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Iterative methods for solving general Trifunction Variational Inequalities

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Abstract

In this paper, we describe and discuss the new concept of trifunctional variational inequality. We propose and examine implicit iterative approaches for solving trifunction variational inequalities using the auxiliary principle methodology and nonlinear operator. The convergence conditions of this novel approach are investigated under the pseudomonotonicity requirement, which is a weaker criterion than monotonicity. Some special cases are also taken into account as well.

Keywords: variational inequalities; auxiliary principle; convergence; iterative methods.
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1. Introduction

Variational inequalities are being utilised as a mathematical tool to study a wide range of issues in the mathematics and engineering sciences. The variational inequality is widely known to specify the optimality requirement for the differentiable convex function on the convex set. Nevertheless, it has been demonstrated that the optimality condition of a directionally differentiable convex function may be described by a type of variational inequality known as the bifunction variational inequality see [1–4, 6, 12, 14, 15, 17–22]. Inspired and motivated by ongoing research in this field, we define and investigate a novel class of variational inequalities known as trifunction variational inequalities.
We find bifunction variational inequalities and various related optimization problems as special instances. There are several numerical approaches for solving variational inequalities, including the projection technique and its variant forms, Wiener-Hopf equations, the auxiliary principle, and resolvent equations. Nevertheless, due to the nature of the issue, projection, Wiener-Hopf equations, proximal and resolvent equations approaches cannot be extended and adapted to suggest and evaluate comparable iterative methods for solving trifunction variational inequalities. This feature has prompted the application of the auxiliary principle technique, owing primarily to Glowinski, Lions,

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and Tremolieres [7]. Noor [9–11] and Noor et al. [23] have extensively used this method to propose and investigate an implicit iterative method for solving variational inequalities and related problems. We propose and analyse an implicit iterative method for solving the trifunction variational inequalities using this method in conjunction with the Bregman function. We also investigate the convergence of this novel approach under the assumption of pseudomonotonicity. It is common knowledge that pseudomonotonicity implies monotonicity, while the opposite is not true. This demonstrates that pseudomonotonicity is a less strong condition than monotonicity. In this way, our result is a refinement of previously known results. These findings represent a fresh and significant use of the auxiliary principle approach. We expect that this paper's ideas and techniques may open further research opportunities and applications of the trifunction variational inequalities. For the recent developments in this direction, see [12–22] and the references therein.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle ., . \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed set in H and T be a nonlinear operator. For a given continuous trifunction function $F : K \times K \times K \to H$ and the non linear operator $g : K \to K$, we consider the problem of finding $u \in K$ such that

$$F(g(u), Tu, g(v) - g(u)) \geq 0, \quad \forall \;\; v \in K \tag{1}$$

which is called an trifunction variational inequality. A number of problems arising in various branches of pure and applied sciences can be studied via the trifunction variational inequalities. If $F(g(u), Tu, g(v) - g(u)) = B(g(u), g(v) - g(u))$, where $B(., .) : K \times K \to H$ is a bifunction, then problem (1) is equivalent to finding $u \in K$ such that

$$(g(u), g(v) - g(u)) \geq 0, \quad \forall \;\; v \in K \tag{2}$$

It is known as the bifunction variational inequality. The bifunction variational inequalities may be used to study a variety of problems in pure and applied sciences. It has been demonstrated [12] that the bifunction variational inequality can characterise the minimum of a directionally differentiable convex function on a convex set (2). Similarly, one may demonstrate that the minimum of a Lipschitz continuous nonconvex meets the bifunction variational inequality (2). See [1–6, 12, 17–22] and the references therein for the formulation, well-posedness, and existence findings for bifunction variational inequalities.

If $F(g(u), Tu, g(v) - g(u)) = \langle Tu, g(v) - g(u) \rangle$, then the trifunction variational inequality (1) is equivalent to finding $u \in K$ such that

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad v \in K$$

and if we take $g = I$ in the above inequality reduces to finding $u \in K$ such that

$$\langle Tu, v-u \rangle \geq 0, \quad v \in K \tag{3}$$

Which is known as the classical variational inequality introduced by Stampacchia [24].

Definition 2.1. *The bifuction $T(.,.) : H \times H \to R$ is said to be monotone type(I) with respect to the operator g, iff*

$$T(u, g(v) - g(u)) + T(v, g(u) - g(v)) \geq 0, \ \forall \ u, v \in H$$

Definition 2.2. *The bifuction $F : H \times H \to R$ is said to be monotone type(II) with respect to the operator g, iff*

$$F(g(u), g(v)) + F(g(v), g(u)) \leq 0, \ \forall \ u, v \in H$$

Definition 2.3. *The trifunction $F(.,.,.)$ and the operator T is said to be jointly pseudomonotone, iff,*

$$F(g(u), Tu, g(v) - g(u)) \geq 0$$
$$\Rightarrow -F(g(v), Tv, g(u) - g(v)) \geq 0, \ \forall \ u, v \in K$$

3. Main Result

In this part, we study at an iterative method for solving the trifunction variational inequality (1) by the auxiliary principle method, which was established by Noor [9–11] and Noor et al. [23]. For a given $u \in K$ satisfying (1), we consider the problem of finding $w \in K$ such that

$$\rho F(g(w), Tw, g(v) - g(w)) + \langle E'(w) - E'(u), g(v) - g(w) \rangle \geq 0 \ \forall \ v \in K \tag{4}$$

It is called the auxiliary trifunction variational inequality. Here $\rho > 0$ is a constant and E′(u) is the differential of a strongly convex function E at $u \in K$. From the strongly convexity of the differentiable function $E(u)$, it follows that problem (4) has a unique solution. It is clear that if $w = u$, then w is a solution of problem (1). This observation enables to suggest and analyze the following iterative method for solving the problems (1).

Algorithm 1. *For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(g(u_{n+1}), Tu_{n+1}, g(v) - g(u_{n+1})) + \langle E'(u_{n+1}) - E'(u_n), g(v) - g(u_{n+1}) \rangle \geq 0 \ \forall \ v \in K \tag{5}$$

where $\rho > 0$ is a constant.

Algorithm 1 is known as the implicit method for solving the trifunction variational inequality (1). If $F(g(u), Tu, g(v) - g(u)) = B(g(u), g(v) - g(u))$, then Algorithm 1 reduces to:

Algorithm 2. *For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme*

$$\rho B(g(u_{n+1}), g(v) - g(u_{n+1})) + \langle E'(u_{n+1}) - E'(u_n), g(v) - g(u_{n+1})\rangle \geq 0 \ \ \forall \ \ v \in K$$

for solving the bifunction variational inequalities (2), see [14,15,17-22] and the references therein.

Note that, if $F(g(u), Tu, g(v) - g(u)) = \langle Tu, g(v) - g(u)\rangle$, then Algorithm 1 reduces to the following iterative scheme for variational inequalities (3) and appears to be a new one.

Algorithm 3. *For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho Tu_{n+1} + E'(u_{n+1}) - E'(u_n), g(v) - g(u_{n+1})\rangle \geq 0 \ \ \forall \ \ v \in K$$

where $\rho > 0$ is a constant.

A number of iterative approaches for addressing bifunction variational inequalities and associated optimization issues may be obtained by selecting the bifunction and the spaces properly. We now study the convergence criteria of Algorithm 1 and this is the main motivation of next result.

Theorem 3.1. *Let the function $F(.,.,.)$ be jointly pseudomonotone with respect to the operator T and let $E(u)$ be strongly convex function with modulus $\beta > 0$. Then the approximate solution $g(u_{n+1})$ obtained from Algorithm 1 converges to a solution $u \in K$ of the trifunction variational inequality (1).*

Proof. Let $u \in K$ be a solution of (1). Then, using the jointly pseudomonotonicity of $F(.,.,.)$, we have

$$-F(g(v), Tv, g(u) - g(v)) \geq 0, \ \ \forall \ \ v \in K \tag{6}$$

Taking $u = u_{n+1}$ in (6) and $v = u$ in (5), we have

$$-F(g(u_{n+1}), Tu_{n+1}, g(u) - g(u_{n+1})) \geq 0, \ \ \forall \ \ v \in K \tag{7}$$

$$\rho F(g(u_{n+1}), Tu_{n+1}, g(u) - g(u_{n+1})) + \langle E'(u_{n+1}) - E'(u_n), g(u) - g(u_{n+1})\rangle \geq 0, \ \ \forall \ \ v \in K \tag{8}$$

Now we consider the generalized Bregman function as

$$B(g(u), g(z)) = E(u) - E(z) - \langle E'(z), u - z\rangle \geq \beta \|g(u) - g(z)\|^2 \tag{9}$$

Where we have used the fact that the function E(u) is strongly convex. Combining (7) - (8), we have

$$\begin{aligned} B(g(u), g(u_n)) - B(g(u), g(u_{n+1})) &= E(u_{n+1}) - E(u_n) - \langle E'(u_n), g(u) - g(u_n)\rangle \\ &\quad + \langle E'(u_{n+1}), g(u) - g(u_{n+1})\rangle \\ &= E(u_{n+1}) - E(u_n) - \langle E'(u_n) - E'(u_{n+1}), g(u) - g(u_{n+1})\rangle \end{aligned}$$

$$
\begin{aligned}
&- \langle E'(u_n), g(u_{n+1}) - g(u_n)\rangle \\
&\geq \beta\|g(u_{n+1}) - g(u_n)\|^2 + \langle E'(u_{n+1}) - E'(u_n), g(u) - g(u_{n+1})\rangle \\
&\geq \beta\|g(u_{n+1}) - g(u_n)\|^2 - \rho F(g(u_{n+1}), Tu_{n+1}, g(u) - g(u_{n+1})) \\
&\geq \beta\|g(u_{n+1}) - g(u_n)\|^2
\end{aligned}
$$

If $g(u_{n+1}) = g(u_n)$, then clearly u_n is a solution of (1). Otherwise, for $\beta > 0$, the sequences $B(g(u), g(u_n)) - B(g(u), g(u_{n+1}))$ is nonnegative and we must have

$$
\lim_{x\to\infty} (| \, g(u_{n+1}) - g(u_n)\|) = 0
$$

As a result, the sequence $g(u_n)$ is bounded. Let $\bar{u}$ be a cluster point of the subsequence un_i, let un_i be a subsequence converging towards $\bar{u}$. Using Zhu and Marcotte's [25] approach, it is now possible to demonstrate that the whole sequence u_n converges to the cluster point $\bar{u}$, satisfying the trifunction variational inequality (1). □

4. Conclusion

We introduced and investigated a new class of variational inequalities known as the trifunction variational inequality in this paper. We proposed and analysed an implicit iterative method to solve the trifunction variational inequalities using the auxiliary principle technique. This approach is quite adaptable. We also investigated the convergence analysis of the suggested implicit iterative approach under appropriate conditions. We hope that the problem addressed in this paper may inspire further research and applications in this field.

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