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Hypercubes - An Overview

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Abstract

Mathematicians and computer scientists both find the idea of hypercubes, commonly referred to as n-dimensional cubes or tesseracts, to be incredibly interesting. Hypercubes are a higher dimensional analog of the familiar three-dimensional cube. They provide novel insights into the nature of spatial dimensions and their applications. This article attempts to delve into the world of hypercubes, looking at its characteristics, mathematical representations and practical uses.

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1. Introduction

In n-dimensional space, a hypercube is a geometrical object. The graph of hypercube is denoted by Q_n, where n stands for the number of dimensions. Q_0 is a single vertex, Q_1 is a single edge, Q_2 is a square and Q_3 is the well-known three-dimensional cube. Q_0 is the simplest hypercube. Graph theorists and computer scientists both find the idea of hypercubes, commonly referred to as n-dimensional cubes or tesseracts, to be incredibly interesting. Hypercubes are analog of the familiar three-dimensional cube in higher dimensions, and they provide novel insights into the nature of spatial dimensions and their practical applications.

A multiprocessor computing system involves many processors, and input is exchanged between them. Through a group of interconnected processors, information is transmitted between them. Evidently, if there is a corresponding rise in the number of 'alternative parallel pathways' available for processing, the transmission would be faster [4]. Accordingly, it is asserted that there should be a rise in the number of parallel paths connecting any two vertices in the graph of an interconnection network and that the distance between any two given vertices in the graph should be modest. Various favorable characteristics of hypercube make it suitable for usage in different kinds of applications.

Graph theorists view hypercubes as a graph that can be defined recursively using the box product or

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Cartesian product. The definition given by Harary [9] is as follows:

$$Q_1 = K_2$$
$$Q_2 = K_2 \Box Q_{n-1}$$

where $\Box$ represents the box product or Cartesian product. It can also be defined using boolean vectors i.e. n-tuples of 0 and 1. It is the graph with the 2^n n-tuples of 0 and 1 as the vertices and two vertices are adjacent whenever they differ in exactly one coordinate (Figure 1). Due to its significance as a

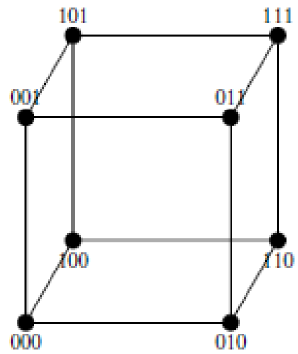

Figure 1: Q_3

stepping stone to higher dimension, hypercubes have attracted interest of many mathematicians. Also as it can be viewed as boolean object many computer scientists have shown great interest in it.The introduction of massively parallel computers with a hypercube-like topology has spurred interest in hypercubes [11, 12]. This not only proposes additional cube-related issues that merit investigation but also suggests possible applications for the current theory. As a consequence, we find a vast literature in this area. In this paper, we try to give an overview of the graph theoretical literature on hypercubes. The detailed survey paper authored by Harary [10] gives a nice review on hypercubes. We include the results from [10] and also some recent developments in this area. For all definitions and notations not mentioned here we refer to Harary [9].

2. Definition and Preliminaries

A graph $G = (V, E)$ with $p = |V|$ vertices and $q = |E|$ edges, and is said to have order p and size q. Thus, the order of Q_n, is 2^n and its size is $n2^{n-1}$.
We have the following theorems regarding the basic graph invariants of hypercube.

Theorem 2.1. *The graph Q_n is connected as well as bipartite.*

Theorem 2.2. *The diameter of Q_n is n.*

Theorem 2.3. *The total distance of Q_n is given by $td(Q_n) = n2^{2n-2}$*

Theorem 2.4. *The average distance of Q_n is given by $\bar{d}(Q_n) = \frac{n2^{n-1}}{2^{n-1}}$*

Theorem 2.5. *In Q_n, any two adjacent edges belong to exactly one cycle of length four.*

Theorem 2.6. *The vertex connectivity $\kappa(Q_n)$ and edge connectivity $\lambda(Q_n)$ are both equal to n*

Theorem 2.7. *The vertex independence number and edge independence number of Q_n are both equal to 2^{n-1}.*

There are various characterizations for hypercubes available in literature. Here we list some of them.

Theorem 2.8 ([7]). *Every hypercube is a bipartite graph such that the number of geodesics between any two vertices x, y is the factorial of the distance between them, i.e., $d(x, y)!$*

Theorem 2.9 ([8]). *A hypercube is a bipartite graph such that the number of vertex-disjoint paths between any two vertices x, y of the graph is $d(x, y)$.*

Theorem 2.10 ([14]). *Let $\mathbf{C_4}$ be the class of connected graphs such that each pair of adjacent edges lies in exactly one 4-cycle. Then we can characterize the n-cube as follows: a graph G in C_4 is an n-cube if and only if its minimum degree δ satisfies $p = 2^\delta$.*

Theorem 2.11 ([5]). *An induced subgraph H of G is convex if for any two vertices of H , every geodesic joining them is in H. A convex subgraph is proper if it is not K_1, K_2 or G. The set of all pairwise non-isomorphic proper convex subgraphs of a graph G is denoted by $PC(G)$. For any $n \geq 3$, $PC(Q_n) = \{Q_2, \ldots Q_n\}$. Conversely, If G is a connected graph such that $PC(G) = \{Q_2, \ldots Q_n\}$ with $n \geq 3$ and $p = 2^n$, then G is isomorphic to Q_n.*

In [15], the concept of magic cubes was introduced by Lin and the relationship between magic cubes and hypercubes were studied. "A magic cube of order n consists of consecutive integers from 1 to n^3, arranged in the form of a cube, so that the sum of numbers in every row, every column, every file, and in each of the four main diagonals is the same. A d-dimensional magic hypercube is called symmetrical if any pair of numbers symmetrically opposite the center add up to the same value." Lin also proved the following in [15].

Theorem 2.12. *A magic hypercube of order 3 is always symmetrical.*

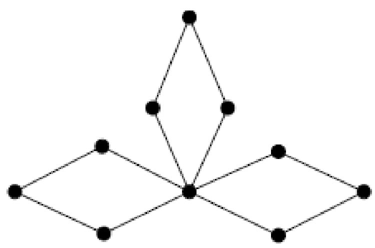

Figure 2: $F_{3,4}$

Decomposition problem of hypercubes were surveyed in detail by A. Roy in [16]. A discussion on how the n dimension hypercube could be decomposed into n-fans (Figure 2) and double n-fans was done in [16] . They also mentioned a systematic construction of how the root vertices as well as the n cycles attached to it generated. They also established that " A hypercube Q_{2n} where n is a positive integer can be decomposed into 4^{n-1} copies of n-fan $F_{n,4}$" (Figure 3).

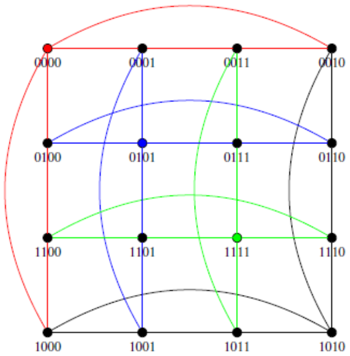

Figure 3: Q_4 decomposed into 4 copies of $F_{2,4}$

In [17], Saad et al. discussed the topological properties of hypercubes and proved the following.

Theorem 2.13. *There are n different ways of tearing an n-cube, i.e, of splitting it into two $(n-1)$-subcubes so that their respective vertices are connected in a one-to-one way.*

Theorem 2.14. *Any two adjacent vertices x and y of an n-cube are such that the nodes adjacent to x and those adjacent to y are connected in a one-to-one fashion.*

The issue of incorporating meshes and rings into the hypercube was discussed in [17]. The following is what is meant when discussing mapping different geometries. Given a graph $G = (V, E)$ with no more than 2^n vertices, we would like to assign the graph's vertices to the vertices of the n-cube such that each vertex adjacencies are preserved. These mappings are significant for primarily two reasons.

1) An algorithm might be created for a different architecture for which it is the ideal fit. Then, with minimal extra programming work, one might want to construct the same algorithm. This will be simple to accomplish if the original architecture can be translated into the hypercube.

2) A certain problem could have a well defined structure that results in a specific communication pattern. The structure may be mapped to cut down on communication time significantly.

The mesh geometries that result from the discretization of elliptic partial differential equations in one, two, or three dimensions are the best illustration. The majority of iterative techniques for solving elliptic PDEs simply call for local communication, or communication between nearby mesh points. Only local communication will be needed between the nodes of the hypercube if the mesh is perfectly mapped onto the cube, resulting in significant transfer time savings. Saad et al. [17] established that rings, linear arrays and grids can be mapped into Hypercubes. The hypercube is the ideal network because of these extremely valuable mapping qualities, which offset some of its inherent disadvantages like the high cost and difficulty of creating massive hypercubes.

Many other authors have also shown interest in this fascinating class of graphs. Sunilchandran et al. [3] computed the pathwidth and tree width of hypercubes. Klavzar, in [13] established that subcubes of a hypercube can be counted in three different ways. This was a break through result as it paved way to yield a new graph theory interpretation of a known combinatorial identity. This result strengthened the bond between algebra and graph theory.

3. Applications

Hypercubes may appear abstract, yet they have real-world uses in plenty of different fields. In parallel computing systems, interconnection networks are designed and analysed using hypercubes. A hypercube's n-dimensional structure allows for effective communication across processors, which makes it useful in distributed computing and supercomputer scenarios. Hypercubes are used in multidimensional data modelling in databases and data warehouses. Hypercubes are the foundation of OLAP (Online Analytical Processing) cubes, which enable effective data aggregation and analysis across multiple dimensions. Complex systems comprising several dimensions are a common topic in theoretical physics and engineering research. In mathematics, hypercubes can be used to model and comprehend these systems. Hypercube-based networks are used to construct fault-tolerant and scalable communication infrastructures in telecommunications and computer networking.

4. Conclusion

In conclusion, hypercubes are useful instruments for resolving practical issues in a variety of domains, not merely as abstract mathematical constructions. They offer a deeper comprehension of higher dimensions and their uses in engineering, physics, data analysis, and computer science. Our capacity to use hypercubes to solve complicated problems will only grow more crucial as technology develops, demonstrating the remarkable mathematical idea's enduring usefulness.

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