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# Common Fixed Point Theorem in Ordered Generalized Cone Metric Spaces over Banach Algebra

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**Abstract**

The purpose of this paper is to generalize and prove some fixed point and common fixed point results in ordered generalized cone metric spaces over Banach algebra and convergence properties of sequences. Also the generalized contractive mapping are introduced here. Our results improve and generalize results of Turkoglu [16].

**Keywords:** Banach algebras; ordered generalized cone metric spaces; generalized Lipschitz conditions.
**2020 Mathematics Subject Classification:** 47H10, 54H25.

## 1. Introduction

A number of advancements in fixed point theory have occurred recently. Many mathematicians studied fixed point theorems over different spaces as cone metric space. Huang and Zhang [4] used an ordered Banach space in place of the real numbers, and defined a cone metric spaces for contractive mapping in addition to providing information about some properties of convergence of sequences. Hao and Shaoyuan [13] created the idea of cone metric spaces over Banach algebras by substituting a Banach algebra for a Banach space. Beg et al. [14] established the idea of generalized cone metric spaces by substituting an ordered Banach space for the set of real numbers, demonstrating the convergence properties of the sequence, and establishing a few fixed point theorems in this space. Generalized cone metric space is more general than metric space and cone metric space. The idea of unique common fixed point for mapping satisfying certain contraction has been center of research activity. Jungck ([21, 22]), proved a common fixed point theorem for commuting maps, generalized the Banach Contraction principal and also defined a pair of self mapping to be weakly compatible if they commute at their coincidence points. Many fixed point theorems have been proved in normal or non normal ordered cone metric space by some authors ([1, 10, 18, 26]). The main purpose of this paper is to prove

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some common fixed point theorems of contraction mapping in setting of ordered generalized cone metric spaces with Banach algebra. Our results is generalization of the result [16].

## 2. Preliminaries

Throughout this paper, we assume that P is a cone in A with int $P \neq \phi(\theta)$, the additive identity element of A and $\preceq$ is the partial ordering with respect to P, where A is a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined, satisfying the following properties [4] (for all $x, y, z \in A, \alpha \in \mathbb{R}$):

1. $(xy)z = x(yz)$;
2. $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. there exists $e \in E$ such that $xe = ex = x$;
5. $||e|| = 1$;
6. $||xy|| \leq ||x||.||y||$;

An element $x \in A$ is called invertible if there exists $x^{-1} \in A$ such that $xx^{-1} = x^{-1}x = e$.

**Proposition 2.1** ([4]). *Let $x \in A$ be a Banach algebra with a unit e, then the spectral radius $\rho(u)$ of $u \in A$ holds*

$$\rho(u) = \lim_{n\to\infty} ||u_n||^{\frac{1}{n}} = inf||u^n||^{\frac{1}{n}} < 1$$

*Further, $(e-u)$ is invertible and $(e-u)^{-1} = \sum_{i=0}^{\infty} u^i$.*

Consider a Banach algebra $A, \theta$ be the null vector, $e$ be the identity element of A and a subset P of A is called a cone if it satisfies the following

1. $\{\theta, e\} \subset P$ and P is closed;
2. $P^2 = PP \subset P$;
3. $\alpha P + \beta P \subset P$, for all non-negative real numbers $\alpha$ and $\beta$;
4. $P \cap (-CP) = \theta$;

With respect to cone P, a partial ordering $\preceq$ is defined as $u \preceq w$ if and only if $w - u \in P$ and $u \prec w$ if $u \preceq w$ and $u \neq w$ whereas $u \ll w$ means $w - u \in int\ P$. If A is a Banach space and $P \subset A$, satisfies the conditions 1, 3 and 4, then P is called a cone of A.

**Remark 2.2** ([4]). *If $\rho(x) < 1$, then $\|x_n\| \to 0$ as $n \to \infty$.*

**Definition 2.3** ([4]). *Consider $X$ is a non-empty set, $A$ be a Banach algebra and $P \subseteq A$ be a cone. Suppose the mapping $d : X \times X \to A$ satisfies the following for all $x, y, z \in X$,*

1. *$d(x,z) = \theta$ if and only if $x = z$, and $\theta \preceq d(x,z)$;*

2. *$d(x,z) = d(z,x)$;*

3. *$d(x,z) \preceq d(x,y) + d(y,z)$ for every $x, y, z \in X$.*

*Here d is called a cone metric and (X, d) is called Cone metric space over a Banach algebra A (in Short CMSBA). Note that $d(x,z) \in P$ for all $x, y \in X$.*

**Definition 2.4** ([6]). *Let $X$ be a non-empty set and $A$ be a Banach algebra and $G : X^3 \to A$ be a function satisfying the following properties:*

1. *$G(x,y,z) = \theta$ if and only if $x = y = z$;*

2. *$\theta \prec G(x,y,z)$ for all $x, y \in X$ with $x \neq y$;*

3. *$G(x,x,y) \preceq G(x,y,z)$ for all $x, y, z \in X$ with $z \neq y$;*

4. *$G(x,y,z) = G(y,z,x) = G(x,z,y) = \dots$ (symmetry);*

5. *$G(x,y,z) \preceq G(x,a,a) + G(a,y,z)$ for all $a, x, y, z \in X$ (rectangle inequality).*

*Then $G$ is called a generalized cone metric over Banach algebra $A$ and the pair $(X, G)$ denotes a G-cone metric space over Banach algebra.*

**Remark 2.5.**

1. *If $A$ is a Banach space in Definition 2.3, then $(X, G)$ becomes a G-cone metric space and if in addition $z = y$, then it becomes a cone metric space as in Huang and Zhang [3].*

2. *If $A = R$ in Definition 2.3, we obtain a G-metric space as in Mustafa and Sims [9] and if in addition, $z = y$ in $G(x,y,z)$, then it becomes a metric space.*

**Definition 2.6** ([4]). *Let $(X, d)$ be a cone metric space over Banach algebra $A$ and $\{x_n\}$ a sequence in X. We say that*

1. *$\{x_n\}$ is a convergent sequence if, for every $c \in B$ with $\theta \ll c$, there is an N such that $d(x_n, x) \ll c$ for all $n \geq N$. Ones write it by $x_n \to x (n \to \infty)$.*

2. *$\{x_n\}$ is a Cauchy sequence if, for every $c \in B$ with $\theta \ll c$, there is an N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.*

3. *$(X, d)$ is a complete cone metric space if every Cauchy sequence in X is convergent.*

**Lemma 2.7** ([4])**.** *Let $A$ be a Banach algebra and $k$, a vector in $A$. If $0 \leq r(k) < 1$, then we have $r((e-k)^{-1}) < (1-r(k))^{-1}$.*

**Lemma 2.8** ([2])**.** *Let $A$ be a Banach algebra and $x$, $y$ be vectors in $A$. If $x$ and $y$ commute, then the following holds:*

1. $r(xy) \leq r(x)r(y)$;
2. $r(x+y) \leq r(x)+r(y)$;
3. $|r(x)-r(y)| \leq r(x-y)$.

**Lemma 2.9** ([2])**.** *If $A$ is real Banach algebra with a solid cone $\mathcal{P}$ and $\{x_n\}$ is a sequence in $A$. Suppose $||x_n|| \to 0 (n \to \infty)$ for any $\theta \ll c$. Then $x_n \ll c$ for all $n > N^1, N^1 \in N$.*

**Lemma 2.10** ([6])**.** *If $E$ is a real Banach space with a solid cone $P$ and if $||x_n|| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $n > N$, we have $x_n \ll c$.*

**Example 2.11** ([2])**.** *Let $A$ be the Banach space of all continuous real-valued functions $C(K)$ on a compact Hausdorff topological space $K$, with multiplication defined pointwise. Then $A$ is a Banach algebra, and the constant function $f(t) = 1$ is the unit of $A$.*
*Let $P = \{f \in A : f(t) \geq 0$ for all $t \in K\}$. Then $\mathcal{P} \subset A$ is a normal cone with a normal constant $M = 1$. Let $X = C(K)$ with the metric $d : X \times X \to A$ defined by $d(f,g) = |f(t)-g(t)|$, where $t \in K$. Then $(X,d)$ is a cone metric space over a Banach algebra $A$.*

**Definition 2.12.** *[6] Let (X,G) be a G-cone metric space over Banach algebra. G is said to be symmetric if:*

$$G(x,y,y) = G(x,x,y) \;\; \forall \;\; x,y,z \in X$$

**Definition 2.13** ([6])**.** *A G-cone metric space over Banach algebra $A$ is said to be G-bounded if for any $x,y,z \in X$, there exists $K \succ \theta$ such that $||G(x,y,z)|| \preceq K$.*

**Definition 2.14** ([6])**.** *Let $(X,G)$ be a G-cone metric space over Banach algebra and $\{x_n\}$ a sequence in $X$, $c \gg \theta$ with $c \in A$. Then*

1. *$\{x_n\}$ converges to $x \in X$ if and only if $G(x_m,x_n,x) \ll c$ for all $n,m > N^1$, $N^1 \in N$.*
2. *$\{x_n\}$ is Cauchy sequence if and only if $G(x_n,x_m,x_p) \ll c$ for all $n,m > p > N^1$, $N^1 \in N$.*
3. *$(X,G)$ is complete G-cone metric space over Banach algebra if every Cauchy sequence converges.*

**Definition 2.15** ([18])**.** *Let $X$ be a nonempty set. Then $(X,G,\sqsubseteq)$ is called an ordered G-cone metric space if:*

1. *$(X,G)$ is a G-cone metric space,*
2. *$(X,\sqsubseteq)$ is a partially ordered set.*

*Let $(X, \sqsubseteq)$ be a partially ordered set. Then $x, y \in X$ are called comparable if $x \sqsubseteq y$ or $y \sqsubseteq x$ holds.*

**Definition 2.16** ([28]). *Let $(X, \sqsubseteq)$ be a partially ordered set. We say that $x, y \in X$ are comparable if $x \sqsubseteq y$ or $y \sqsubseteq x$ holds. Similarly, $f : X \to X$ is said to be comparable if for any comparable pair $x, y \in X, f(x), f(y)$ are comparable.*

**Definition 2.17** ([24]). *Let $(X, \sqsubseteq)$ be a partially ordered set. Two maps $f, g : X \to X$ are said to be weakly comparable if both $f(x)$, $gf(x)$ and $g(x)$, $fg(x)$ are comparable for all $x \in X$.*

**Lemma 2.18** ([17]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space $(X, G)$ is complete. Let $f : X \to X$ be a continuous and nondecreasing mapping with $\sqsubseteq$. Suppose that the following two assertions hold:*

1. *there exists $k \in (0, 1)$ such that $G(Tx, Ty, Tz) \preceq hG(x, y, z)$ for each $x, y, z \in X$ with $y \sqsubseteq x$;*
2. *there exists $x_0 \in X$ such that $x_0 \sqsubseteq f(x_0)$.*

*Then f has a fixed point $x^* \in X$.*

**Lemma 2.19** ([28]). *Let $(X, \sqsubseteq)$ be a partially ordered set. Two maps f and g be weakly compatible of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

**Lemma 2.20** ([7]). *If X is a symmetric G-cone metric space, then $d_G(x, y) = 2G(x, y, y)$.*

## 3. Main Result

**Theorem 3.1.** *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that $(X, G)$ be a complete G-cone metric space over a Banach algebra A and P be a non normal cone. Suppose that the mapping $T : X \to X$ is continuous and comparable and the following two assertions holds:*

1. *there exists $a, b, c \in P$ with $\rho(a) + 2\rho(b) + 2\rho(c) < 1$ such that*

$$G(Tx, Ty, z) \preceq aG(x, y, z) + b[G(x, Tx, z) + G(y, Ty, z)] + c[G(x, Ty, z) + G(y, Tx, z)] \tag{1}$$

   *for any comparable pair $x, y, z \in X$;*

2. *there exists $x_0 \in X$ such that $x_0, f(x_0)$ are comparable.*

*Then f has a fixed point $x^* \in X$.*

*Proof.* We have assuming that T satisfies the inequality (1)

$$G(Tx, Ty, y) \preceq aG(x, y, y) + b[G(x, Tx, y) + G(y, Ty, y)] + c[G(x, Ty, y) + G(y, Tx, y)] \tag{2}$$

and

$$G(Ty, Tx, x) \preceq aG(y, x, x) + b[G(y, Ty, x) + G(x, Tx, x)] + c[G(y, Tx, x) + G(x, Ty, x)] \tag{3}$$

Since X is a symmetric G - cone metric space, by adding (2) and (3) we have

$$d_G(T(x), T(y)) \preceq a d_G(x, y) + b[d_G(x, T(x)) + d_G(y, T(y))] + c[d_G(x, T(y)) + d_G(y, T(x))] \quad (4)$$

If $T(y_0) = y_0$ then the proof is completed. Let $T(y_0) \neq y_0$ from condition second and T is comparable, we deduce that $T^i(y_0)$ and $T^{i+1}$ are comparable for any $i \geq 0$. Replacing $y_n = T^n(y_0)$, we recover $y_i, y_{i+1}$ are comparable by condition (4)

$$d_G(y_{n+1}, y_n) \preceq a d_G(y_n, y_{n-1}) + b[d_G(y_n, y_{n+1}) + d_G(y_{n-1}, y_n)] + c d_G(y_{n+1}, y_{n-1}) \quad (5)$$

$$d_G(y_{n+1}, y_n) \preceq a d_G(y_n, y_{n-1}) + b[d_G(y_n, y_{n+1}) + d_G(y_{n-1}, y_n)] + c[d_G(y_{n-1}, y_n) + d_G(y_n, y_{n+1}] \quad (6)$$

that is

$$(e - b - c) d_G(y_{n+1}, y_n) \preceq (a + b + c) d_G g(y_n, y_{n-1})$$

Since $\rho(a) + 2\rho(b) + 2\rho(c) < 1$, then $\rho(b + c) \leq \rho(b) + \rho(c) < 1$, and $(e - b - c)$ is invertible by Proposition 2.1. Then multiplying both sides with $\frac{1}{(e-b-c)}$, it follows that

$$d_G(y_{n+1}, y_n) \preceq \frac{(a + b + c)}{(e - b - c)} d_G(y_n, y_{n-1})$$

for all $n \geq 1$. Repeating this relation we get

$$d_G(y_{n+1}, y_n) \preceq h^n d_G(y_1, y_0)$$

Where, $h = \frac{(a+b+c)}{(e-b-c)}$.

Now We claim that $\rho(k) < 1$. By Lemma 2.8, we get

$$\rho(a + b + c) + \rho(b + c) \leq \rho(a) + \rho(b) + \rho(c) + \rho(b) + \rho(c) = \rho(a) + 2\rho(b) + 2\rho(c) < 1$$

then $\rho(a + b + c) < 1 - \rho(b + c)$, that is, $\frac{\rho(a+b+c)}{1-\rho(b+c)} < 1$. From Lemma 2.7 and 2.8 that

$$\begin{aligned} \rho(h) &= \rho[(e - b - c)^{-1}(a + b + c)] \\ &\leq \rho[(e - b - c)^{-1}] \rho(a + b + c) \\ &\leq [1 - \rho(b + c)]^{-1} \rho(a + b + c) \\ &= \frac{\rho(a + b + c)}{1 - \rho(b + c)} < 1 \end{aligned}$$

Let $m > n$, then

$$d_G(y_m, y_n) \preceq d_G(y_m, y_{m-1}) + \cdots + d_G(y_{n+1}, y_n)$$

$$\begin{aligned}
&\preceq (h^{m-1} + \dots h^n) d_G(y_1, y_0) \\
&= (e + h + \cdots + h^{m-n-1}) h^n d_G(y_1, y_0) \\
&\preceq (\sum_{i=0}^{\infty} h^i) h^n d_G(y_1, y_0) \\
&\preceq \left[\frac{h^n}{e-h}\right] d_G(y_1, y_0)
\end{aligned}$$

Using Lemmas 2.6 and 2.7, we have

$$\begin{aligned}
\rho\left[\frac{h^n}{e-h}\right] &\leq \rho(h^n).\rho[(e-h)^{-1}] \\
&\leq \frac{(\rho(h))^n}{(e-\rho(h))}
\end{aligned}$$

By Remark 2.2 and Lemma 2.10,

$$\|\frac{h^n}{e-h}\| \to 0 \text{ as } n \to \infty$$

It follows that for any $c \in A$ with $\theta \ll c$, there exist $N \in \mathbb{N}$ such that $m > n > N$, we have that

$$d_G(y_m, y_n) \preceq \left[\frac{h^n}{e-h}\right] d_G(y_1, y_0) \ll c$$

for all $n > n_0$. It implies that $\{y_n\}$ is a Cauchy sequence. By completeness of X, there exists $y^* \in X$ such that $y_n \to y^*$ as $n \to \infty$. Since T is continuous, it follows that $y_{n+1} = Ty_n \to Ty^*$ as $n \to \infty$. By the uniqueness of limit we get $y^* = Ty^*$, that is $y^*$ is a fixed point of T. □

**Theorem 3.2.** *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that $(X, G)$ be a complete G-cone metric space over a Banach algebra A. Suppose that the mapping $T : X \to X$ is comparable and the following assertions holds:*

1. *There exists $h \in \mathcal{P}$ with $\rho(h) \in (0, 1)$ such that*

$$G(Tx, Ty, z) \preceq aG(x, y, z) + b[G(x, Tx, z) + G(y, Ty, z)] + c[G(x, Ty, z) + G(y, Tx, z)] \tag{7}$$

   *for any comparable pair $x, y, z \in X$;*

2. *There exists $y_0 \in X$ such that $y_0, f(y_0)$ are comparable;*

3. *If a sequence $\{y_n\}$ converges to $y$ in X and $y_i, y_{i+1}$ are comparable for all $i \geq 0$ then $y_i$,$y$ are comparable.*

*Then T has a fixed point $y^* \in X$.*

*Proof.* Assume $y_n = T(y_0)$, we get that $y_n, y_{n+1}$ are comparable for all $n \geq 0$ and $\{y_n\}$ converges to $y^*$ as in the proof of Theorem 3.1. Now the condition 3 implies $y_n, y^*$ are comparable. Therefore, the condition 1 gives that

$$G(Ty^*, y^*, y^*) \preceq aG(Ty^*, Ty_n, Ty) + b[G(y_n, y_{n+1}, y^*) + G(y^*, Ty^*, y_n)] + c[G(y_n, Ty^*, y^*) + G(y^*, y_n, y^*)] \quad (8)$$

Hence for each $c \gg \theta$ we have $G(Ty^*, y^*, y^*) \ll c$, so $G(Ty^*, y^*, y^*) = \theta$, which implies that $y^*$ is a fixed point of T. □

Now we give some common fixed point results on ordered generalized cone metric spaces over Banach algebra:

**Theorem 3.3.** *Let $(X, \sqsubseteq)$ be a partially ordered set and $(X, G)$ be a complete G - cone metric space over a Banach algebra A. Suppose that the mapping $S, T : X \to X$ be two weakly comparable maps and the following two assertions hold:*

1. *There exist $a, b, c \in P$ with $\rho(a) + 2\rho(b) + 2\rho(c) < 1$ such that*

$$G(Sx, Ty, z) \preceq aG(x, y, z) + b[G(x, Sx, z) + G(y, Ty, z)] + c[G(x, Ty, z) + G(y, Sx, z)] \quad (9)$$

   *for all $x, y, z \in X$;*

2. *S or T is continuous.*

*Then S and T have a common fixed point in X.*

*Proof.* Suppose that S and T satisfies condition (9) then for all $x, y \in X$

$$G(Sx, Ty, y) \preceq aG(x, y, y) + b[G(x, Sx, y) + G(y, Ty, y)] + c[G(x, Ty, y) + G(y, Sx, y)] \quad (10)$$

and

$$G(Sy, Tx, x) \preceq aG(y, x, x) + b[G(y, Sy, x) + G(x, Tx, x)] + c[G(y, Tx, x) + G(x, Sy, x)] \quad (11)$$

If X is symmetric G-cone metric space , then inequalities (10) and (11) give

$$d_G(S(x), T(y)) \preceq ad_G(x, y) + b[d_G(x, S(x)) + d_G(y, T(y))] + c[d_G(x, T(y)) + d_G(y, S(x))] \quad (12)$$

Let $y_0 \in X$ and a sequence $\{y_n\}$ in X define as $y_{2n+1} = S(y_{2n})$ and $y_{2n+2} = T(y_{2n+1})$ for all $n \geq 0$. Mapping S and T are weakly comparable, we have $y_1 = S(y_0)$ and $y_2 = T(y_1) = TS(y_0)$ are comparable, by similar this $y_3 = S(y_2) = ST(y_1)$ are also comparable, we have that $y_n, y_{n+1}$ are comparable for all $n \geq 1$. From condition (12) that

$$\begin{aligned} d_G(y_{2n+1}, y_{2n+2}) &= d_G(y_{2n}, y_{2n+2}) \\ &\preceq ad_G(y_{2n}, y_{2n+1}) + b[d_G(y_{2n}, y_{2n+1}) + d_G(y_{2n+1}, y_{2n+2})] + cd_G(y_{2n}, y_{2n+2}) \end{aligned}$$

$$\begin{aligned} &\preceq a d_G(y_{2n}, y_{2n+1}) + b[d_G(y_{2n}, y_{2n+1}) + d_G(y_{2n+1}, y_{2n+2})] \\ &+ c[d_G(y_{2n}, y_{2n+1}) + d_G(y_{2n+1}, y_{2n+2})] \end{aligned} \tag{13}$$

That is

$$(e - b - c)d_G(y_{2n+1}, y_{2n+2}) \preceq (a + b + c)d_G(y_{2n}, y_{2n+1}) \tag{14}$$

Since $\rho(a) + 2\rho(b) + 2\rho(c) < 1$, then $\rho(b + c) \leq \rho(b) + \rho(c) < 1$, and $(e - b - c)$ is invertible by Proposition 2.1. Then multiplying both sides with $\frac{1}{(e-b-c)}$, it follows that

$$d_G(y_{2n+1}, y_{2n+2}) \preceq \frac{(a + b + c)}{(e - b - c)}(a + b + c)d_G(y_{2n}, y_{2n+1}) \tag{15}$$

For all $n \geq 1$, repeated this we have

$$d_G(y_{n+1}, y_n) \preceq h^n d_G(y_1, y_0) \tag{16}$$

Where $h = \frac{(a+b+c)}{(e-b-c)}$. Let $m > n$,

$$\begin{aligned} d_G(y_m, y_n) &\preceq d_G(y_m, y_{m-1}) + \cdots + d_G(y_{n+1}, y_n) \\ &\preceq (h^{m-1} + \ldots h^n)d_G(y_1, y_0) \\ &= (e + h + \cdots + h^{m-n-1})h^n d_G(y_1, y_0) \\ &\preceq (\sum_{i=0}^{\infty} h^i)h^n d_G(y_1, y_0) \\ &\preceq \Big[\frac{h^n}{e - h}\Big]d_G(y_1, y_0) \end{aligned} \tag{17}$$

It is implies that $\{y_n\}$ is Cauchy sequence and X is complete, there exists $y^* \in X$ such that $y_n \to y^*(n \to \infty)$. We have assumed that S is continuous then it is clear $y^*$ is fixed point of S. Now we show that $y^*$ is also fixed point of T. Since $y^*$ is comparable, then by condition first, we have

$$\begin{aligned} d_G(S(y^*), T(y^*)) &\preceq a d_G(y^*, y^*) + b[d_G(y^*, S(y^*)) + d_G(y^*, T(y^*))] \\ &+ c[d_G(y^*, T(y^*)) + d_G(y^*, S(y^*))] \end{aligned} \tag{18}$$

And $d_G(y^*, T(y^*)) \preceq (b + c)d_G(y^*, T(y^*))$ that is $(e - b - c)d_G(y^*, S(y^*)) \preceq \theta$. Multiplying both sides by $\frac{1}{(e-b-c)}$, we get $d_G(y^*, T(y^*)) = \theta$. Hence $y^* = T(y^*)$. Therefore, it is proved that S and T have a common fixed point. □

**Theorem 3.4.** *Let $(X, \sqsubseteq)$ be a partially ordered set and $(X, G)$ be complete G-cone metric space over Banach algebra A. Suppose that the mapping $S, T : X \to X$ be two weakly comparable maps suppose that the two assertions hold:*

1. *There exist $a, b, c \in P$ with $\rho(a) + 2\rho(b) + 2\rho(c) < 1$ such that*

$$G(Sx, Ty, z) \preceq aG(x, y, z) + b[G(x, Sx, z) + G(y, Ty, z)] + c[G(x, Ty, z) + G(y, Sx, z)] \tag{19}$$

*for all $x, y, z \in X$;*

2. *if a sequence $\{y_n\}$ converges to $y$ in $X$ and $y_i, y_{i+1}$ are comparable for all $i \geq 0$ then $y_i, y$ are comparable.*

*Then $S$ and $T$ have a common fixed point in $X$.*

*Proof.* This theorem can be proved in same way as 3.2 and 3.3. So omit it. □

## 4. Conclusions

The aim of this paper is to introduce the concept of ordered generalize cone metric space with Banach algebra, which generalizes cone metric space with Banach algebra and we explain some properties of such metric spaces. In addition, we provide fixed point and common fixed theorems for generalize contraction mappings in such spaces. Also presented example is constructed to support our result. Our results extend and unify many existing results in the rescent literature ([1, 4, 16, 24]).

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