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On Schur and pseudo-Schur complements

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Abstract

The primary goal of this instructive paper is to present various useful identities and explore basic properties of the two concepts: *Schur* and Pseudo-*Schur* complements (named in honor of the mathematician *Isaai Schur*) which have been paid attention by some researchers in many fields of mathematics. *Schur* and Pseudo-*Schur* complements play a central role and they serve as a rich and powerful tool by many authors. I will go over numerous main formulas related with these two concepts and some of their applications (I will omit the proofs). This paper is intended to familiarize the reader with the *Schur* and Pseudo-*Schur* complements, and this by using a lucid style that will attract readers from diverse backgrounds. The literature on the subject is vast, and its applications far reaching. It is highly recommended that one examine more rigorously the references of this paper.

Keywords: *Schur* complement; pseudo-*Schur* complement; *Haynsworth* inertia formula; *Sylvester*'s Law of Inertia; matrix inequalities.

1. Introduction

The name *Schur* is associated with many terms and concepts that are widely used in a number of diverse fields of mathematics and engineering [34], in particular *Schur* and Pseudo-*Schur* complements are basic and powerful tools (but still quite unknown and not most frequently encountered) and very useful notions which play a fundamental role in several branches of mathematics including:

- Linear algebra (for instance: for solving linear systems) [100].
- Numerical analysis (*Schur* complement techniques could be beneficial in sequence transformation related to fixed point iterations for systems of linear and nonlinear equations and to numerical methods for differential equations) [16].
- Optimization (in particular, in approximation for *PDE* constrained optimization, the *Schur* complement appeared (in a special symmetric form) as the Hessian of the Lagrangian [66]).

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- Statistics (for example: the *Schur* complement arises in computation of the probability density function of multivariate normal distribution [70]).
- Operator theory and applied mathematics. A great deal of work on the topic has been done by a number of authors [125].

Also, thanks to *Schur* complement, a class of convex nonlinear inequalities that appears regularly in control problems is converted into the so-called linear matrix inequalities (*LMI*) (recognized in the early 1960's) [14,109,124]. Likewise, *Schur* complement occurs frequently in the theory of systems engineering. Considerable interest in recent work on *Schur* and Pseudo-*Schur* complements has been witnessed. There were some researchers of the 19^{th} and 20^{th} centuries who, implicitly, dealt with these subjects [89,125]. For further reading on the rich history of the development of the subject and related topics, several classical references are recommend, from which we mention only [4,5,7,8,10,13,17,22,29, 37,43,45,71,76,79,83,89,91,95,97,99,103,128,129].
The implicit manifestations of the *Schur* complement appeared early (first published in 1851 as an idea to study in more detail entries of t he matrix that appears in block Gaussian elimination process [89]) in the paper (On the relation between the minor determinants of linearly equivalent quadratic functions) of the British mathematician *James Joseph Sylvester* (1814—1897) [128], but the concept of *Schur* complement which was introduced in 1968 by the American mathematician *Emilie Virginia Haynsworth* (1916–1985) [23] has been since repeatedly discussed [7,49,94].

2. Preliminaries and Mainstays

Let A, B, C, D denote four matrices (complex or real, a greater algebraic generality is also possible) and $p, q \in \mathbb{N}^*$ such that A is a square matrix of order p, B is a matrix of size $p \times q$, C is a matrix of size $q \times p$ and D is a square matrix of order q. Let $\mathcal{M}$ be the 2×2 matrix block (partitioned) defined by

$$\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{1}$$

Throughout the paper, the matrix $\mathcal{M}$ will be assumed to be partitioned as in (2.1). In what follows, we give a famous definition [7,8,17,22,29,37,40,43,45,71,76,89,91,95,99,128,129]:

Definition 2.1.

- *Suppose the square matrix A is invertible. We call the Schur complement of the block entry A in $\mathcal{M}$, the square matrix $\mathcal{S}_{(A,\mathcal{M})}$ of order q defined by*

$$\mathcal{S}_{(A,\mathcal{M})} = D - CA^{-1}B. \tag{2}$$

Analogously

- *Assume that the square matrix D is invertible. We call the Schur complement of the block entry D in* $\mathcal{M}$*, the square matrix* $\mathcal{S}_{(D,\mathcal{M})}$ *of order p defined by*

$$\mathcal{S}_{(D,\mathcal{M})} = A - BD^{-1}C. \tag{3}$$

- *Suppose the matrix B is invertible (thus* $p = q$*). We call the Schur complement of the block entry B in* $\mathcal{M}$*, the square matrix* $\mathcal{S}_{(B,\mathcal{M})}$ *of order p defined by*

$$\mathcal{S}_{(B,\mathcal{M})} = C - DB^{-1}A. \tag{4}$$

- *Suppose the matrix C is invertible (thus* $p = q$*). We call the Schur complement of the block entry C in* $\mathcal{M}$*, the square matrix* $\mathcal{S}_{(C,\mathcal{M})}$ *of order p defined by*

$$\mathcal{S}_{(C,\mathcal{M})} = B - AC^{-1}D. \tag{5}$$

Remark 2.2. *The square matrix* $\mathcal{M}$ *is of order* $(p+q)$.

Remark 2.3. *When the matrices A, B, C and D are invertible, their Schur complements are respectively commonly denoted (in linear algebra) by*

$$\mathcal{M}/A, \mathcal{M}/B, \mathcal{M}/C, \mathcal{M}/D. \tag{6}$$

3. Elementary Properties

- Suppose the square matrix A is invertible and that $C = 0$ or $B = 0$ (this is the case where the square matrix $\mathcal{M}$ is upper or lower triangular). Then $\mathcal{S}_{(A,\mathcal{M})} = D$ and $\mathcal{S}_{(D,\mathcal{M})} = A$.
- ($\mathcal{M}$ is positive definite) $\Longrightarrow$ ($\mathcal{S}_{(A,\mathcal{M})}$ is positive definite) [37,99].
- Assume that $\mathcal{M}$ is real and positive definite. Then the following statements hold [7]
  (1) $\mathcal{M}^{-1}$ is positive definite.
  (2) $\mathcal{S}_{(A,\mathcal{M})}$ and $\mathcal{S}_{(D,\mathcal{M})}$ are both positive definite.
- Assume that the square matrix D is invertible. Then [97,129]
  (1) $det\mathcal{M} = detD.det\mathcal{S}_{(D,\mathcal{M})}$ (*Schur*'s determinantal formula).
  (2) ($\mathcal{M}$ is invertible) $\Longleftrightarrow$ ($\mathcal{S}_{(D,\mathcal{M})}$ is invertible).
- Likewise, if the matrix A is invertible. Then [17,30,37,89,129]
  (1) $det\mathcal{M} = detA.det\mathcal{S}_{(A,\mathcal{M})}$ (*Schur*'s determinantal formula).
  (2) ($\mathcal{M}$ is invertible) $\Longleftrightarrow$ ($\mathcal{S}_{(A,\mathcal{M})}$ is invertible).

- Suppose both A and D are invertible. If $\mathcal{M}$ is invertible then we have an alternative expression of $\mathcal{M}^{-1}$ [7,47]

$$\mathcal{M}^{-1} = \begin{pmatrix} \mathcal{S}^{-1}_{(D,\mathcal{M})} & -A^{-1}B\mathcal{S}^{-1}_{(A,\mathcal{M})} \\ -D^{-1}C\mathcal{S}^{-1}_{(D,\mathcal{M})} & \mathcal{S}^{-1}_{(A,\mathcal{M})} \end{pmatrix} (\textit{Banachiewicz-Frazer-Duncan-Collar}) \tag{7}$$

3.1 Gaussian factorization

Assume that the square matrix A is invertible. Then the block matrix $\mathcal{M}$ is factorized as follows

$$\mathcal{M} = \begin{pmatrix} I_p & 0 \\ CA^{-1} & I_q \end{pmatrix} \cdot \begin{pmatrix} A & B \\ 0 & \mathcal{S}_{(A,\mathcal{M})} \end{pmatrix}$$

from which the *Schur*'s determinantal formula immediately holds.

Corollary 3.1 ([1,99,128])**.** *Assume that*

- $p = q$.
- *The square matrices A and C commute.*
- *The four square matrices A, B, C and D (which are of order p in this case) are invertible.*

Then the square matrix $\mathcal{M}$ is also invertible, and we have:

$$det\mathcal{M} = det(AD - CB)$$

and

$$\mathcal{M}^{-1} = \begin{pmatrix} \mathcal{S}^{-1}_{(D,\mathcal{M})} & \mathcal{S}^{-1}_{(B,\mathcal{M})} \\ \mathcal{S}^{-1}_{(C,\mathcal{M})} & \mathcal{S}^{-1}_{(A,\mathcal{M})} \end{pmatrix} \textit{(Banacheiwicz inversion formula, 1937).}$$

Theorem 3.2 (Crabtree-Haynsworth [30,49,52,93,97,99])**.** *Suppose the matrix A (the principal submatrix (of the matrix block $\mathcal{M}$) situated in its upper left-hand corner) is partitioned (two-way block) as follows*

$$A = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

where the four matrices E, F, G and H (complex or real, a greater algebraic generality is also possible), $p', q' \in \mathbb{N}^*$ such that $p = p' + q'$ and:
E is a square matrix of order p', F is a matrix of size $p' \times q'$, G is a matrix of size $q' \times p'$ and H is a square matrix of order q'. If A and E are invertible, then:

$$\mathcal{S}_{(A,\mathcal{M})} = \mathcal{S}_{(\mathcal{S}_{(E,A)},\mathcal{S}_{(E,\mathcal{M})})} \textit{ (quotient property) [20].}$$

Definition 3.3 (Hadamard product: Schur product [41,64,104,105]). *Let E and F be two matrices of size $m \times n$. The Hadamard product (Schur product) of E and F, denoted by $E \circ F$ is a matrix of size $m \times n$, with elements given by*

$$(E \circ F)_{ij} = (E)_{ij}(F)_{ij}$$

(this product is obtained by multiplying the corresponding entries in each matrix).

Remark 3.4. *For matrices E and F of different sizes $m \times n$ and $p \times q$, where $m \neq p$ or $n \neq q$, then the Hadamard product of E and F is undefined.*

3.2 Properties of *Hadamard* product

The *Hadamard* product is:

- Associative.
- Distributive (over addition).
- Commutative.
- Linear.

3.3 *Schur* Complement and Matrix Inequalities

We present some inequalities on generalized *Schur* complements [125]. The matrix inequalities are in elegant forms and have many interesting applications [78]. *Schur* complements are a powerful tool for deriving matrix inequalities and further in deducing determinant, trace, norm, eigenvalue, singular value, majorization, and other matrix inequalities [127].

Definition 3.5. *Let A and B be two Hermitian matrices of the same size. If $A - B$ is positive semidefinite, we write*

$$A \geq B \qquad or \qquad B \leq A.$$

4. Pseudo-*Schur* Complement

My purpose in this section is to present a formal treatment covering the Pseudo-*Schur* complement. Similarly to *Schur* complement, the Pseudo-*Schur* complement is defined as follows:

Definition 4.1. *Suppose the matrix A is singular and/or rectangular. We call the Pseudo-Schur complement of the block entry A in $\mathcal{M}$ (the concept reduces to the classical Schur complement), the matrix $\tilde{\mathcal{S}}_{(A,\mathcal{M})}$ defined by [80,91]*

$$\tilde{\mathcal{S}}_{(A,\mathcal{M})} = D - CA^{\dagger}B. \tag{8}$$

where, $A^\dagger$ is the Moore-Penrose inverse (or pseudo-inverse) of the matrix A [9,18,28,97,118,120,121]. It is the unique matrix satisfying all of the following four criteria (Penrose's conditions) [2,3,9,18,27,28,46,55,87]:

$$\begin{aligned} AA^\dagger A &= A \\ A^\dagger AA^\dagger &= A^\dagger \\ (AA^\dagger)^* &= AA^\dagger \\ (A^\dagger A)^* &= A^\dagger A \end{aligned} \tag{9}$$

(where * denotes the conjugate transpose of a matrix). These four equations are called the Moore-Penrose equations and the order in which they are written is crucial for our subsequent development [97].

Definition 4.2 ([33,106])**.** *A matrix which satisfies the first two Penrose's conditions is called by C. A. Rohde "a reflexive generalized inverse".*

Remark 4.3 ([62])**.** *Similarly, we also define*

$$\begin{aligned} \tilde{\mathcal{S}}_{(B,\mathcal{M})} &= C - DB^\dagger A. \\ \tilde{\mathcal{S}}_{(C,\mathcal{M})} &= B - AC^\dagger D. \\ \tilde{\mathcal{S}}_{(D,\mathcal{M})} &= A - BD^\dagger C. \end{aligned} \tag{10}$$

4.1 General properties of the pseudo-inverse matrix

We now turn our attention to an exploration of the most important properties of $A^\dagger$ [2,3,27,46,87,97]

$$\begin{aligned} A : m \text{ by } n &\Longrightarrow A^\dagger : \ n \text{ by } m \\ A = 0 &\Longrightarrow A^\dagger = 0 \\ (A^\dagger)^\dagger &= A \\ (A^\dagger)^t &= (A^t)^\dagger \\ A^\dagger (A^\dagger)^t &= (A^t A)^\dagger \\ A^\dagger &= (A^t A)^\dagger A^t \\ A^\dagger &= A^t (AA^t)^\dagger \\ rank(A^\dagger) &= rank(A). \end{aligned} \tag{11}$$

4.2 Famous application of the pseudo-inverse matrix

Let $n, m \in \mathbb{N}^*$, $N \in \mathbb{R}^{n\times m}$, $X \in \mathbb{R}^m$ and $U \in \mathbb{R}^n$. Let us consider the rectangular linear system $N.X = U$. Then the least squares solution of minimum Euclidian norm [53,118] of the problem

$$\underset{X\in\mathbb{R}^m}{argmin} \|N.X - U\| \tag{12}$$

is given by $X = N^{\dagger}.U$ ($N^{\dagger}$ is the pseudo-inverse of the matrix N).

Notation 4.4. *We denote by $\mathbb{H}_n$ the set of $n \times n$ Hermitian matrices.*

5. Special Cases

One of the most elegant results in matrix theory was derived for real symmetric matrices by *Cauchy* (or *Rayleigh*) [26,42,59,75,114]

Theorem 5.1 (*Cauchy* (eigenvalue) interlacing theorem []). *Suppose $C = B^*$. if $\mathcal{M} \in \mathbb{H}_n$ is partitioned as*

$$\mathcal{M} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \tag{13}$$

in which A is $p \times p$ principal submatrix, then for each $i = 1, 2, ..., p$:

$$\lambda_i(\mathcal{M}) \geq \lambda_i(A) \geq \lambda_{i+n-p}(\mathcal{M}). \tag{14}$$

Theorem 5.2 (*Fischer*'s inequality [38,57,86,88]). *In matrix theory, the well-known Fischer's inequality asserts that if the complex matrix $\mathcal{M}$ given by (1) is Hermitian and positive semidefinite, then*

$$det\mathcal{M} \leq detA.detD. \tag{15}$$

6. The Inertia Formula

Definition 6.1 ([12,31,57,58,60,92,120]). *Assume that the (generally complex) matrix $\mathcal{M}$ is Hermitian. The inertia of the matrix $\mathcal{M}$ (denoted by $In(\mathcal{M})$) is the ordered triple of nonnegative integers*

$$In(\mathcal{M}) = (n_+(\mathcal{M}), n_-(\mathcal{M}), n_0(\mathcal{M})) \tag{16}$$

in which $n_+(\mathcal{M})$, $n_-(\mathcal{M})$ and $n_0(\mathcal{M})$) give the number of positive, negative and zero eigenvalues of $\mathcal{M}$ counted with multiplicities, respectively.
(Recall that all eigenvalues of a Hermitian matrix are real).
Notice that [42,61,63,120]

- *The rank of $\mathcal{M}$ is equal to*

$$rank(\mathcal{M}) = n_+(\mathcal{M}) + n_-(\mathcal{M}). \tag{17}$$

- *The signature of $\mathcal{M}$ is equal to*

$$sgn(\mathcal{M}) = n_+(\mathcal{M}) - n_-(\mathcal{M}). \tag{18}$$

Proposition 6.2 ([42,97]). *Let $n_+(\mathcal{M})$ and $n_-(\mathcal{M})$ denote the number of positive and negative eigenvalues of*

$\mathcal{M}$ *respectively (and likewise for A). Then:*

$$n_+(A) \leq n_+(\mathcal{M}),$$
$$n_-(A) \leq n_-(\mathcal{M}).$$

6.1 The *Haynsworth* additivity formula for inertia

Assume that the matrix A is invertible. The *Haynsworth* formula for inertia (1968) of the matrix $\mathcal{M}$ is given by [40,43,51,60,89,95,97,121]

$$In(\mathcal{M}) = In(A) + In(\mathcal{S}_{(A,\mathcal{M})}). \tag{19}$$

6.2 *Sylvester*'s Law of Inertia

Let A and B be two Hermitian square matrices of order n. Then we have the following statement [12,31,54,55,63,65,74,107,110]

$$\left(\begin{array}{c} \exists G \text{ a square invertible matrix of order } n \text{ such that:} \\ B = G^* A G \end{array} \right) \Longleftrightarrow (In(A) = In(B)).$$

(The inertia of a matrix A is preserved under conjugation).

For further examples and properties, the reader may consult [55].

7. *Guttman* rank

Assume that the matrix $\mathcal{M}$ is Hermitian. Then the *Louis Guttman* rank additivity formula (first published in 1946) for the matrix $\mathcal{M}$ is defined by [21,39,43,95,97]

$$\begin{aligned} rank(\mathcal{M}) &= rank(A) + rank(\mathcal{S}_{(A,\mathcal{M})}) \text{ (when the matrix } A \text{ is invertible)} \\ &= rank(D) + rank(\mathcal{S}_{(D,\mathcal{M})}) \text{ (when the matrix } D \text{ is invertible).} \end{aligned} \tag{20}$$

8. *Schur* complements-linear systems

If both A and D are square and invertible, and if we consider the system

$$\mathcal{M}. \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \tag{21}$$

then, the solution of 21 is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mathcal{S}^{-1}_{(D,\mathcal{M})}.(u - BD^{-1}v) \\ \mathcal{S}^{-1}_{(A,\mathcal{M})}.(v - CA^{-1}u) \end{pmatrix}.$$

9. Other concepts

Among other concepts that include *Schur* complement and Pseudo–*Schur* complement, we can mention: Eigenvalue inequalities of *Schur* complements (see [125]). A great number of inequalities on eigenvalues and singular values of matrices can be found in the literature (see [123]), Ordinary products and *Schur* complements (see [125]), Inequalities related to partitioned positive semi-definite matrices (one can see [88]), Inequality relating the *Schur* complement of the *Hadamard* product and the *Hadamard* product of *Schur* complements for positive definite matrices (see [81]), Inequalities for inertias of block Hermitian matrices (see [120]) and Inequalities for positive semi-definite Hermitian matrices (see [20]).

10. Conclusion

This paper highlights and emphasizes on *Schur*'s and Pseudo-*Schur*'s complements, it is geared toward a wide audience of readers, from graduate students to professional mathematicians, interested in the topic. Several significant results (that characterize *Schur*'s and Pseudo-*Schur*'s complements) are proposed. The reader may examine rigorously the large bibliography (accompanying this paper) that contains a major contribution to *Schur*'s and Pseudo-*Schur*'s complements.

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