International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **11(4)**(2023), 33–39
Available Online: http://ijmaa.in

Common Fixed Point Theorem for Four Weakly Compatible Self Maps with E.A Property of a Fuzzy G - Metric Space

P. Senthil Kumar[1,*], P. Thiruveni[1]

[1]*Department Of Mathematics, Rajah Serfoji Government College (Autonomous) (Affiliated To Bharathidasan University), Thanjavur, Tamilnadu , India*

Abstract

In this paper we analyse the common fixed point theorem for four weakly compatible self maps with (E.A) property of a G - fuzzy metric space. Some of the inclusions is given to prove the new way of approach for common fixed point theorem.

Keywords: G - Metric space; fixed point; fuzzy logic; weakly compatible self maps; (E.A) property; Implicit relation.

2020 Mathematics Subject Classification: 54H25, 46S40, 03B52, 26B10.

1. Introduction

Many mathematician scholars [2,4,6,7,9–11,14,18,19] generalized metric spaces in different approaches. On the line of these the G-metric space given by Zead Mustafa and Brailey Sims [15,16] is interesting. The eminent researcher G. Jungck [5,13] first gave the frame of compatible mappings and later Jungck and Rhoades [17] introduced the conviction of weakly compatible mappings as a generalization of weakly commuting mappings. Recently, Aamri and Moutawakil [1] foreword the concept of (E.A) property. In present paper we prove a common fixed point theorem for four weakly compatible self maps with (E.A) property in a G- fuzzy metric space.

2. Preliminaries

Definition 2.1 ([3]). *Let $\mathbb{X}$ be a non empty set and let $G : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$ be a function satisfying the following properties*

1. *$G(a, b, c) = 0$ if $a = b = c$.*
2. *$0 < G(a, a, b)$ for all $a, b \in \mathbb{X}$ with $a \neq b$.*

*Corresponding author (smile.thiruveni@gmail.com)

3. $G(a,a,b) \leq G(a,b,c)$ *for all* $a,b,c \in \mathbb{X}$ *with* $b \neq c$.

4. $G(a,b,c) = G(a,c,b) = G(b,c,a) = \dots$, *symmetry all three variables.*

5. $G(a,b,c) \leq G(a,x,x) + G(x,b,c)$ *for all* $a,b,c,x \in \mathbb{X}$.

Then the function G is called a generalized metric or a G- metric on $\mathbb{X}$ *and the pair* $(\mathbb{X}, G)$ *is called a G - metric space.*

Definition 2.2 ([3]). *The G - metric space* $(\mathbb{X}, G)$ *is called symmetry if* $G(a,a,b) = G(a,b,b)$ *for all* $a,b \in \mathbb{X}$.

Definition 2.3 ([8]). *A 3-tuple* $(\mathbb{X}, G, *)$ *is called a G - fuzzy metric space if* $\mathbb{X}$ *is an arbitrary non empty set,* $*$ *is a continuous t-norm and G is a Fuzzy set on* $\mathbb{X}^3 \times (0,\infty)$ *satisfying the following conditions for each* $t,s > 0$.

1. $G(a,a,b,t) > 0$ *for all* $a,b \in \mathbb{X}$ *with* $a \neq b$.

2. $G(a,a,b,t) \geq G(a,b,c,t)$ *for all* $a,b,c \in \mathbb{X}$ *with* $b \neq c$.

3. $G(a,b,c,t) = 1$ *if and only if* $a = b = c$.

4. $G(a,b,c,t) = G(p(a,b,c),t)$, *where p is a permutation function.*

5. $G(a,b,c,t+s) \geq G(x,b,c,t) * G(a,x,x,s)$ *for all* $a,b,c,x \in \mathbb{X}$.

6. $G(a,b,c,.) : (0,\infty) \to [0,1]$ *is continuous.*

Definition 2.4 ([12]). *A G - fuzzy metric space* $(\mathbb{X}, G, *)$ *is said to be symmetric if* $G(a,a,b,t) = G(a,b,b,t)$, *for all* $a,b \in \mathbb{X}$ *and for each* $t > 0$.

Example 2.5. *Let* $\mathbb{X}$ *be a non - empty set and let G be a G - fuzzy metric on* $\mathbb{X}$. *Denote* $x * y = xy$ *for all* $x,y \in [0,1]$ *for each* $t > 0$. $G(a,b,c,t) = \frac{t}{t+G(a,b,c,t)}$ *is a G - fuzzy metric on* $\mathbb{X}$. *Let* $(\mathbb{X}, G, *)$ *be a G - fuzzy metric space. For* $t > 0$, $0 < r < 1$ *and* $a \in \mathbb{X}$, *the set* $B_G(a,r,t) = \{b \in \mathbb{X},\ G(a,b,b,t) > 1-r\}$ *is called an open ball with centre a and radius r.*

A subset A of $\mathbb{X}$ is called an open set if for each $a \in \mathbb{X}$, there exist $t > 0$ and $0 < r < 1$ such that $B_0 = (a,r,t) \subseteq A$. A sequence $\{a_0\}$ in G - fuzzy metric space $\mathbb{X}$ is said to be G - convergent to $a \in \mathbb{X}$ if $G(a_n, a_n, a, t) \to 1$ as $n \to \infty$ or each $t > 0$. It is called a G - Cauchy sequence if $G(a_n, a_n, a_m, a, t) \to 1$ as $n, m \to \infty$ for each $t > 0$. $\mathbb{X}$ is called G - complete if every G - Cauchy sequence in $\mathbb{X}$ is G - convergent in $\mathbb{X}$.

Lemma 2.6. *Let* $(\mathbb{X}, G, *)$ *be a G - fuzzy metric space, then* $G(a,b,c,t)$ *is non - decreasing with respect to t for all* $a,b,c \in \mathbb{X}$.

Lemma 2.7. *Let* $(\mathbb{X}, G, *)$ *be a G - fuzzy metric space then* $G(a,b,b,t) \leq 2G(b,a,a,t)$ *for all* $a,b,t \in \mathbb{X}$.

Definition 2.8. *Let f and g mapping from a G - fuzzy metric space* $(\mathbb{X}, G, *)$ *into itself. A pair of map* $\{f,g\}$ *is said to be compatible if* $\lim_{n\to\infty} G(fga_n, gfa_n, t) = 1$, *whenever* $\{a_n\}$ *is a sequence in* $\mathbb{X}$ *such that* $\lim_{n\to\infty} fa_n = \lim_{n\to\infty} ga_n = u$ *for some* $u \in \mathbb{X}$ *and for all* $t > 0$.

Definition 2.9. *Let $(\mathbb{X}, G, *)$ be a G - fuzzy metric space, suppose f and g be self maps on $\mathbb{X}$. A point a in $\mathbb{X}$ is called a coincidence point of f and g iff $fa_n = ga_n$. In this case, $w = fa_n = ga_n$ is called a point of coincidence of f and g.*

Definition 2.10. *A pair of self mapping $\{f, g\}$ of a G - fuzzy metric space $G(\mathbb{X}, G, *)$ is said to be weakly compatible if they commute at the coincidence point. That is, if $fu = gu$ for some $u \in \mathbb{X}$, then $fgu = gfu$.*

It is to see that two compatible maps are weakly compatible but converse is not true.

Definition 2.11. *A pair of self - mappings (f, g) of a fuzzy metric spaces $(\mathbb{X}, M, *)$ is said to satisfy E.A property, if there exists a sequence $\{a_n\}$ in $\mathbb{X}$ such that $\lim\limits_{n\to\infty} M(fa_n, ga_n, t) = 1$ for some $t \in \mathbb{X}$.*

Example 2.12. *Let $\mathbb{X} = [0, 1]$ with the usual metric space d, that is $d(a, b) = |a - b|$. Define $M(a, b, t) = \left[\frac{t}{t+d(a,b)}\right]$ for all $a, b \in \mathbb{X}$ and for all $t > 0$ and also define*

$$fa = \begin{cases} 1 - a, & \text{if } a \in [0, 1/2]; \\ 0, & \text{if } a \in (1/2, 1]. \end{cases}$$

$$ga = \begin{cases} \frac{1}{2}, & \text{if } a \in [0, 1/2]; \\ \frac{3}{4}, & \text{if } a \in (1/2, 1]. \end{cases}$$

consider the sequence $\{a_n\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}$, $n \geq 2$, we have $\lim\limits_{n\to\infty} f\left(\frac{1}{2} - n\right) = \frac{1}{2} \lim\limits_{n\to\infty} g\left(\frac{1}{2} - \frac{1}{n}\right)$. Thus the pair (f, g) satisfies E.A property. Further f and g are weakly compatible since $a = \frac{1}{2}$ is their unique coincidence point $fg\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = g\left(\frac{1}{2}\right) = gf\left(\frac{1}{2}\right)$, we further observe that $\lim\limits_{n\to\infty} d\left(fg\left(\frac{1}{2} - \frac{1}{n}\right), gf\left(\frac{1}{2} - \frac{1}{n}\right)\right) \neq 0$, showing that $\lim\limits_{n\to\infty} M(fga_n, gfa_n, t) \neq 1$, therefore the pair (f, g) is non-compatible.

Definition 2.13 (A class of implicit relation [12]). *Let φ be the set of all real continuous functions. $F : (R^+)^5 \to R$ non-decreasing in the first argument satisfying the following conditions:*

a) For $u, v \geq 0$, $u(u, v, u, v, 1) \geq 0$ then $u \geq v$.

b) $F\{u, 1, 1, u, 1\} \geq 0$ or $F\{u, u, 1, 1, u\} \geq 0$ or $F\{u, 1, u, 1, u\} \geq 0$ Implies that $u \geq 1$.

For example, define $F(t_1, t_2, t_3, t_4, t_5) \geq 20t_1 - 18t_2 + 10t_3 - 12t_4 - t_5 + 1$ then $F \in \Phi$.

3. Main Results

Theorem 3.1. *Let A, B, S and T be self maps of a G - metric space $(\mathbb{X}, G, *)$ satisfying the following conditions:*

i) $A(X) \subseteq S(X)$ and $B(X) \subseteq T(X)$ one of $A(X), B(X), S(X)$ and $T(X)$ is closed subset of $\mathbb{X}$.

ii) $G(Aa, Bb, kt) \geq \phi\{G(Ta, Sb, t), G(Aa, Ta, t), G(Sb, Bb, t), G(Ta, Bb, t), G(Aa, Sb, t)\}$ for every $a, b \in \mathbb{X}$ and $\phi \in \Phi$.

iii) The pairs (A, T) and (B, S) are weakly compatible.

iv) The pairs (A, T) or (B, S) satisfies the property (E.A).

Then A, B, S and T have a unique common fixed point in $\mathbb{X}$.

Proof. We first show beyond doubt the existence of a common fixed point in one of the two cases of the condition (v) and the other case follows similarly with some suitable changes. In case (B, S) satisfies the property (E.A). Then there exists a sequence $\{a_n\}$ in $\mathbb{X}$ such that $\lim_{n\to\infty} Ba_n = \lim_{n\to\infty} Sa_n = w$ for some $w \in \mathbb{X}$. Since $B(X) \subseteq T(X)$, there exists a $\{b_n\}$ in $\mathbb{X}$ such that,

$$\lim_{n\to\infty} Bb_n = \lim_{n\to\infty} Tb_n = w$$

We now prove that $\lim_{n\to\infty} Ab_n = w$. Let $\lim_{n\to\infty} Ab_n = l$ and if $l \neq w$ then $G(l, w, w) \neq 0$. From (iii), we have,

$$G(Ab_n, Bb_n, kt) \geq \phi(G(Tb_n, Sa_n, t), G(Ab_n, Tb_n, t), G(Sa_n, Ba_n, t), G(Tb_n, Ba_n, t), G(Ab_n, Sa_n, t))$$

on letting $n \to \infty$ we get,

$$\begin{aligned} G(l, w, kt) &\geq \phi(G(w, wt), G(l, w, t), G(w, w, t), G(w, w, t), G(l, w, t)) \\ &= \phi(0, G(l, w, t), 0, 0, G(l, w, t)) \\ &\geq \phi(G(l, w, t), G(l, w, t), G(l, w, t), G(l, w, t), G(l, w, t)) \\ &> G(l, w, t) \end{aligned}$$

This is contradiction. Since $l = w$. Hence we have $\lim_{n\to\infty} Ab_n = w$. Suppose $T(X)$ is closed subset of $\mathbb{X}$ then there exists u in $\mathbb{X}$ such that $tu = w$. Therefore we have $\lim_{n\to\infty} Ab_n = \lim_{n\to\infty} Ba_n = \lim_{n\to\infty} Sa_n = \lim_{n\to\infty} Tb_n = w = Tu$. Now from (iii), we have,

$$G(Au, Ba_n, kt) \geq \phi(G(Tu, Sa_n, t), G(Au, Tu, t), G(Sa_n, Ba_n, t), G(Tu, Ba_n, t), G(Au, Sa_n, t))$$

On letting $n \to \infty$ we get,

$$\begin{aligned} G(Au, w, w) &\geq \phi(G(w, w, t), G(Au, w, w), G(w, w, w), G(w, w, w), G(Au, w, w)) \\ &= \phi(0, G(Au, w, w), 0, 0, G(Au, w, w)) \\ &\geq \phi(G(Au, w, w), G(Au, w, w), G(Au, w, w), G(Au, w, w), G(Au, w, w)) \\ &> G(Au, w, w). \end{aligned}$$

This is a contradiction. Hence $Au = w$. Therefore $Au = Tu = w$. Since $A(X) \subseteq S(X)$, then there exists a point v in $\mathbb{X}$ such that $Au = Su = w$.

Now we claim that $Bv = w$. If $Bv \neq w$, then $G(w, Bv, t) > 0$. From (iii), we have

$$\begin{aligned} G(Au, Bv, kt) &\geq \phi(G(Tu, Sv, t), G(Au, Tu, t), G(Sv, Bv, t), G(Tu, Bv, t), G(Au, Sv, t)) \\ G(w, Bv, kt) &\geq \phi(G(w, w, t), G(w, w, t), G(w, Bv, t), G(w, Bv, t), G(w, w, t)) \\ &= \phi(0, 0, G(w, Bv, Bv), G(w, Bv, Bv), 0) \\ &> \phi(G(w, Bv, t), G(w, Bv, t), G(w, Bv, t), G(w, Bv, t), G(w, Bv, t)) \\ &> G(w, Bv, t) \end{aligned}$$

This is a contradiction. Hence $Bv = w$. Thus $Sv = Bv = w$. Since the pair (A, T) is weakly compatible then $\int Tu = T \int u$ implies $Aw = w$. We now show that $Aw = w$. If $Aw \neq w$ then $G(Aw, w, t) > 0$. From (iii), we have,

$$\begin{aligned} G(Aw, Bv, kt) &\geq \phi(G(Tw, Sv, t), G(Aw, Tw, t), G(Sv, Bv, t), G(Tw, Bv, t), G(Aw, Sv, t)) \\ G(Aw, w, kt) &\geq \phi(G(Aw, w, t), G(Aw, Aw, t), G(w, w, t), G(Aw, w, t), G(Aw, w, t)) \\ &= \phi(G(Aw, w, t), G(Aw, Aw, t), G(w, w, t), G(Aw, w, t), G(Aw, w, t)) \\ &\geq \phi(G(Aw, w, t), G(Aw, w, t), G(w, w, t), G(Aw, w, t), G(Aw, w, t)) \\ &> G(Aw, w, t). \end{aligned}$$

This is a contradiction. Hence $Aw = w$. Thus $Aw = Tw = w$. Proving that w is common fixed point of A & T. Since the pair (B, S) is weakly compatible then $BSv = SBv \Rightarrow Bw = Sw$. We now prove that $Bw = w$. If $Bw \neq w$, then $G(w, Bw, t) > 0$. From (iii), we have,

$$\begin{aligned} G(Aw, Bw, kt) &\geq \phi(G(Tw, Sw, t), G(Aw, Tw, t), G(Sw, Bw, t), G(Tw, Bv, t), G(Aw, Sw, t)) \\ G(w, Bw, kt) &\geq \phi(G(w, Bw, t), G(w, w, t), G(Bw, Bw, t), G(w, Bw, t), G(w, Bw, t)) \\ &= \phi(G(w, Bw, t), 0, 0, G(Bw, Bw, t), G(w, Bw, t)) \\ &\geq \phi(G(w, Bw, t), G(w, Bw, t), G(w, Bw, t), G(w, Bw, t), G(w, Bw, t)) \\ &> G(w, Bw, t) \end{aligned}$$

This is a contradiction, hence $Bw = w$. Thus $Bw = Sw = w$. Proving that w is common fixed point of B and S. Showing that w is a common fixed point of A, B, S and T. The proof is similar in the other cases of the condition (ii) with appropriate changes.

Uniqueness: Let w be the another common fixed point of A, B, S and T. If $w \neq x$ then $G(w, x, t) > 0$. From (iii), we have $G(w, x, t)$,

$$\begin{aligned} G(Aw, Bx, kt) &\geq \phi(G(Tw, Sx, t), G(Aw, Tw, t), G(Sx, Bx, t), G(Tw, Bx, t), G(Aw, Sx, t)) \\ &\geq (G(w, x, t), G(w, w, t), G(x, x, t), G(w, x, t), G(w, x, t)) \end{aligned}$$

$$
\begin{aligned}
&= \phi(G(w,x,t), 0, 0, G(w,x,t), G(w,x,t)) \\
&\geq \phi(G(w,Bw,t), G(w,Bw,t), G(w,Bw,t), G(w,Bw,t), G(w,Bw,t)) \\
&> G(w,Bw,kt)
\end{aligned}
$$

This is a contradiction, hence $x = w$. Therefore w is the unique common fixed point of A, B, S and T. □

Corollary 3.2. *Let A, B and S be self maps of a G-metric space* $(\mathbb{X}, G)$ *satisfying the following points*

(i) $A(X) \subseteq S(X)$ *and* $B(X) \subseteq S(X)$;

(ii) one of $A(X), B(X)$ *and* $S(X)$ *is closed subset of* $\mathbb{X}$;

(iii) $G(Aa, Bb, kt) \geq \phi(G(Sa, Sb, t), G(Aa, Sa, t), G(Sb, Bb, t), G(Sa, Bb, t), G(Sa, Bb, t), G(Aa, Sb, t))$ *for every* $a, b \in \mathbb{X}$ *and* $\phi \in \Phi$;

(iv) The pairs (A, S) *and (B,S) are weakly compatible;*

(v) The pairs (A, S) *or* (B, S) *satisfies the property (E.A).*

Then A, B *and S have a peculiar common fixed point in* $\mathbb{X}$.

Proof. On taking $T = S$ in the Theorem 3.1, the corollary follows. □

4. Conclusion

From the above theorems and proof we conclude that the new way of approach for common fixed point theorem using G - fuzzy metric space have some unique manner for self mappings.

References

[1] M. Aamri and D. El Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, Journal of Mathematics Analysis Applications, 270(2002), 181-188.

[2] D. Damodharan and Doaa Rizk, *A Novel Approach to Fixed Point Theorems in 2 - Cone Normed Spaces over Banach Algebras*, Lobachevskii Journal of Mathematics, 42(12)(2021), 2754-2763.

[3] V. Kiran, *Common fixed point theorem for four weakly compatible self maps with (E.A) property of a G - metric space*, Malaya Journal of Matematik, 7(3)(2019), 436-439.

[4] A. Mohamed Ali and G. RanjithKanna, *Intuitionistic Fuzzy Cone Metric Spaces and Fixed Point Theorems*, International Journal of Mathematics And its Applications, 5(1-A)(2017), 25-36.

[5] G. Junck, *Compatible mappings and Common fixed points*, Int. J. Math and Math. Sci., 9(4)(1986), 771-779.

[6] S. Yahya Mohamed and A. Mohamed Ali, *Fixed point theorems in intuitionistic fuzzy graph metric space*, Annals of Pure and Applied Mathematics, 118(6)(2017), 67-74.

[7] A. Mohamed Ali, S. Senthil and T. Chendralekha, *Intuitionistic Fuzzy Sequences in Metric Space*, International Journal of Mathematics and its Applications, 4(1-B)(2016), 155-159.

[8] H. Shojaei, K. Banaei and N. Shoaei, *Fixed Point Theorems for Weakly Compatible Maps Under E.A Prooerty in Fuzzy 2 - Metric Spaces*, Journal of Mathematics and Computer Science, 6(2013), 118-128.

[9] R. Krishnakumar and D. Dhamodharan, *Common fixed point of contractive modulus on 2 cone Banach spaces*, Malaya Journal of Mathematics, 5(3)(2017), 608-618.

[10] A. Mohamed Ali and I. Athal, *Extended Intuitionistic fuzzy metric and fixed point theorems*, Int. J. of Pure and Engg. Mathematics, 3(II)(2015), 111-123.

[11] R. Krishnakumar, K. Dinesh and D. Dhamodharan, *Some Fixed point (α, β)-weak contraction on Fuzzy Metric space*, International Journal Sci. Res. in Mathematical and Statistical Science, 5(3)(2018), 1-6.

[12] Sanjay Kumar, *Fixed Point Theorems For Weakly Compatible Maps Under E.A Property in Fuzzy Metric Spaces*, J. Appl. Math. & Informatics, 29(1-2)(2011), 395-405.

[13] G. Junck, *Common Fixed Points for Non-Continuous non self maps on non metric spaces*, Far East J. Math. Sci., 4(2)(1996), 199-215.

[14] R. Krishnakumar, T. Mani and D. Dhamodharan, *Fixed point (α, β, γ)- contraction Mapping under simulation functions in Banach Spaces*, Malaya Journal of Matematik, 9(1)(2021), 745-750.

[15] Zead Mustafa and B. Sims, *Some Remarks Concerning D-Metric Spaces*, Proc. of the Int. Conference on Fixed Point Theory and Appl., (2003), 189-198.

[16] Zead Mustafa and B. Sims, *A new Approach to Generalized Metric Spaces*, Journal of Non linear and Convex Analysis, 7(2006), 289-297.

[17] G. Junck and B. E. Rhoads, *Fixed Point for Set Valued Functions without Continuity*, Indian Journal Pure Appl. Math. Sci., 29(1998), 227-238.

[18] P. Senthil Kumar and P. Thiruveni, *Fuzzy Fixed Point Theorems in Normal Cone Metric Space*, Asian Research journal of Mathematics, 19(10)(2023), 57-66.

[19] P. Senthil Kumar and P. Thiruveni, *Estimation of fuzzy metric spaces from metric spaces*, Ratio Mathematica, 41(2021), 64-70.