star coloring. Thus $\chi_s(B(K_n, \ K_n)) =$ n+1 $\forall \ n \geq 3$. □

Example 2.6.

$c(v_1) = 1, \ c(v_2) = 2, \ldots., \ c(v_{13}) = 13, \ c(u_1) = 1, \ c(u_2) = 2, \ldots., \ c(u_{12}) = 12, \ c(u_{13}) = 14$. *Here no path on four vertices is bicolored. Therefore* $\chi_s(B(K_{13}, \ K_{13})) = 14 (13+1)$.



Theorem 2.7 (Star coloring of windmill graph)**.** *Let* $W_n^{(m)}$ *be the windmill graph then the star chromatic number of* $W_n^{(m)}$ *is given by* $\chi_s\left(W_n^{(m)}\right) = n \; \forall \; n \geq 3$ *and* $\forall \; m \geq 1$.

Proof. From the definition of windmill graph $W_n^{(m)}$ and we know that it is formed by taking m- copies of the complete graph K_n with a vertex in common [7]. Since all the vertices of K_n are adjacent to each other, each vertex receives distinct colors (i.e.) for a complete graph K_i, receives i colors for i. Let us star color the graph ${W_n}^{(m)}$.

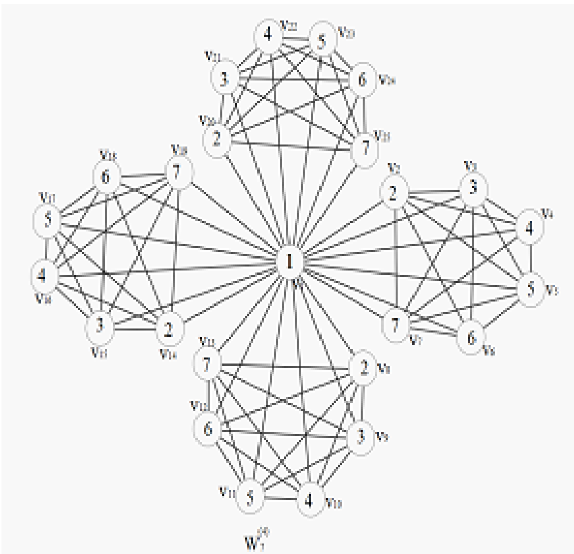
- First assign color 1 to the vertex in common of $W_n^{(m)}$
- Assign colors from 2 to n to the remaining vertices of one copy K_n

Here the vertex with color 1 is the common vertex to all the copies of K_n.

- We shall assign colors from 2 to n to the vertices of the remaining copies of K_n.

Here for any vertex v_i with color i its neighboring vertices are assigned with distinct colors so there is no possibility for the paths on four vertices to be bicolored. Thus is a valid star coloring. Hence $\chi_s\left(W_n^{(m)}\right) = $ n, $\forall \; n \geq 3$ and $\forall \; m \geq 1$. ☐

Example 2.8.



$c(v_1) = 1$, $c(v_2) = 2, \ldots ., c(v_{25}) = 7$. *Here no path on four vertices is bicolored.* $\chi_s\left(W_7^{(4)}\right) = 7$.

Theorem 2.9 (Star coloring of fan graph)**.** *Let* $F_{m,n}$ *be the fan graph, then the star chromatic number of* $F_{m,n}$ *are given as follows: When* $m = 1$, $\forall \; n \geq 2$.

$$\chi_s(F_{m,n}) = \begin{cases} 3, & \text{when } n = 3 \\ 4, & \text{otherwise.} \end{cases}$$

when $m \geq 2$, $\forall\ n \geq 2$, $\chi_s(F_{m,n}) = n + 1$.

Proof. From the definition of fan graph we know that $F_{m,n} = \overline{K}_m$? P_n [5] where $\overline{K}_m$ is the empty graph [10] on m vertices and P_n is the path graph on n vertices. Let us now star color the graph $F_{m,n}$.

Case 1: when $m = 1$.

If $m = 1$ then $\overline{K}_1$ is the empty graph with one vertex, thus $F_{1,n} = \overline{K}_1 \Box P_n$. On star coloring $F_{1,n}$ we have two sub cases.

Sub case 1: $\chi_s(F_{1,n})$=3, $n = 3$

If $n = 2$ then $F_{1,2}$ is clearly the cycle graph C_3 (i.e.) $F_{1,2}$= C_3 w.k.t $\chi_s(C_3) = 3$ [4] therefore $\chi_s(F_{1,2})$=3 $\rightarrow$ (1), if n=3 we get $F_{1,3}$ we have to assign color 1 the vertex of the empty graph then the remaining three vertices of the path of $F_{1,3}$ is assigned with colors 2,3 consecutively thus is a proper star coloring hence $\chi_s(F_{1,3})$=3$\rightarrow$ (2). From (1) and (2)

$$\chi_s(F_{1,n}) = 3,\ n = 3 \tag{3}$$

Sub case 2: $\chi_s(F_{1,n})$=4, $n > 3$

Assign color 1 to the vertex of the empty graph then the remaining vertices are the vertices of the path graph and w.k.t $\chi_s(P_n)$= 3 [4]. $\forall\ n \geq 4$ now assign colors 2, 3, 4 consecutively to all the vertices of the path. Clearly we say that this coloring is a valid star coloring therefore

$$\chi_s(F_{1,n}) = 4,\ n > 3 \tag{4}$$

From (3), (4) we have

$$\chi_s(F_{m,n}) = \begin{cases} 3, & \text{when } n = 3 \\ 4, & \text{otherwise} \end{cases} \tag{5}$$

Case 2: when $m = 2$

Since all the vertices of $\overline{K}_m$ are non adjacent we can assign color 1 to all the vertices of $\overline{K}_m$. now we are left with the vertices of the path. Suppose we star color the vertices of the path with repetition of colors 2, 3, 4 then there is a possibility of having certain paths to be bicolored. Therefore without repetition of colors we star color the n vertices of the path with n+1colors so any vertex v_i on the path graph its neighborhood vertices are assigned with color 1 along with two distinct colors. Therefore no path on four vertices is bicolored. Hence this is a valid star coloring.

$$\chi_s(F_{m,n}) = n + 1 \text{ when } m = 2,\ \forall\ n = 2 \tag{6}$$

hence case(2) from (5), (6) we have When $m = 1$, $\forall\ n \geq 2$

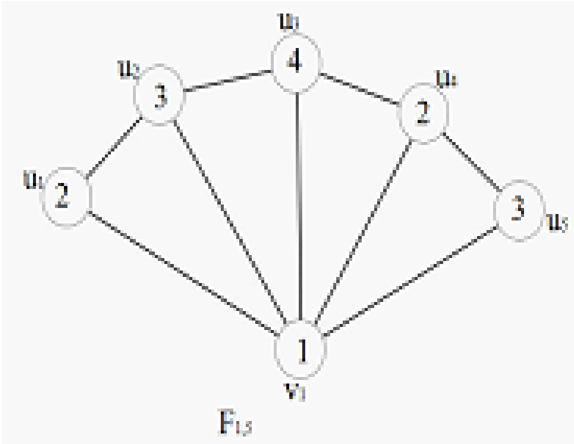
$$\chi_s(F_{m,n}) = \begin{cases} 3, & \text{when } n = 3 \\ 4, & \text{otherwise} \end{cases}$$

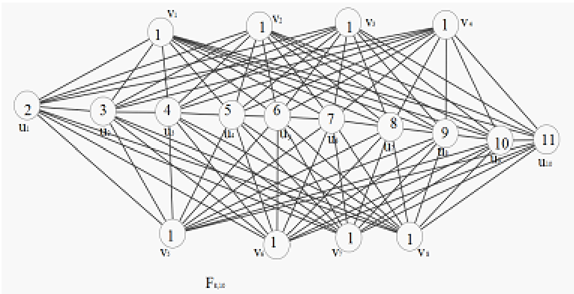
and when $m \geq 2$, $\forall\ n \geq 2$, $\chi_s(F_{m,n})$=n+1. □

Example 2.10.

$c(v_1) = 1$, $c(u_1) = 2$, $c(u_2) = 3$, $c(u_3) = 4$, $c(u_4) = 2$, $c(u_5) = 3$. *Here no path on four vertices is bicolored.* $\chi_s(F_{1,5}) = 4$.

$c(v_1) = c(v_2) = c(v_3) = \ldots = c(v_8) = 1$, $c(u_1) = 2$, $c(u_2) = 3, \ldots$, $c(u_{10}) = 11$. *Here no path on four vertices is bicolored* $\chi_s(F_{8,10}) = 11$.





3. Conclusion

In this paper we determined the star chromatic number of various graphs formed from the Cartesian product of simple graphs. This work can be further extended for various graphs formed from various graph products.

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